Chain Transitivity and Variations of the Shadowing Property

Jonathan Meddaugh

Will Brian and Brian Raines

Baylor University

The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications
July 10th, 2014

Outline

Outline

Basic Terminology

- A *dynamical system* is a continuous map f on a compact metric space (X, d).

Basic Terminology

- A *dynamical system* is a continuous map f on a compact metric space (X, d).
- An *orbit* for f is a sequence of the form $\langle f^i(x)\rangle_{i\in\mathbb{N}}$ for some $x \in X$.

Chain Transitivity and Variations of the Shadowing Property
Preliminaries
Definitions

Basic Terminology

- A *dynamical system* is a continuous map f on a compact metric space (X, d).
- An *orbit* for f is a sequence of the form $\langle f^i(x)\rangle_{i\in\mathbb{N}}$ for some $x \in X$.
- For $\delta > 0$, a δ-pseudo-orbit is a sequence $\langle z_i\rangle_{i\in\mathbb{N}}$ in X satisfying $d(z_{i+1}, f(z_i)) < \delta$ for $i \in \mathbb{N}$.

Shadowing

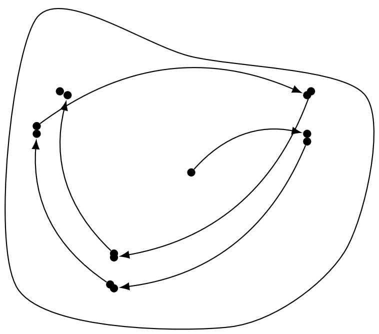
- A map f has shadowing provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for every δ-pseudo-orbit $\langle z_i\rangle$ there exists $x \in X$ such that $d(z_i, f^i(x)) < \epsilon$ for all $i \in \mathbb{N}$.

Chain Transitivity and Variations of the Shadowing Property
Preliminaries
Definitions

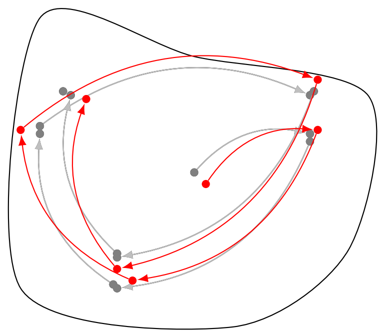
Shadowing

- A map f has shadowing provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for every δ-pseudo-orbit $\langle z_i\rangle$ there exists $x \in X$ such that $d(z_i, f^i(x)) < \epsilon$ for all $i \in \mathbb{N}$.
- The point x is said to ϵ-shadow the pseudo-orbit $\langle z_i\rangle$.

Shadowing



Shadowing



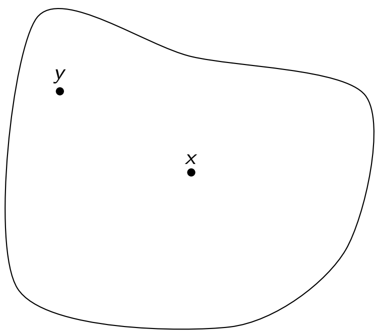
Chain Transitivity

- A δ-chain from x to y is a sequence $x = z_0, z_1, \ldots z_n = y$ in X which satisfies $d(z_{i+1}, f(z_i)) < \delta$ for $i < n$.

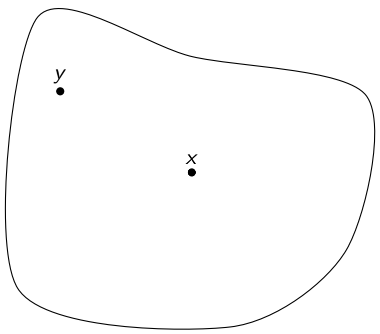
Chain Transitivity

- A δ-chain from x to y is a sequence $x = z_0, z_1, \ldots z_n = y$ in X which satisfies $d(z_{i+1}, f(z_i)) < \delta$ for $i < n$.
- A map f is *chain transitive* provided that for all $\delta > 0$ and all $x, y \in X$, there exists a δ-chain from x to y.

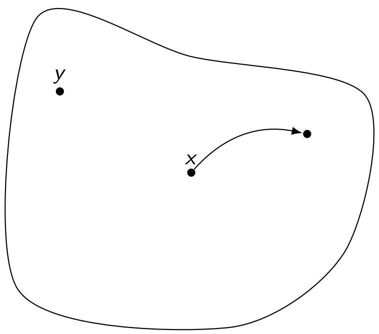
Chain Transitivity



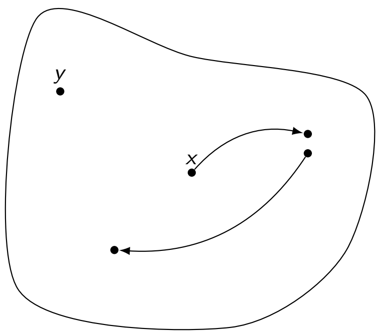
Chain Transitivity



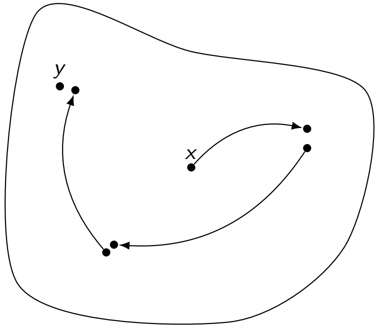
Chain Transitivity



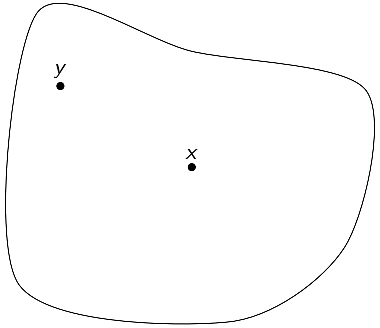
Chain Transitivity



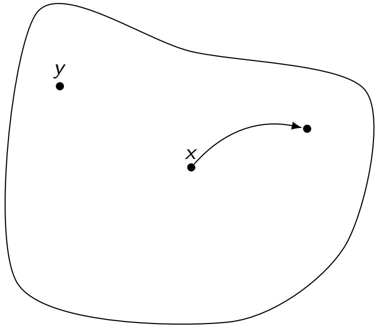
Chain Transitivity



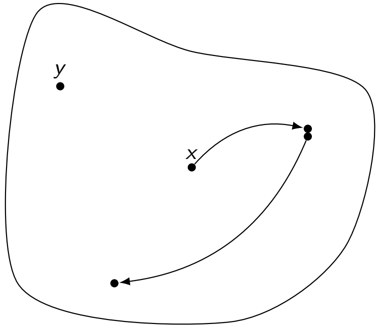
Chain Transitivity



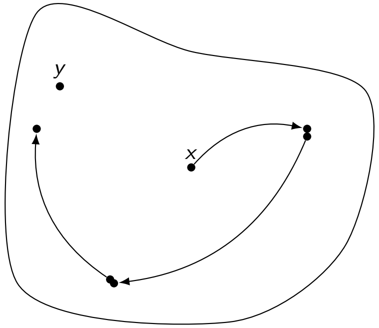
Chain Transitivity



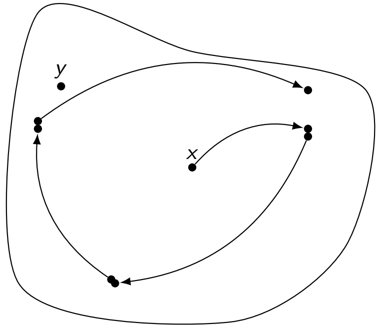
Chain Transitivity



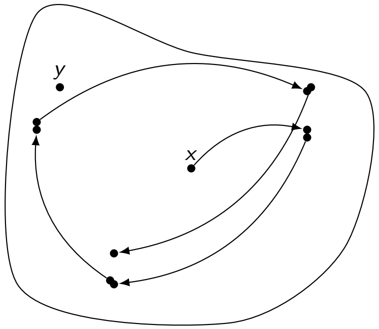
Chain Transitivity



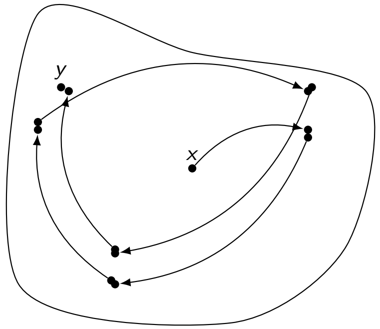
Chain Transitivity



Chain Transitivity



Chain Transitivity



Terminology

- A sequence $\langle z_i \rangle$ is a *δ-pseudo-orbit on A* provided that $A \subseteq \{i \in \mathbb{N} : d(z_{i+1}, f(z_i)) < \delta\}$.

Chain Transitivity and Variations of the Shadowing Property
Preliminaries
Variations on Shadowing

Terminology

- A sequence $\langle z_i \rangle$ is a *δ-pseudo-orbit on A* provided that $A \subseteq \{i \in \mathbb{N} : d(z_{i+1}, f(z_i)) < \delta\}$.
- A point $x \in X$ *ϵ-shadows $\langle z_i \rangle$ on B* provided that $B \subseteq \{i \in \mathbb{N} : d(z_i, f^i(x)) < \epsilon\}$.

$(\mathcal{F}, \mathcal{G})$-shadowing

- A *family* $\mathcal{F}$ is a collection of subsets of $\mathbb{N}$ for which $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.

Chain Transitivity and Variations of the Shadowing Property
Preliminaries
Variations on Shadowing

$(\mathcal{F}, \mathcal{G})$-shadowing

- A *family* $\mathcal{F}$ is a collection of subsets of $\mathbb{N}$ for which $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.
- For families $\mathcal{F}$ and $\mathcal{G}$, a map f has $(\mathcal{F}, \mathcal{G})$-shadowing provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\langle z_i \rangle$ is a δ-pseudo-orbit on a set $A \in \mathcal{F}$ then there exists a point $x \in X$ which ϵ-shadows $\langle z_i \rangle$ on a set $B \in \mathcal{G}$.

$(\mathcal{F}, \mathcal{G})$-shadowing

- A *family* $\mathcal{F}$ is a collection of subsets of $\mathbb{N}$ for which $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.
- For families $\mathcal{F}$ and $\mathcal{G}$, a map f has $(\mathcal{F}, \mathcal{G})$-shadowing provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\langle z_i \rangle$ is a δ-pseudo-orbit on a set $A \in \mathcal{F}$ then there exists a point $x \in X$ which ϵ-shadows $\langle z_i \rangle$ on a set $B \in \mathcal{G}$.

Theorem [BMR]

Suppose that $\mathcal{F} \supseteq \mathcal{F}'$ and that $\mathcal{G} \subseteq \mathcal{G}'$. Then every space with $(\mathcal{F}, \mathcal{G})$-shadowing has $(\mathcal{F}', \mathcal{G}')$-shadowing.

Variations on Shadowing

- Many commonly used variations on shadowing are of this form for appropriate families $\mathcal{F}$ and $\mathcal{G}$.

Variations on Shadowing

- Many commonly used variations on shadowing are of this form for appropriate families $\mathcal{F}$ and $\mathcal{G}$.
- Let $\mathcal{T}$ denote the family of thick subsets of $\mathbb{N}$, i.e. those sets $A \subseteq \mathbb{N}$ containing arbitrarily long intervals.

Variations on Shadowing

- Many commonly used variations on shadowing are of this form for appropriate families $\mathcal{F}$ and $\mathcal{G}$.
- Let $\mathcal{T}$ denote the family of thick subsets of $\mathbb{N}$, i.e. those sets $A \subseteq \mathbb{N}$ containing arbitrarily long intervals.
- Let $\mathcal{D}$ denote the family of subsets of $\mathbb{N}$ with lower density equal to 1.

Variations on Shadowing

- Immediately, $(\{\mathbb{N}\}, \{\mathbb{N}\})$-shadowing is the usual shadowing.

Variations on Shadowing

- Immediately, $(\{\mathbb{N}\}, \{\mathbb{N}\})$-shadowing is the usual shadowing.
- $(\mathcal{D}, \mathcal{T})$-shadowing is thick shadowing [Dastjerdi, Hosseini 2010]

Variations on Shadowing

- Immediately, $(\{\mathbb{N}\}, \{\mathbb{N}\})$-shadowing is the usual shadowing.
- $(\mathcal{D}, \mathcal{T})$-shadowing is thick shadowing [Dastjerdi, Hosseini 2010]
- $(\mathcal{D}, \mathcal{D})$-shadowing is ergodic shadowing [Fakhari, Gane 2010]

Variations on Shadowing

- Immediately, $(\{\mathbb{N}\}, \{\mathbb{N}\})$-shadowing is the usual shadowing.
- $(\mathcal{D}, \mathcal{T})$-shadowing is thick shadowing [Dastjerdi, Hosseini 2010]
- $(\mathcal{D}, \mathcal{D})$-shadowing is ergodic shadowing [Fakhari, Gane 2010]
- Several other shadowing subtypes fit this framework (though not all.)

Outline

Chain transitivity

Lemma [Richeson, Wiseman 2008]

Let $f : X \to X$ be chain transitive and let $\delta > 0$. Then there exists $k_\delta \in \mathbb{N}$ such that for any $x \in X$, k_δ is te greatest common denominator of the lengths of δ-chains from x to x.

Chain transitivity

Lemma [Richeson, Wiseman 2008]

Let $f : X \to X$ be chain transitive and let $\delta > 0$. Then there exists $k_\delta \in \mathbb{N}$ such that for any $x \in X$, k_δ is te greatest common denominator of the lengths of δ-chains from x to x.

- Define the relation $\sim_\delta$ on x by $x \sim_\delta y$ provided that there is a δ-chain from x to y of length a multiple of k_δ.

Chain transitivity

Lemma [Richeson, Wiseman 2008]

Let $f : X \to X$ be chain transitive and let $\delta > 0$. Then there exists $k_\delta \in \mathbb{N}$ such that for any $x \in X$, k_δ is te greatest common denominator of the lengths of δ-chains from x to x.

- Define the relation $\sim_\delta$ on x by $x \sim_\delta y$ provided that there is a δ-chain from x to y of length a multiple of k_δ.
- The are precisely k_δ many equivalence classes of $\sim_\delta$ which are clopen and are permuted cyclicly by f.

Chain Lengths

Lemma [BMR]

Let $f : X \to X$ be chain transitive. For each $\delta > 0$ there exists $M \in \mathbb{N}$ such that for any $m \geq M$, and any $x, y \in X$ with $x \sim_\delta y$, there is a δ-chain from x to y of length exactly mk_δ.

Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitiivity
Lemmas

Chain Lengths

Lemma [BMR]

Let $f : X \to X$ be chain transitive. For each $\delta > 0$ there exists $M \in \mathbb{N}$ such that for any $m \geq M$, and any $x, y \in X$ with $x \sim_\delta y$, there is a δ-chain from x to y of length exactly mk_δ.

- This is a straightforward application of the fact that δ-chains can be concatenated and Schur's Theorem.

Main Theorem

Theorem [BMR]

For a chain transitive dynamical system, the following are equivalent:

1. shadowing, i.e. $(\{\mathbb{N}\}, \{\mathbb{N}\})$-shadowing,
2. $(\mathcal{T}, \mathcal{T})$-shadowing,
3. thick shadowing, i.e. $(\mathcal{D}, \mathcal{T})$-shadowing, and
4. $(\{\mathbb{N}\}, \mathcal{T})$-shadowing.

Sketch of Proof

- First, note that $\{\mathbb{N}\} \subset \mathcal{D} \subset \mathcal{T}$ so as an application of the earlier theorem, (2) implies (3) and (3) implies (4).

Sketch of Proof

- First, note that $\{\mathbb{N}\} \subset \mathcal{D} \subset \mathcal{T}$ so as an application of the earlier theorem, (2) implies (3) and (3) implies (4).
- So, we need only establish that (1) implies (2) and (4) implies (1).

(4) implies (1)

- It is sufficient to show that for any $\epsilon > 0$ we can find $\delta > 0$ such that any δ-chain in X can be ϵ-shadowed.

(4) implies (1)

- It is sufficient to show that for any $\epsilon > 0$ we can find $\delta > 0$ such that any δ-chain in X can be ϵ-shadowed.
- Let $\epsilon > 0$ and let $\delta > 0$ be given by $(\{\mathbb{N}\}, \mathcal{T})$-shadowing.

(4) implies (1)

- It is sufficient to show that for any $\epsilon > 0$ we can find $\delta > 0$ such that any δ-chain in X can be ϵ-shadowed.
- Let $\epsilon > 0$ and let $\delta > 0$ be given by $(\{\mathbb{N}\}, \mathcal{T})$-shadowing.
- Fix a δ-chain $z_0, z_1, \ldots z_n$. Since f is chain transitive we can find a δ-chain $z_n, y_1, y_2, \ldots y_m, z_0$ from z_n to z_0.

(4) implies (1)

- It is sufficient to show that for any $\epsilon > 0$ we can find $\delta > 0$ such that any δ-chain in X can be ϵ-shadowed.
- Let $\epsilon > 0$ and let $\delta > 0$ be given by $(\{\mathbb{N}\}, \mathcal{T})$-shadowing.
- Fix a δ-chain $z_0, z_1, \ldots z_n$. Since f is chain transitive we can find a δ-chain $z_n, y_1, y_2, \ldots y_m, z_0$ from z_n to z_0.
- Then $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ is a δ-pseudo-orbit.

(4) implies (1)

- Let $x \in X$ shadow $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ on a set $A \in \mathcal{T}$.

(4) implies (1)

- Let $x \in X$ shadow $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ on a set $A \in \mathcal{T}$.
- Since A is thick, it contains arbitrarily long sequences of consecutive integers.

(4) implies (1)

- Let $x \in X$ shadow $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ on a set $A \in \mathcal{T}$.
- Since A is thick, it contains arbitrarily long sequences of consecutive integers.
- In particular, one long enough to gaurantee that x shadows the pseudo-orbit on some segment coinciding with $z_0, z_1, \ldots z_n$.

(4) implies (1)

- Let $x \in X$ shadow $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ on a set $A \in \mathcal{T}$.
- Since A is thick, it contains arbitrarily long sequences of consecutive integers.
- In particular, one long enough to gaurantee that x shadows the pseudo-orbit on some segment coinciding with $z_0, z_1, \ldots z_n$.
- Then, the approriate iterate of x shadows the δ-chain $z_0, z_1, \ldots z_n$.

(1) implies (2)

- We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that for any δ-pseudo-orbit $\langle z_i \rangle$ on a set $A \in \mathcal{T}$, there is an $x \in X$ that ϵ-shadows it on a set $B \in \mathcal{T}$.

(1) implies (2)

- We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that for any δ-pseudo-orbit $\langle z_i \rangle$ on a set $A \in \mathcal{T}$, there is an $x \in X$ that ϵ-shadows it on a set $B \in \mathcal{T}$.
- Our strategy is to construct a proper δ-pseudo-orbit $\langle q_i \rangle$ which agrees with $\langle z_i \rangle$ on a thick set and then find a point x that shadows this modified pseudo-orbit.

(1) implies (2)

- We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that for any δ-pseudo-orbit $\langle z_i \rangle$ on a set $A \in \mathcal{T}$, there is an $x \in X$ that ϵ-shadows it on a set $B \in \mathcal{T}$.
- Our strategy is to construct a proper δ-pseudo-orbit $\langle q_i \rangle$ which agrees with $\langle z_i \rangle$ on a thick set and then find a point x that shadows this modified pseudo-orbit.
- The point x will then shadow the original pseudo-orbit on a thick set as desired.

(1) implies (2)

- Let $\epsilon > 0$ and fix $\delta > 0$ as given by shadowing. Let $\langle z_i \rangle$ be a δ-pseudo-orbit on T where $T \in \mathcal{T}$.

Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitiivity
Theorem

(1) implies (2)

- Let $\epsilon > 0$ and fix $\delta > 0$ as given by shadowing. Let $\langle z_i \rangle$ be a δ-pseudo-orbit on T where $T \in \mathcal{T}$.
- Let $K = k_\delta$ and let $X_0, X_1, \ldots X_K$ be the equivalence classes of $\sim_\delta$ named so that $f(X_i) = X_{i+1}$ mod K.

Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitiivity
Theorem

(1) implies (2)

- Let $\epsilon > 0$ and fix $\delta > 0$ as given by shadowing. Let $\langle z_i \rangle$ be a δ-pseudo-orbit on T where $T \in \mathcal{T}$.
- Let $K = k_\delta$ and let $X_0, X_1, \ldots X_K$ be the equivalence classes of $\sim_\delta$ named so that $f(X_i) = X_{i+1}$ mod K.
- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$.

Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitiivity
Theorem

(1) implies (2)

- Let $\epsilon > 0$ and fix $\delta > 0$ as given by shadowing. Let $\langle z_i \rangle$ be a δ-pseudo-orbit on T where $T \in \mathcal{T}$.
- Let $K = k_\delta$ and let $X_0, X_1, \ldots X_K$ be the equivalence classes of $\sim_\delta$ named so that $f(X_i) = X_{i+1}$ mod K.
- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$.
- If $d(z_{i+1}, f(z_i)) < \delta$ it follows that $m(i) = m(i+1)$.

(1) implies (2)

- Let $\epsilon > 0$ and fix $\delta > 0$ as given by shadowing. Let $\langle z_i \rangle$ be a δ-pseudo-orbit on T where $T \in \mathcal{T}$.
- Let $K = k_\delta$ and let $X_0, X_1, \ldots X_K$ be the equivalence classes of $\sim_\delta$ named so that $f(X_i) = X_{i+1} \bmod K$.
- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$.
- If $d(z_{i+1}, f(z_i)) < \delta$ it follows that $m(i) = m(i+1)$.
- Let $A = \{i \in \mathbb{N} : m(i) = m(i+1)\}$, and notice that this contains T and is hence thick.

(1) implies (2)

- Let $A_k = \{i \in A : m(i) = k\}$ and notice that for some k, A_k is thick. Without loss, A_0.

(1) implies (2)

- Let $A_k = \{i \in A : m(i) = k\}$ and notice that for some k, A_k is thick. Without loss, A_0.
- By previous lemma, fix $M \in \mathbb{N}$ such that for all $m \geq M$, and any $x, y \in X_0$ there is a δ-chain of length mK from x to y.

(1) implies (2)

- Let $A_k = \{i \in A : m(i) = k\}$ and notice that for some k, A_k is thick. Without loss, A_0.
- By previous lemma, fix $M \in \mathbb{N}$ such that for all $m \geq M$, and any $x, y \in X_0$ there is a δ-chain of length mK from x to y.
- We can then leverage the thickness to replace segments of $\langle z_i \rangle$ for which $m(i) \neq 0$ (and some parts where $m(i) = 0$ as well) with δ-chains of lengths mK.

(1) implies (2)

- Let $A_k = \{i \in A : m(i) = k\}$ and notice that for some k, A_k is thick. Without loss, A_0.
- By previous lemma, fix $M \in \mathbb{N}$ such that for all $m \geq M$, and any $x, y \in X_0$ there is a δ-chain of length mK from x to y.
- We can then leverage the thickness to replace segments of $\langle z_i \rangle$ for which $m(i) \neq 0$ (and some parts where $m(i) = 0$ as well) with δ-chains of lengths mK.
- In particular, do this in such a way that we retain subintervals of A_0 of arbitrary length.

Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitiivity
Theorem

(1) implies (2)

- Let $A_k = \{i \in A : m(i) = k\}$ and notice that for some k, A_k is thick. Without loss, A_0.
- By previous lemma, fix $M \in \mathbb{N}$ such that for all $m \geq M$, and any $x, y \in X_0$ there is a δ-chain of length mK from x to y.
- We can then leverage the thickness to replace segments of $\langle z_i \rangle$ for which $m(i) \neq 0$ (and some parts where $m(i) = 0$ as well) with δ-chains of lengths mK.
- In particular, do this in such a way that we retain subintervals of A_0 of arbitrary length.
- The modified sequence $\langle q_i \rangle$ is now a proper δ-pseudo-orbit and agrees with $\langle z_i \rangle$ on a thick set.

Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitiivity
Theorem

Thank you

Thank you!