Chain Transitivity and Variations of the Shadowing Property

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Outline

1 Preliminaries

2 Shadowing and Chain Transitivity
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1 Preliminaries
   Definitions
   Variations on Shadowing

2 Shadowing and Chain Transitivity
Basic Terminology

- A *dynamical system* is a continuous map $f$ on a compact metric space $(X, d)$. 
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- An *orbit* for $f$ is a sequence of the form $\langle f^i(x) \rangle_{i \in \mathbb{N}}$ for some $x \in X$.
- For $\delta > 0$, a $\delta$-pseudo-orbit is a sequence $\langle z_i \rangle_{i \in \mathbb{N}}$ in $X$ satisfying $d(z_{i+1}, f(z_i)) < \delta$ for $i \in \mathbb{N}$. 
A map $f$ has shadowing provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\delta$-pseudo-orbit $\langle z_i \rangle$ there exists $x \in X$ such that $d(z_i, f^i(x)) < \epsilon$ for all $i \in \mathbb{N}$. 
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The point $x$ is said to $\epsilon$-shadow the pseudo-orbit $\langle z_i \rangle$. 
Shadowing
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A $\delta$-chain from $x$ to $y$ is a sequence $x = z_0, z_1, \ldots z_n = y$ in $X$ which satisfies $d(z_{i+1}, f(z_i)) < \delta$ for $i < n$. 
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A map $f$ is chain transitive provided that for all $\delta > 0$ and all $x, y \in X$, there exists a $\delta$-chain from $x$ to $y$. 
Chain Transitivity
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![Diagram showing chain transitivity with points x and y connected by a path through a region.]

Preliminaries
Definitions
Chain Transitivity
Chain Transitivity

![Diagram of Chain Transitivity]

- **Chain Transitivity**

- Preliminaries
- Definitions
Chain Transitivity

\[ x \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow y \]
Chain Transitivity
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Terminology

- A sequence \( \langle z_i \rangle \) is a \( \delta \)-pseudo-orbit on \( A \) provided that \( A \subseteq \{ i \in \mathbb{N} : d(z_{i+1}, f(z_i)) < \delta \} \).
A sequence $\langle z_i \rangle$ is a $\delta$-pseudo-orbit on $A$ provided that $A \subseteq \{ i \in \mathbb{N} : d(z_{i+1}, f(z_i)) < \delta \}$.

A point $x \in X$ $\epsilon$-shadows $\langle z_i \rangle$ on $B$ provided that $B \subseteq \{ i \in \mathbb{N} : d(z_i, f^i(x)) < \epsilon \}$.
A family $\mathcal{F}$ is a collection of subsets of $\mathbb{N}$ for which $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.
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For families $\mathcal{F}$ and $\mathcal{G}$, a map $f$ has $(\mathcal{F}, \mathcal{G})$-shadowing provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\langle z_i \rangle$ is a $\delta$-pseudo-orbit on a set $A \in \mathcal{F}$ then there exists a point $x \in X$ which $\epsilon$-shadows $\langle z_i \rangle$ on a set $B \in \mathcal{G}$. 

Theorem [BMR]

Suppose that $\mathcal{F} \supseteq \mathcal{F}'$ and that $\mathcal{G} \subseteq \mathcal{G}'$. Then every space with $(\mathcal{F}, \mathcal{G})$-shadowing has $(\mathcal{F}', \mathcal{G}')$-shadowing.
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Variations on Shadowing

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- Let $\mathcal{T}$ denote the family of thick subsets of $\mathbb{N}$, i.e. those sets $A \subseteq \mathbb{N}$ containing arbitrarily long intervals.
- Let $\mathcal{D}$ denote the family of subsets of $\mathbb{N}$ with lower density equal to 1.
Variations on Shadowing

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- $(\mathcal{D}, \mathcal{D})$-shadowing is ergodic shadowing [Fakhari, Gane 2010]
Variations on Shadowing

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- $(\mathcal{D}, \mathcal{D})$-shadowing is ergodic shadowing [Fakhari, Gane 2010]
- Several other shadowing subtypes fit this framework (though not all.)
Outline

1 Preliminaries

2 Shadowing and Chain Transitivity
   Lemmas
   Theorem
Lemma [Richeson, Wiseman 2008]

Let $f : X \rightarrow X$ be chain transitive and let $\delta > 0$. Then there exists $k_\delta \in \mathbb{N}$ such that for any $x \in X$, $k_\delta$ is the greatest common denominator of the lengths of $\delta$-chains from $x$ to $x$. 
Chain transitivity

Lemma [Richeson, Wiseman 2008]
Let \( f : X \to X \) be chain transitive and let \( \delta > 0 \). Then there exists \( k_\delta \in \mathbb{N} \) such that for any \( x \in X \), \( k_\delta \) is the greatest common denominator of the lengths of \( \delta \)-chains from \( x \) to \( x \).

- Define the relation \( \sim_\delta \) on \( x \) by \( x \sim_\delta y \) provided that there is a \( \delta \)-chain from \( x \) to \( y \) of length a multiple of \( k_\delta \).
Chain transitivity

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- Define the relation $\sim_\delta$ on $x$ by $x \sim_\delta y$ provided that there is a $\delta$-chain from $x$ to $y$ of length a multiple of $k_\delta$.
- There are precisely $k_\delta$ many equivalence classes of $\sim_\delta$ which are clopen and are permuted cyclicly by $f$. 
Lemma [BMR]

Let $f : X \to X$ be chain transitive. For each $\delta > 0$ there exists $M \in \mathbb{N}$ such that for any $m \geq M$, and any $x, y \in X$ with $x \sim_\delta y$, there is a $\delta$-chain from $x$ to $y$ of length exactly $mk_\delta$. 
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- This is a straightforward application of the fact that $\delta$-chains can be concatenated and Schur’s Theorem.
Main Theorem

Theorem [BMR]

For a chain transitive dynamical system, the following are equivalent:

1. shadowing, i.e. \((\mathbb{N}, \mathbb{N})\)-shadowing,
2. \((\mathcal{T}, \mathcal{T})\)-shadowing,
3. thick shadowing, i.e. \((\mathcal{D}, \mathcal{T})\)-shadowing, and
4. \((\mathbb{N}, \mathcal{T})\)-shadowing.
First, note that $\{\mathbb{N}\} \subset D \subset T$ so as an application of the earlier theorem, (2) implies (3) and (3) implies (4).
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So, we need only establish that (1) implies (2) and (4) implies (1).
Chain Transitivity and Variations of the Shadowing Property
Shadowing and Chain Transitivity
Theorem

(4) implies (1)

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- Let \( \epsilon > 0 \) and let \( \delta > 0 \) be given by \( (\mathbb{N}, T) \)-shadowing.
- Fix a \( \delta \)-chain \( z_0, z_1, \ldots, z_n \). Since \( f \) is chain transitive we can find a \( \delta \)-chain \( z_n, y_1, y_2, \ldots, y_m, z_0 \) from \( z_n \) to \( z_0 \).
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- Let $\epsilon > 0$ and let $\delta > 0$ be given by $({\mathbb N}, T)$-shadowing.
- Fix a $\delta$-chain $z_0, z_1, \ldots z_n$. Since $f$ is chain transitive we can find a $\delta$-chain $z_n, y_1, y_2, \ldots y_m, z_0$ from $z_n$ to $z_0$.
- Then $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ is a $\delta$-pseudo-orbit.
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- Let \( x \in X \) shadow
  \[ z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots \] on a set \( A \in \mathcal{T} \).
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- Let $x \in X$ shadow $z_0, z_1, \ldots, z_n, y_1, \ldots, y_m, z_0, \ldots, z_n, y_1, \ldots, y_m, \ldots$ on a set $A \in \mathcal{T}$.
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- Since $A$ is thick, it contains arbitrarily long sequences of consecutive integers.
- In particular, one long enough to guarantee that $x$ shadows the pseudo-orbit on some segment coinciding with $z_0, z_1, \ldots z_n$.
- Then, the appropriate iterate of $x$ shadows the $\delta$-chain $z_0, z_1, \ldots z_n$. 
(1) implies (2)

- We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that for any $\delta$-pseudo-orbit $\langle z_i \rangle$ on a set $A \in \mathcal{T}$, there is an $x \in X$ that $\epsilon$-shadows it on a set $B \in \mathcal{T}$. 
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- Our strategy is to construct a proper $\delta$-pseudo-orbit $\langle q_i \rangle$ which agrees with $\langle z_i \rangle$ on a thick set and then find a point $x$ that shadows this modified pseudo-orbit.
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The point $x$ will then shadow the original pseudo-orbit on a thick set as desired.
(1) implies (2)

- Let $\epsilon > 0$ and fix $\delta > 0$ as given by shadowing. Let $\langle z_i \rangle$ be a $\delta$-pseudo-orbit on $T$ where $T \in \mathcal{T}$. 
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Let $K = k_\delta$ and let $X_0, X_1, \ldots X_K$ be the equivalence classes of $\sim_\delta$ named so that $f(X_i) = X_{i+1} \text{ mod } K$. 

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- Let $K = k_\delta$ and let $X_0, X_1, \ldots X_K$ be the equivalence classes of $\sim_\delta$ named so that $f(X_i) = X_{i+1} \mod K$.
- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$. 
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- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$.
- If $d(z_{i+1}, f(z_i)) < \delta$ it follows that $m(i) = m(i + 1)$.
- Let $A = \{i \in \mathbb{N} : m(i) = m(i + 1)\}$, and notice that this contains $T$ and is hence thick.
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- Let $A_k = \{ i \in A : m(i) = k \}$ and notice that for some $k$, $A_k$ is thick. Without loss, $A_0$. 
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- By previous lemma, fix $M \in \mathbb{N}$ such that for all $m \geq M$, and any $x, y \in X_0$ there is a $\delta$-chain of length $mK$ from $x$ to $y$. 

We can then leverage the thickness to replace segments of $\langle z_i \rangle$ for which $m(i) \neq 0$ (and some parts where $m(i) = 0$ as well) with $\delta$-chains of lengths $mK$.

In particular, do this in such a way that we retain subintervals of $A_0$ of arbitrary length.

The modified sequence $\langle q_i \rangle$ is now a proper $\delta$-pseudo-orbit and agrees with $\langle z_i \rangle$ on a thick set.
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- The modified sequence $\langle q_i \rangle$ is now a proper $\delta$-pseudo-orbit and agrees with $\langle z_i \rangle$ on a thick set.
Thank you!