Complexity and Simplicity in the dynamics of Totally Transitive graph maps

Lluís Alsedà

in collaboration with Liane Bordignon and Jorge Groisman

Centre de Recerca Matemàtica

Departament de Matemàtiques
Universitat Autònoma de Barcelona

http://www.mat.uab.cat/∼alseda

MURPHYS 2018 Barcelona, May 28th, 2018

Outline

1. Introduction – Statement of the problem
2. Examples
3. Results for the circle and the σ graph.

Introduction – Statement of the problem

- Transitivity,
- the existence of infinitely many periods and
- positive topological entropy

often characterize the complexity in dynamical systems.

Definition

A map $f: X \longrightarrow X$ is *transitive* if for every pair of open subsets $U, V \subset X$ there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.
A map f is called *totally transitive* if all iterates of f are transitive.

Introduction – Statement of the problem

A transitive map on a graph has positive topological entropy and dense set of periodic points (except for an irrational rotation on the circle).

A. M. Blokh.
On transitive mappings of one-dimensional branched manifolds.
In *Differential-difference equations and problems of mathematical physics (Russian)*, pages 3–9, 131. Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1984.

A. M. Blokh.
The connection between entropy and transitivity for one-dimensional mappings.
Uspekhi Mat. Nauk, 42(5(257)):209–210, 1987.

Ll. Alsedà, M. A. Del Río, and J. A. Rodríguez.
A survey on the relation between transitivity and dense periodicity for graph maps.
J. Difference Equ. Appl., 9(3-4):281–288, 2003.
Dedicated to Professor Alexander N. Sharkovsky on the occasion of his 65th birthday.

Ll. Alsedà, M. A. del Río, and J. A. Rodríguez.
Transitivity and dense periodicity for graph maps.
J. Difference Equ. Appl., 9(6):577–598, 2003.

Introduction – Statement of the problem

Thus, in view of

J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey.
On Devaney's definition of chaos.
Amer. Math. Monthly, 99(4):332–334, 1992.

every transitive map on a graph is chaotic in the sense of Devaney (except, again, for an irrational rotation on the circle).

Moreover, a totally transitive map on a graph which is not an irrational rotation on the circle has *cofinite set of periods* (meaning that the complement of the set of periods is finite or, equivalently, that it contains all positive integers larger than a given one).

Ll. Alsedà, M. A. del Río, and J. A. Rodríguez.
A note on the totally transitive graph maps.
Internat. J. Bifur. Chaos Appl. Sci. Engrg., 11(3):841–843, 2001.

Introduction – Statement of the problem

Summarizing

Totally transitive maps on graphs are complicate since they have *positive topological entropy* and *cofinite set of periods*.

Introduction – Statement of the problem

However, for every graph that is not a tree and for every $\varepsilon > 0$, there exists a totally transitive map *with periodic points* such that its topological entropy is positive but smaller than ε.

Ll. Alsedà, M. A. del Río, and J. A. Rodríguez.
A splitting theorem for transitive maps.
J. Math. Anal. Appl., 232(2):359–375, 1999.

Summarizing again

The complicate totally transitive maps on graphs may be relatively simple because they may have *arbitrarily small positive topological entropy*.

Introduction – Statement of the problem

In this talk we consider the question whether the simplicity phenomenon that happens for the topological entropy can be extended to the set of periods. More precisely,

is it true that when a totally transitive graph map with periodic points has small positive topological entropy it also has small "cofinite part" of the set of periods?

To measure the size of the "cofinite part" of the set of periods we introduce the notion of *boundary of cofiniteness*.

Boundary of Cofiniteness definition

The *boundary of cofiniteness of* a totally transitive map f is defined as the largest positive integer $L \in \text{Per}(f)$, $L > 2$ such that $L - 1 \notin \text{Per}(f)$ but there exists $n \geq L$ such that $\text{Per}(f) \supset \{n, n+1, n+2, \dots\}$ and

$$\frac{\text{Card}(\{1, \dots, L-2\} \cap \text{Per}(f))}{L-2} \leq \frac{2\log_2(L-2)}{L-2}$$

That is, the cofinite par of the set of periods is beyond the boundary of cofiniteness and the density of the low periods is small.

The boundary of cofiniteness of f is denoted by $\text{BdCof}(f)$.

Introduction – Statement of the problem

Then, we can state precisely what do we mean by extending the entropy simplicity phenomenon to the set of periods: *there exist relatively simple maps such that the boundary of cofiniteness is arbitrarily large (simplicity) which are totally transitive (and hence robustly complicate)*.

We illustrate the above statement with three examples for arbitrary graphs which are not trees and we present the corresponding theorems for the circle and the σ graph.

Examples

Example I (with persistent fixed low periods)

Theorem

For every $n \in \{4k+1, 4k-1 \colon k \in \mathbb{N}\}$ there exists a totally transitive continuous circle map of degree one such that: $Rot(f_n) = \left[\frac{1}{2}, \frac{n+2}{2n}\right]$,

$$\text{Per}(f_n) = \{2\} \cup \{p \text{ odd} \colon 2k+1 \leq p \leq n-2\} \\ \cup \{n, n+1, n+2, \dots\}.$$

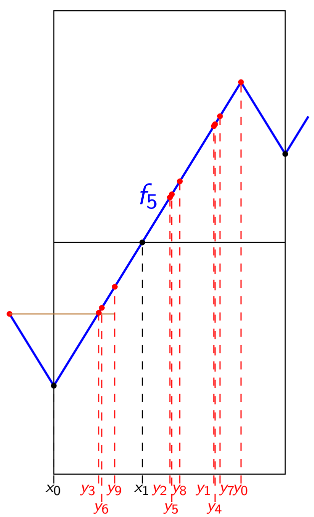
Moreover $\lim_{n\to\infty} h(f_n) = 0$. Furthermore, given any graph G with a circuit, the maps f_n can be extended to continuous totally transitive maps $\varphi_n \colon G \longrightarrow G$ so that $\text{Per}(g_n) = \text{Per}(f_n)$ but still $\lim_{n\to\infty} h(g_n) = 0$.

Remark

- $2k+1 \leq \text{BdCof}(g_n) = \text{BdCof}(f_n) < n$ and, hence, $\lim_{n\to\infty} \text{BdCof}(g_n) = \infty$.
- The density of "lower" periods outside the cofinite part converges to $\frac{1}{4}$ and there is a very small period 2.
- Despite of the fact that still $\lim_{n\to\infty} h(g_n) = 0$, in general, $h(g_n)$ is slightly larger than $h(f_n)$.

Examples — Example I: Idea of the construction

$n = 5$, $\text{Rot}(f_5) = \left[\frac{1}{2}, \frac{7}{10}\right]$, $\text{Per}(f_5) = \{2,3\} \cup \{5,6,7,\dots\}$ and $h(f_5) = \log 1.61960\dots$.



Note that

$$\{\dots, -x_1, x_0, x_1, 1+x_0, 1+x_1, \dots\}$$

and

$$\{\dots, -y_9, y_0, y_1, \dots, y_9, 1+y_0, 1+y_1, \dots\}$$

are *twist*, orbits with rotation numbers 1/2 and 7/10, respectively, since the map f_5 is globally increasing on each of them.

Hence, $\text{Rot}(f_5) = \left[\frac{1}{2}, \frac{7}{10}\right]$.

Thanks to the way the points of the two orbits are intertwined, the map f_5 is the one with the smallest entropy among the maps having rotation interval $\left[\frac{1}{2}, \frac{7}{10}\right]$. Thus, this construction gives the formula to mix both orbits so that the entropy is the smallest possible.

This is the key point in obtaining

$$\lim_{n\to\infty} h(f_n) = 0.$$

The set of periods is determined by the rotation interval (and the fact that f_5 minimizes the entropy).

Examples — Example II (with non-constant low periods)

Theorem

For every $n \in \mathbb{N}$ there exists a totally transitive continuous circle map of degree one such that: $\operatorname{Rot}(f_n) = \left[\frac{2n-1}{2n^2}, \frac{2n+1}{2n^2}\right] = \left[\frac{1}{n} - \frac{1}{2n^2}, \frac{1}{n} + \frac{1}{2n^2}\right]$,

$$\operatorname{Per}(f_n) = \{n\} \cup \{tn + k : t \in \{2,3,\dots,\nu-1\} \text{ and } -\tfrac{t}{2} < k \le \tfrac{t}{2},\ k \in \mathbb{Z}\} \cup \{L \in \mathbb{N} : L \ge n\nu + 1 - \tfrac{\nu}{2}\}$$

with

$$\nu = \begin{cases} n & \text{if } n \text{ is even, and} \\ n-1 & \text{if } n \text{ is odd;} \end{cases}$$

and $n \le BdCof(f_n) \le n\nu - 1 - \frac{\nu}{2}$.

Moreover $\lim_{n\to\infty} h(f_n) = 0$ *and* $\lim_{n\to\infty} BdCof(g_n) = \infty$. *Furthermore, given any graph G with a circuit, the maps f_n can be extended to continuous totally transitive maps* $\varphi_n \colon G \longrightarrow G$ *so that* $\operatorname{Per}(g_n) = \operatorname{Per}(f_n)$ *but still* $\lim_{n\to\infty} h(g_n) = 0$.

Examples — Example II (with non-constant low periods)

Remark

- The density of "lower" periods outside the cofinite part converges to $\frac{1}{2}$ but the smallest period is n.
- As before, despite of the fact that still $\lim_{n\to\infty} h(g_n) = 0$, in general, $h(g_n)$ is slightly larger than $h(f_n)$.

Examples — Example III (the dream example)

Theorem

For every $n \in \mathbb{N}$ there exists a totally transitive continuous circle map of degree one such that: $\operatorname{Rot}(f_n) = \left[\frac{1}{2n-1}, \frac{2}{2n-1}\right]$,

$$\operatorname{Per}(f_n) = \{n, n+1, n+2, \dots\}.$$

Moreover $\lim_{n\to\infty} h(f_n) = 0$. *Furthermore, given any graph G with a circuit, the maps f_n can be extended to continuous totally transitive maps* $\varphi_n \colon G \longrightarrow G$ *so that* $\operatorname{Per}(g_n) = \operatorname{Per}(f_n)$ *but still* $\lim_{n\to\infty} h(g_n) = 0$.

Remark

- $\operatorname{BdCof}(g_n) = \operatorname{BdCof}(f_n) = n$ and, hence, $\lim_{n\to\infty} \operatorname{BdCof}(g_n) = \infty$.
- There are no "lower" periods outside the cofinite part.
- As in the previous two cases, despite of the fact that still $\lim_{n\to\infty} h(g_n) = 0$, in general, $h(g_n)$ is slightly larger than $h(f_n)$.

Results for the circle and the σ graph.

Theorem

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of totally transitive circle maps of degree one with periodic points such that $\lim_{n\to\infty} h(f_n) = 0$. *For every n let $F_n \in \mathcal{L}_1$ be a lifting of f_n. Then,*

- $\lim_{n\to\infty} \operatorname{len}(Rot(F_n)) = 0$,
- *there exists $N \in \mathbb{N}$ such that $BdCof(f_n)$ exists for every $n \ge N$, and*
- $\lim_{n\to\infty} BdCof(f_n) = \infty$.

Results for the circle and the σ graph.

For *σ-maps* (continuous self maps of the space σ) we use the extension of *lifting*, *degree* and *rotation interval* $\mathrm{Rot}_{\mathbb{R}}$ developed in

Ll. Alsedà and S. Ruette.
Rotation sets for graph maps of degree 1.
Ann. Inst. Fourier (Grenoble), 58(4):1233–1294, 2008.

to get the same result:

Theorem

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of totally transitive σ-maps of degree one with periodic points such that $\lim_{n\to\infty} h(f_n) = 0$. For every n let $F_n \in \mathcal{L}_1$ be a lifting of f_n. Then,

- $\lim_{n\to\infty} \operatorname{len}(Rot(F_n)) = 0$,
- *there exists $N \in \mathbb{N}$ such that $BdCof(f_n)$ exists for every $n \geq N$, and*
- $\lim_{n\to\infty} BdCof(f_n) = \infty$.

Idea of the proof of the Theorem for the circle

Fix $M \in \mathbb{N}$, $M > 8$. Since $\lim_{n\to\infty} h(f_n) = 0$, there exists $N \in \mathbb{N}$ such that

$$h(f_n) < \frac{3\log\sqrt{2}}{M}.$$

for every $n \geq N$.

Let q be a denominator of a rational in the interior of the rotation interval. Recall that, by Misiurewicz's Theorem, if $\frac{r}{s} \in \operatorname{Int}(\operatorname{Rot}(f_n))$ with r, s coprime,

$$s\mathbb{N} = \{sl : l \in \mathbb{N}\} \subset \operatorname{Per}(f_n) \quad \text{and} \quad h(f_n) \geq \frac{\log 3}{s}.$$

Hence, $q = sl$ with $l \geq 1$ and

$$\frac{\log 3}{q} \leq \frac{\log 3}{s} \leq h(f_n) < \frac{3\log\sqrt{2}}{M} < \frac{\log 3}{M}$$

and, consequently, $q > M$.

So, $\operatorname{Int}(\operatorname{Rot}(f_n))$ does not intersect $\mathcal{F}_M$, the Farey sequence of order M. Since $\{i/M \colon i = 0, 1, 2, \dots, M\} \subset \mathcal{F}_M$ it follows that $\operatorname{diam}(\operatorname{Rot}(f_n)) \leq 1/M$ and, thus, $\lim_{n\to\infty} \operatorname{diam}(\operatorname{Rot}(f_n)) = 0$.

Idea of the proof of the Theorem for the circle

We will prove now that $\operatorname{BdCof}(f_n) \geq M - 2$. Again by Misiurewicz's Theorem, and the part already proven

$$\operatorname{Per}(f_n) \subset \{M, M+1, M+2\dots\} \cup S(c_n) \cup S(d_n),$$

where c_n and d_n denote the endpoints of the rotation interval and

$$S(\rho) = \begin{cases} \emptyset & \text{if } \rho \notin \mathbb{Q}, \text{ and} \\ qS & \text{if } \rho = p/q \text{ with } p \text{ and } q \text{ coprime,} \end{cases}$$

where S is an initial segment of the Sharkovskii Ordering.

To prove that $\operatorname{BdCof}(f_n) \geq M - 2$ it is enough to show that

$$\operatorname{Card}(\{M-3, M-2, M-1\} \cap S(c_n) \leq 1$$

and

$$\operatorname{Card}(\{M-3, M-2, M-1\} \cap S(d_n) \leq 1$$

(then $\{M-3, M-2, M-1\} \not\subseteq \operatorname{Per}(f_n)$ and, hence, $\operatorname{BdCof}(f_n) \geq M - 2$).

Idea of the proof of the Theorem for the circle

- If $c_n \notin \mathbb{Q}$ or $c_n = \frac{p}{q}$ with p, q coprime and $q \geq M$, then $S(c_n) \subset q\mathbb{N} \subset \{M, M+1, M+2\dots\}$. Therefore, $\{M-3, M-2, M-1\}$ does not intersect $S(c_n)$ and we are done.
- If $c_n = \frac{p}{q}$ with p, q coprime and $3 \leq q < M$ then, since $S(c_n) \subset q\mathbb{N} = \{q, 2q, 3q, \dots\}$, two consecutive elements of $S(c_n)$ are at distance $q \geq 3$. Hence, $\operatorname{Card}(\{M-3, M-2, M-1\} \cap S(c_n) \leq 1$.
- Assume now that $S(c_n)$ contains an element of the form $q \cdot t \cdot 2^m$ with $t \geq 3$ odd and $m \geq 1$. This means that the map f_m^q has a periodic point of period $t \cdot 2^m$ (as a map of the real line) and hence, from [BGMY],

$$h(f_n) = \frac{1}{q} h(f_n^q) \geq \frac{1}{q}\frac{1}{2^m} \log \lambda_t$$

where λ_t is the largest root of $x^t - 2x^{t-2} - 1$. It is well known that $\lambda_t > \sqrt{2}$. So,

$$\frac{3\log\sqrt{2}}{M} > h(f_n) > \frac{\log\sqrt{2}}{q2^m}$$

which is equivalent to $q \cdot t \cdot 2^m \geq q \cdot 3 \cdot 2^m > M$. Hence,

$$\{M-3, M-2, M-1\} \cap S(c_n) \subset \{M-3, M-2, M-1\} \cap q\{1, 2, 4, \dots, 2^n, \dots\}.$$

Idea of the proof of the Theorem for the circle

It remains to show that

$$\mathrm{Card}(\{M-3, M-2, M-1\} \cap q\{1,2,4,\dots,2^n,\dots\}) \le 1$$

when $q \in \{1,2\}$.

Let m be such that $2^m < M \le 2^{m+1}$.

The assumption $M > 8$ implies $2^m \ge 8$ and, hence, $2^{m-1} \ge 4$. Consequently,

$$M-3 \ge 2^m - 2 > 2 \cdot 2^{m-1} - 4 \ge 2^{m-1}.$$

So,

$$\mathrm{Card}(\{M-3, M-2, M-1\} \cap q\{1,2,4,\dots,2^n,\dots\}) \subset \{M-3, M-2, M-1\} \cap \{1,2,4,\dots,2^n,\dots\} \subset \{2^m\}.$$

Idea of the proof of the Theorem for the σ graph

It goes along the same lines except that very few theory is available. Essentially we only know that if 0 is in the interior of the rotation interval, then $\mathrm{Per}(f_n) \supset \mathbb{N} \setminus \{1,2\}$.

Ll. Alsedà and S. Ruette.
On the set of periods of sigma maps of degree 1.
Discrete Contin. Dyn. Syst., 35(10):4683–4734, 2015.

Ll. Alsedà and S. Ruette.
Periodic orbits of large diameter for circle maps.
Proc. Amer. Math. Soc., 138(9):3211–3217, 2010.

All the above results that are known for circle maps must be either extended or make a detour to avoid using them by getting similar conclusions. This is interesting by itself.