On the sets of periods of continuous tree maps

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Aims and summary

We aim at characterising the set of periods of continuous self maps on trees.

More precisely, we show that the set of periods of any continuous self map from a tree into itself is the union of a finite number of initial segments of Baldwin's orderings ${}_p\geq$, and a finite set $\mathcal{F}$. The possible values of p are described as well as explicit bounds of the set $\mathcal{F}$ in terms of the combinatorial properties of the tree.

Conversely, given a set $\mathcal{A}$ which is union of a finite set of initial segments of Baldwin's orderings ${}_p\geq$ (with the numbers p determined in a precise way) and a finite set $\mathcal{F}$, there exists a continuous self map from a tree into itself whose set of periods is precisely $\mathcal{A}$.

Sketch of the talk

- An introductory example: the interval case
  - Notation
  - Sharkovskii's Theorem
  - Idea of the proof of Sharkovskii's Theorem
- Sets of periods of star maps
  - General Notation
  - Baldwin's partial orderings
  - Baldwin's Theorem
- The case of tree maps
  - General strategy (in 4 steps)
  - Idea of the proof

An introductory example: the interval case

Notation

The Sharkovskii Ordering ${}_{\text{Sh}}\!\geq$:

$$3 \,{}_{\text{Sh}}\!> 5 \,{}_{\text{Sh}}\!> 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> 2\cdot 3 \,{}_{\text{Sh}}\!> 2\cdot 5 \,{}_{\text{Sh}}\!> 2\cdot 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!>$$
$$4\cdot 3 \,{}_{\text{Sh}}\!> 4\cdot 5 \,{}_{\text{Sh}}\!> 4\cdot 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!>$$
$$2^n\cdot 3 \,{}_{\text{Sh}}\!> 2^n\cdot 5 \,{}_{\text{Sh}}\!> 2^n\cdot 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> 2^\infty \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!>$$
$$2^n \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> 16 \,{}_{\text{Sh}}\!> 8 \,{}_{\text{Sh}}\!> 4 \,{}_{\text{Sh}}\!> 2 \,{}_{\text{Sh}}\!> 1.$$

is defined on the set

$$\mathbb{N}_{\text{Sh}} = \mathbb{N} \cup \{2^\infty\}$$

(we have to include the symbol 2^∞ to assure the existence of supremum for certain sets).

In the ordering ${}_{\text{Sh}}\!>$ the least element is 1 and the largest is 3. The supremum of the set $\{1, 2, 4, \ldots, 2^n, \ldots\}$ is 2^∞.

The Sharkovskii Ordering formal definition

If $k = k' \cdot 2^p$ where p is non negative and k' is odd:

(1) $k \;_{\text{Sh}}\!> 2^\infty$ if $k' > 1$,
(2) $2^\infty \;_{\text{Sh}}\!> k$ if $k' = 1$,

and if $n = n' \cdot 2^q$ where q is non negative and n' is odd, then $n \;_{\text{Sh}}\!> k$ if and only if one of the following next statements holds:

(3) $k' > 1$, $n' > 1$ and $p > q$,
(4) $k' > n' > 1$ and $p = q$,
(5) $k' = 1$ and $n' > 1$,
(6) $k' = 1$, $n' = 1$ and $p < q$.

Initial segments for the Sharkovskii Ordering

For $s \in \mathbb{N}_{\mathrm{Sh}}$, $S(s)$ denotes the set $\{k \in \mathbb{N} \colon s \ {}_{\mathrm{Sh}}\!\geq k\}$. Examples of sets of the form $S(s)$ are:

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Note: $S(s)$ is finite if and only if $s \in S(2^\infty)$.

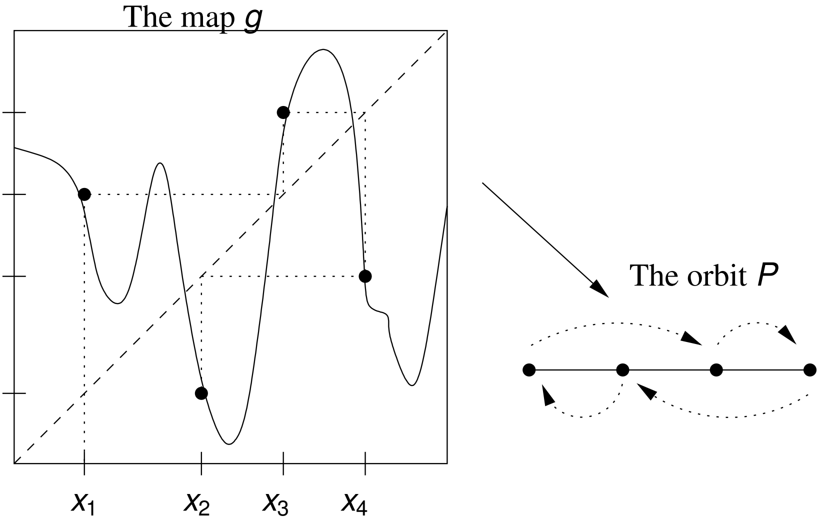
Sharkovskii's Theorem

Theorem (Sharkovskii)

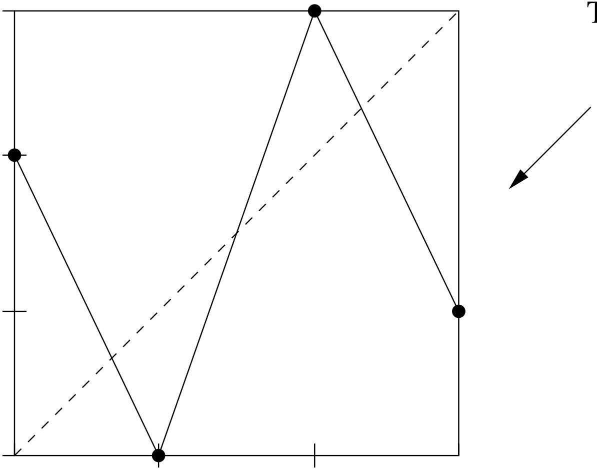
For each continuous map g *from a closed interval of the real line into itself, there exists* $s \in \mathbb{N}_{\text{Sh}}$ *such that* $\text{Per}(g) = S(s)$*. Conversely, for each* $s \in \mathbb{N}_{\text{Sh}}$ *there exists a continuous map* g *from a closed interval of the real line into itself such that* $\text{Per}(g) = S(s)$*.*

$\text{Per}(g)$ denotes the set of (least) periods of all periodic points of g.

Idea of the proof of Sharkovskii's Theorem



The minimal map f_P



The pattern of P

$(1, 3, 4, 2)$

One has:

$$\mathrm{Per}(g) \supset \mathrm{Per}(f_P).$$

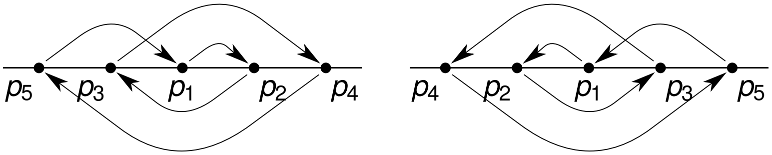
The set of periods of the minimal model

Let us suppose, *for example*, that P is an orbit of Stefan type of period n. That is, of the following type:

$$p_n < p_{n-2} < \cdots < p_5 < p_3 < p_1 < p_2 < p_4 < \cdots < p_{n-3} < p_{n-1},$$

o

$$p_{n-1} < p_{n-3} < \cdots < p_4 < p_2 < p_1 < p_3 < p_5 < \cdots < p_{n-2} < p_n.$$



Then:

Lemma

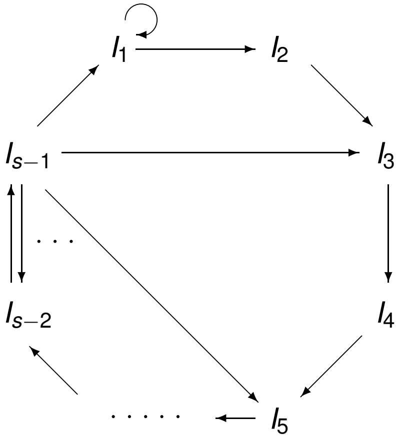
The vertices of the f_P-(combinatorial) Markov graph of f associated to P can be labelled so that their arrows are

(a) $I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_{s-1} \longrightarrow I_1$,

(b) $I_1 \longrightarrow I_1$,

(c) $I_{s-1} \longrightarrow I_1,\ I_{s-1} \longrightarrow I_3,\ I_{s-1} \longrightarrow I_5,\ \dots,\ I_{s-1} \longrightarrow I_{s-2}$.

That is:



Conclusion

It is easy to see that the previous Markov Graph gives loops of length equals to any positive integer contained in $S(n)$. Consequently, $S(n) \subset \mathrm{Per}(f_P)$, since:

Lemma

Let $f \in \mathcal{C}^0(I, I)$, let $P \subset I$ be a finite set and let $\alpha = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_0$ a loop in the f-Markov graph associated to P. Then, there exists a fixed point x of f^n, such that $f^i(x) \in I_i$ for $i = 0, 1, \ldots, n-1$. By choosing the loop in an appropriate way one can contain a point x whose (least) period is precisely n. Consequently, $n \in \mathrm{Per}(f)$.

Finally one gets $\mathrm{Per}(g) = S(s)$ by taking

$$s = \max_{\mathrm{Sh}\geq} \mathrm{Per}(g).$$

A *(topological) graph* is a connected Hausdorff space G, which is a finite union of subspaces G_i, each of them homeomorphic to a closed, non-degenerate interval of the real line and $G_i \cap G_j$ is finite for all $i \neq j$. Clearly any graph is compact. The points from a graph which do not have a neighbourhood homeomorphic to an open interval are called *vertices*. The set of vertices of a graph G is denoted by $\mathrm{V}(G)$ and is clearly finite (or empty — when G is homeomorphic to to the circle).

The closure of any connected component of $G \setminus \mathrm{V}(G)$ is called an *edge of G*. Clearly, a graph has finitely many edges and each of them is homeomorphic to a closed interval or to the circle.

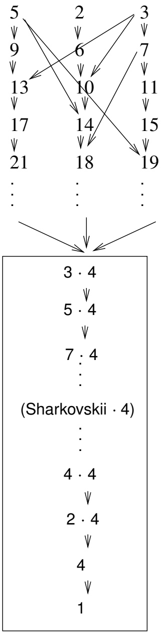
Trees and stars

A *tree* is a graph which is uniquely arcwise connected.

Let G be a graph, let $z \in G$ and let U be an open neighboorhood (in G) of z such that $\mathrm{Cl}(U)$ is a tree. The number of connected components of $U \setminus \{z\}$ is called *the valence of z* and is denoted by $\mathrm{Val}(z)$. This definition is independent of the choice of U and $\mathrm{Val}(z) \neq 2$ if and only if $z \in \mathrm{V}(G)$. A vertex of valence 1 is called an *endpoint of G* whereas a point of valence larger than 2 is called a *branching point of G*.

Let $n \in \mathbb{N} \setminus \{1\}$. A *n-star* is a tree with n endpoints and at most one branching point. Note that a 2-star is homeomorphic to an interval (an thus it has no branching point) while an n-star with $n \geq 3$ has a unique branching point b with $\mathrm{Val}(b) = n$. X_n will denote a n star and $\mathcal{X}_n$ the class of all continuous maps from X_n into X_n.

Baldwin partial orderings. The structure of ${}_4\geq$



$\mathbb{N}_t = (\mathbb{N} \cup \{t \cdot 2^\infty\}) \setminus \{2, 3, \ldots, t-1\}$

$\mathbb{N}_t^\vee$

Baldwin partial orderings. Formal definition

For each integer $t \geq 2$ we denote:

$$\mathbb{N}_t = (\mathbb{N} \cup \{t \cdot 2^\infty\}) \setminus \{2, 3, \ldots, t-1\} \text{ and}$$
$$\mathbb{N}_t^\vee = \{mt \colon m \in \mathbb{N}\} \cup \{1, t \cdot 2^\infty\}.$$

Then, the ordering ${}_t{\geq}$ is defined in $\mathbb{N}_t$ as follows: for $k, m \in \mathbb{N}_t$ we have $m \;{}_t{\geq}\; k$ if one of the following holds:

(i) $k = 1$ o $k = m$,
(ii) $k, m \in \mathbb{N}_t^\vee \setminus \{1\}$ and $m/t \;{}_{\mathrm{Sh}}{>}\; k/t$,
(iii) $k \in \mathbb{N}_t^\vee$ and $m \notin \mathbb{N}_t^\vee$,
(iv) $k, m \notin \mathbb{N}_t^\vee$ and $k = im + jt$ with $i, j \in \mathbb{N}$,

where, in case (ii) we use the following arithmetic rule $t \cdot 2^\infty$: $t \cdot 2^\infty / t = 2^\infty$.

Note: By identifying $2 \cdot 2^\infty$ with 2^∞ we have ${}_2{\geq} = {}_{\mathrm{Sh}}{\geq}$.

Initial segments

A set $S \subset \mathbb{N}_t \cap \mathbb{N}$ is an *initial segment of the ordering ${}_t\geq$* if for every $m \in S$ we have $\{k \in \mathbb{N} \colon m \ {}_t\geq k\} \subset S$ (that is, S is closed under predecessors).

Also we set

$$\mathcal{S}_t(s) := \{n \in \mathbb{N} : n \leq_t s\},$$

which is a particular case of an initial segment. Indeed, any initial segment of the $\leq_t$ ordering can be expressed as the union of at most $t-1$ sets of the form $\mathcal{S}_t(s_i)$ because the set $\mathbb{N}_t$ splits in at most $t-1$ branches by the ordering $\leq_t$.

Baldwin's Theorem

Theorem

Let $f \in \mathcal{X}_n$. Then, $\mathrm{Per}(f)$ *is a finite union of initial segments of the orderings ${}_t\!\geq$ with $2 \leq t \leq n$. Conversely, given a set A that is a finite union of initial segments of the orderings ${}_t\!\geq$ with $2 \leq t \leq n$, there exists a map $f \in \mathcal{X}_n$ such that $f(b) = b$ and* $\mathrm{Per}(f) = A$*.*

In a similar way to the interval case, the basic implication to prove is of the following kind: *Assume that f has a periodic orbit with period n and type t. Then* $\mathrm{Per}(f) \supset \mathcal{S}_t(n)$.

Sets of periods of continuous tree maps
General strategy – I

Let S be a tree and let $g: S \longrightarrow S$ be continuous. To characterise the structure of the set $\text{Per}(g)$ we use the following *strategy*: We fix a periodic orbit P of g:

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Step 2. Let us consider the *canonical* (minimal) model (T, A, f) of (S', P', g'). For this model we calculate (or we get a good estimate) Λ_P, of the set of periods of f.

Step 3. Since (T, A, f) is the *canonical* (minimal) model of (S', P', g'), we prove that the "essential part" of $\mathrm{Per}(f)$ is contained in the set of periods of any map having the same *pattern* as (S, P, g). In particular, $\Lambda_P \subset \mathrm{Per}(g)$.

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Step 4. Let us consider the set of *all* periodic orbits P of g. The structure of $\mathrm{Per}(g)$ can be obtained by organising the unions of all the sets Λ_P in an appropriate way.

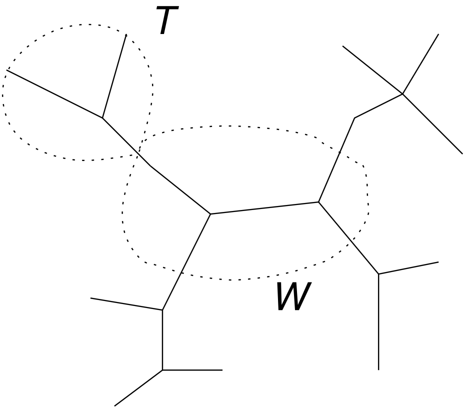
Step 4: Structure of the set of periods

[AJM2005] L. Alsedà, D. Juher, and P. Mumbrú,
Periodic behavior on trees,
Ergodic Theory Dynam. Systems **25(5)** (2005), 1373–1400.

Definition

Given S and T trees, and $p \geq 2$ we write $S \sqsupset pT$ when S contains a subtree W with p endpoints, such that T is homeomorphic to a connected component of $S \setminus \mathrm{Int}(W)$, and the number of endpoints of each connected component of $S \setminus \mathrm{Int}(W)$ is larger than or equal to the number of endpoints of T.

Example



$$S \sqsupset 4T$$

Definition (continued)

Let Σ be the set of all finite sequences of positive integers $\underline{s} = (p_1, p_2, \ldots, p_m)$ with $p_i \geq 2$ for $1 \leq i < m$.

Given a tree S, Σ_S denotes the set of all sequences $(p_1, p_2, \ldots, p_m) \in \Sigma$ for which there exists a sequence of trees $(S_1, S_2, \ldots, S_m)$ satisfying:

(i) $S \supset S_1$, $S_i \sqsupset p_i S_{i+1}$ and $\text{En}(S_m) \geq p_m$, where $\text{En}(\cdot)$ denotes the number of endpoints of a tree.

(ii) S_i it is *not* a star for $1 \leq i < m$.

Note: Σ_S *is finite* since $m \leq 1 + \log_2(\text{En}(S) - 1)$.

Examples

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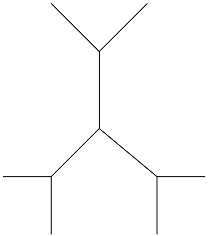


- **Admissible sequences:**
  $\Sigma_S = \{(1), (2), (3), (4), (5), (6),$
  $(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3),$
  $(4, 1), (4, 2), (5, 1), (5, 2), (6, 1), (6, 2)\}$

The characterisation of the set of periods

Theorem (Direct Implication)

Let $g\colon S \longrightarrow S$ be a tree map. Then there exists a (finite) set $\mathsf{S} \subset \Sigma_S$ such that

$$\operatorname{Per}(g) = \bigcup_{\underline{\mathsf{s}} \in \mathsf{S}} \left(\mathcal{K}_{\underline{\mathsf{s}}} \cup \mathcal{F}_{\underline{\mathsf{s}}} \cup \left(\mathcal{I}_{\underline{\mathsf{s}}} \setminus \lceil \underline{\mathsf{s}} \rceil \{2, 3, \dots, \lambda_{\underline{\mathsf{s}}}\} \right) \right)$$

where, for each $\underline{\mathsf{s}} = (p_1, p_2, \dots, p_m) \in \mathsf{S}$, $\lambda_{\underline{\mathsf{s}}}$ is a positive integer, $\lceil \underline{\mathsf{s}} \rceil = p_1 p_2 \cdots p_m$ and

(a) $\mathcal{K}_{\underline{\mathsf{s}}} = \{p_1, p_1 p_2, \dots, p_1 p_2 \cdots p_{m-1}\}$

(b) *$\mathcal{I}_{\underline{\mathsf{s}}}$ is an initial segment of the $\lceil \underline{\mathsf{s}} \rceil$-Baldwin ordering whose maximal elements belong to $\{1\} \cup p_1 p_2 \cdots p_{m-1}(\mathbb{N} \cup 2^\infty)$.*

(c) *If $\mathcal{I}_{\underline{\mathsf{s}}} \subsetneq \{1\} \cup \lceil \underline{\mathsf{s}} \rceil \mathbb{N}$ then $\lambda_{\underline{\mathsf{s}}} = 0$ and $\mathcal{F}_{\underline{\mathsf{s}}} = \emptyset$.*

The characterisation of the set of periods–continued

Theorem (Direct Implication–continued)

(d) $\mathcal{F}_{\underline{s}}$ *is disjoint from* $\mathcal{K}_{\underline{s}} \cup \mathcal{I}_{\underline{s}} \setminus \lceil \underline{s} \rceil \{2, 3, \dots, \lambda_{\underline{s}}\}$.

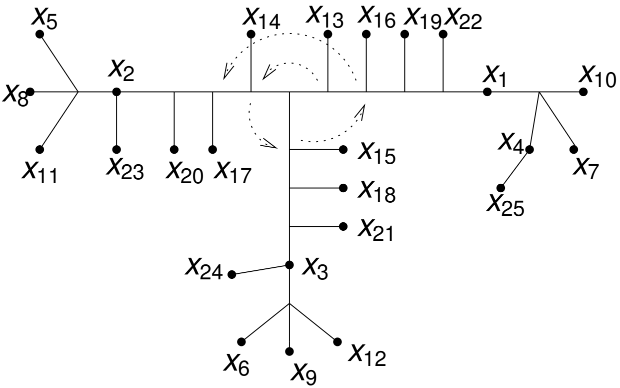
(e) $\mathcal{F}_{\underline{s}}$ *is finite (or empty). When* $\mathcal{F}_{\underline{s}} \neq \emptyset$*, we have* $\min \mathcal{F}_{\underline{s}} \geq \lambda_{\underline{s}} \lceil \underline{s} \rceil / 2$ *and* $|\mathcal{F}_{\underline{s}}|$ *is bounded in terms of* $\operatorname{En}(S)$.

Theorem (Converse Implication)

Given a finite set $\mathrm{S} \subset \Sigma$ *and a family* $\{\mathcal{F}_{\underline{s}}, \mathcal{I}_{\underline{s}}, \lambda_{\underline{s}}\}_{\underline{s} \in \mathrm{S}}$ *verifying (a–e) of the* Direct Theorem*, there exists a tree* S *and a continuous map* $g\colon S \longrightarrow S$ *such that* $\mathrm{S} \subset \Sigma_S$ *and*

$$\operatorname{Per}(g) = \bigcup_{\underline{s} \in \mathrm{S}} \left(\mathcal{K}_{\underline{s}} \cup \mathcal{F}_{\underline{s}} \cup \left(\mathcal{I}_{\underline{s}} \setminus \lceil \underline{s} \rceil \{2, 3, \dots, \lambda_{\underline{s}}\} \right) \right).$$

Example

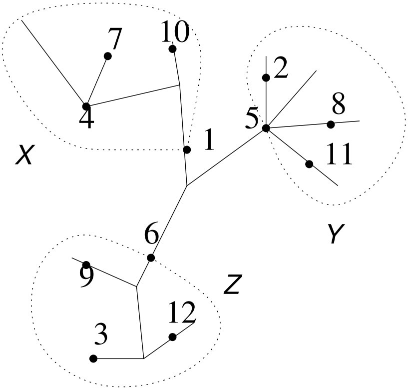


$$\mathrm{Per}(g) = \{25, 28, 31, 62, 65, 68\} \cup \mathcal{S}_3(34) \setminus \{8\}.$$

In the notation of the theorem we have $\underline{s} = (3)$, $S = \{\underline{s}\}$, $\mathcal{F}_{\underline{s}} = \{25, 28, 31, 62, 65, 68\}$, $\mathcal{I}_{\underline{s}} = \mathcal{S}_3(34)$ and $\lambda_{\underline{s}} = 2$.

Step 1: Reduction

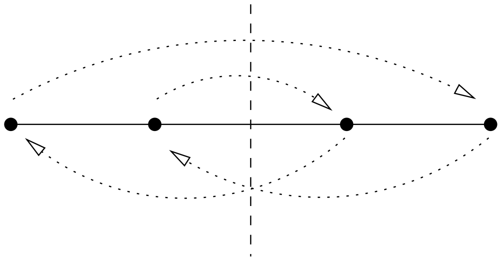
Example of a periodic orbit *3–twist* of period 12 (this notion generalises the notion of a *division* in the interval).



$g' = r_X \circ g \circ r_Z \circ g \circ r_Y \circ g : X \longrightarrow X;$ $\qquad$ $\mathrm{Per}(g) \supset \{1\} \cup 3 \cdot \mathrm{Per}(g').$

Notation: $r_Y : S \longrightarrow y$ denotes the natural retraction from S to Y.

The above construction generalises the notion of a *division* in the interval:



Step 1: Formalisation

Proposition

For a model (S, P, g), the following statements hold:

(a) *There exists a finite sequence of models $\{(S_i, P_i, g_i), p_i\}_{i=1}^{m}$ such that:*

- (i) $(S_1, P_1, g_1) = (S, P, g)$
- (ii) *P_i is a periodic orbit of g_i such that the endpoints of S_i are contained in P_i for $i > 1$.*
- (iii) *for each $i < m$, (S_i, P_i, g_i) is p_i–twist and $(S_{i+1}, P_{i+1}, g_{i+1})$ is a reduction of (S_i, P_i, g_i).*
- (iv) *(S_m, P_m, g_m) is either not twist or S_m is a star.*
- (v) *$|P| = p_1 p_2 \cdots p_{m-1} |P_m|$.*

(b)

$$\mathrm{Per}(g) \supset \{1, p_1, p_1 p_2, \ldots, p_1 p_2 \cdots p_{m-1}\} \cup p_1 p_2 \ldots p_{m-1}\, \mathrm{Per}(g_m)$$

Step 1: Conclusion

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- If S_m is a star, then this set if given by Baldwin Theorem (stated before).
- In the other case, (S_m, P_m, g_m) is not twist and the computation of its set of periods is done in the Steps 2 and 3.

Step 2: Computation of the set of periods in canonical models

One of the crucial notions in this theory is the concept of *pattern* for tree maps:

[AGLMM] Ll. Alsedà, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú,
Canonical representatives for patterns of tree maps,
Topology **36** (1997), 1123-1153.

As in the interval case we need a definition of *pattern* for which it always *exists* a minimal model (canonical — in the interval is the “connect–the–dots” map) (T, A, f) with the following properties of dynamical minimality:

Basic requirements on a canonical (minimal) model

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- f minimises the topological entropy among all the tree maps having a periodic orbit with the same pattern as (T, A, f) (this is essentially due to the fact that any such tree map will have $\mathcal{G}$ as a subgraph — as in the interval case).

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- The dynamics of f can be coded by means of a combinatorial (Markov) graph $\mathcal{G}$. Essentially, there exists a bijection between the periodic orbits of f and the loops of $\mathcal{G}$. Moreover, the topological entropy of f is the logarithm of the spectral radius of $\mathcal{G}$.

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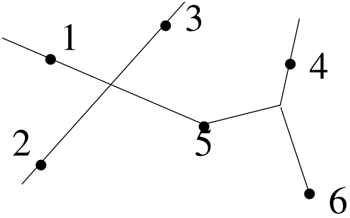
- f minimises the topological entropy among all the tree maps having a periodic orbit with the same pattern as (T, A, f) (this is essentially due to the fact that any such tree map will have $\mathcal{G}$ as a subgraph — as in the interval case).

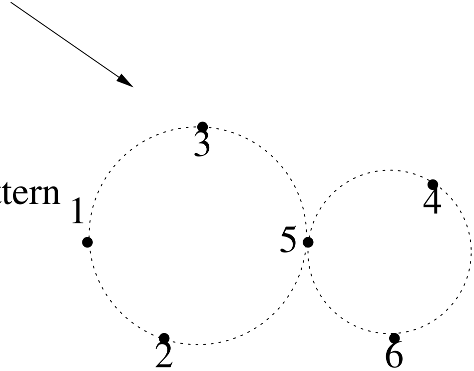
- The dynamics of f can be coded by means of a combinatorial (Markov) graph $\mathcal{G}$. Essentially, there exists a bijection between the periodic orbits of f and the loops of $\mathcal{G}$. Moreover, the topological entropy of f is the logarithm of the spectral radius of $\mathcal{G}$.

Note: Due to the existence of branch points, in this context there does not exist the "connect–the–dots" map. The price to pay for having minimal models in this context is that the space must be not fixed (homotopy relative to P)!!!.

Example: Pattern – 1st part

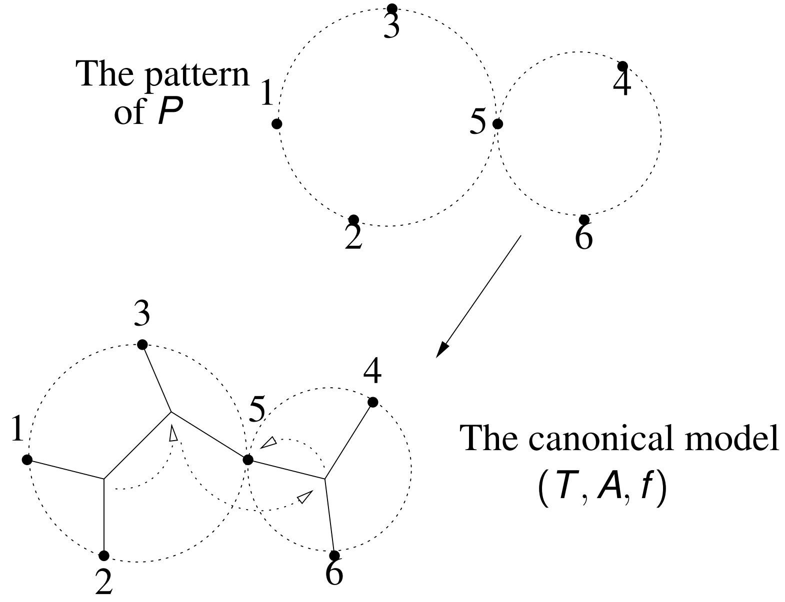


The original map
(S, P, g)



The pattern
of P

Example: Pattern – 2nd part

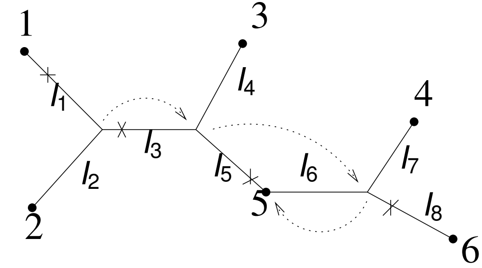


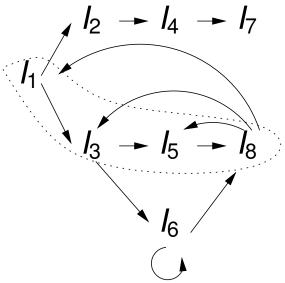
Properties of a canonical model

(1) f is *A-monotone*: if $\{a, b\} \subset A$ and $(a, b) \cap A = \emptyset$, then f maps $[a, b]$ monotonely "onto" $[f(a), f(b)]$.
(2) $f(V(T)) \subset V(T) \cup A$. Thus $A \cup V(T)$ is f–invariant and $(T, A \cup V(T), f)$ is a *Markov* model.
(3) In general $T \neq S$!!

Despite of Property (3), Properties (1) and (2) allow us to compute the periods associate to loops in the Markov Graph that are *simple and extern*:

Example





Sets of periods for canonical models
Notation

Given $t \geq 2$ and $r \in \mathbb{N}_t$ we denote:

$$\mathcal{S}_t^*(r) = \begin{cases} \{k \in \mathbb{N} \colon n \leq_t r\} & \text{si } r \notin \mathbb{N}_t^{\vee}, \\ \{1\} \cup t\mathbb{N} & \text{si } r \in \mathbb{N}_t^{\vee}. \end{cases}$$

Sets of periods for canonical models

Theorem

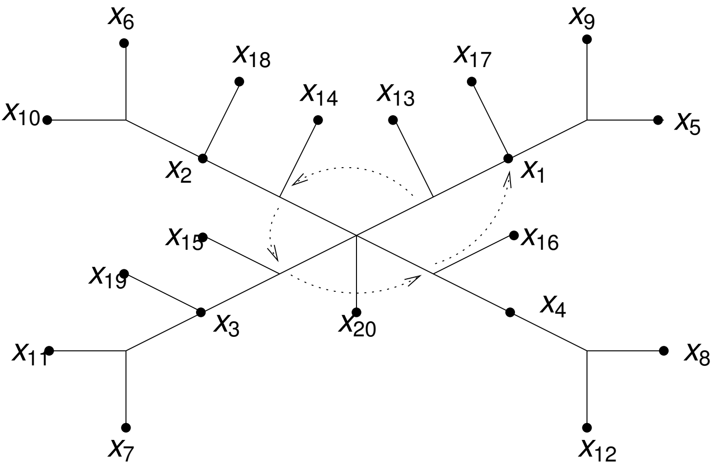
Let (T, A, f) be a non twist canonical model. Then,

$$\mathrm{Per}(f) \supset \mathcal{S}_p^*(|A| + lp) \setminus \{2p, 3p, \ldots, \lambda p\},$$

Where p is the type of the model (a generalisation of the corresponding notion introduced by Baldwin for the stars) and l and λ are bounded constants in terms of the combinatorial properties of T.

[AJM2003] L. Alsedà, D. Juher, and P. Mumbrú,
Sets of periods for piecewise monotone tree maps,
Int. J. of Bifurcation and Chaos **13** (2003), 311–341.

Example



A model with $|A| = 20$, $p = 4$, $\lambda = 2$ and $l = 1$

$\mathrm{Per}(f) = \mathcal{S}_4^*(24) \setminus \{8\} = \{1\} \cup 4\mathbb{N} \setminus \{8\}$

Step 3: Minimality of canonical models relative to the set of periods

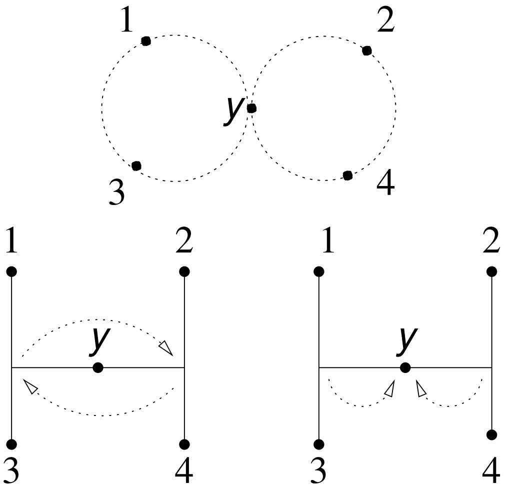
Let $g\colon S \longrightarrow S$ be a tree map, let P be a periodic orbit of g and let (T, A, f) be a canonical representative of the pattern (S, P, g). *Is it true that* $\text{Per}(f) \subset \text{Per}(g)$*?*

In

[AGLMM] Ll. Alsedà, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú,
Canonical representatives for patterns of tree maps,
Topology **36** (1997), 1123-1153.

it is proved that *if* $n \in \text{Per}(f)$ *then* g^{2n} *has a fixed point* x*.* However it is not made explicit which is really the least period of x.

In general the answer to this question is **NO**



Canonical model
$\mathrm{Per}(f) = \{1, 2, 4\}$

$\mathrm{Per}(g) = \{1, 4\}$

When the answer is positive?

A periodic point of (T, A, f) will be called *significant* if it does not travel together with a vertex of T.

Note: the periods computed in the Characterisation of the set of periods (Direct Implication) correspond to significant orbits.

When the answer is positive?

Theorem

Let $g\colon S \longrightarrow S$ be a tree map exhibiting a periodic orbit P with pattern $\mathcal{P}$. Let (T, A, f) be the canonical model of $\mathcal{P}$. If there is a significant n-periodic orbit of f, then $n \in \mathrm{Per}(g)$.

[AJM2005b] L. Alsedà, D. Juher, and P. Mumbrú,
On the preservation of combinatorial types for maps on trees,
Annales de l'Institut Fourier **55(7)** (2205) 2375–2398.

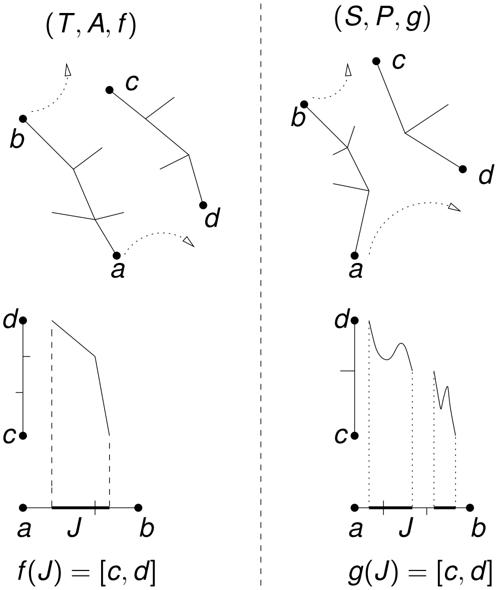
[AJM2006] L. Alsedà, D. Juher, and P. Mumbrú,
Periodic behaviour on trees,
In preparation.

Idea of the proof

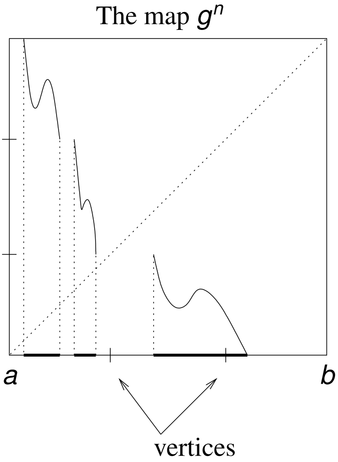
Let x be a significant n-periodic point of f. There exists a unique simple loop β in the $\mathcal{P}$-path graph such that x and β are associated:

$$\beta \; = \; \pi_0 \to \pi_1 \to \ldots \to \pi_{n-1} \to \pi_0$$

Set $\pi_0 = \{a, b\}$ and $\pi_1 = \{c, d\}$.

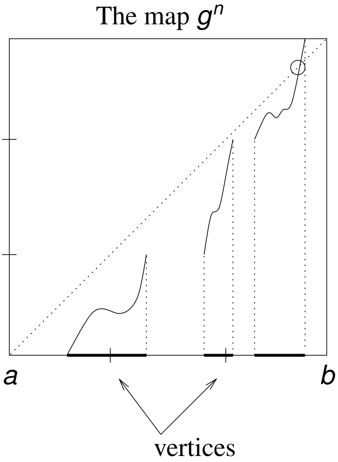

(T, A, f)
(S, P, g)
a
b
c
d
J
f(J) = [c, d]
g(J) = [c, d]

There exists a finite union $J(\beta) \subset [a, b]$ of closed intervals with pairwise disjoint interiors such that $g^i(J) \subset \langle \pi_i \rangle$ for $0 \leq i < n$ and $g^n(J) = \langle \pi_0 \rangle$:



It may happen that there are no fixed points of g^n in $[a, b]$. In fact there are several situations where this does not happen. Namely:

* When the loop β is positive i.e. g^n is “increasing” $(g^n(a) < a < b < g^n(b))$:



* When $n = |\beta|$ is bigger than $K \cdot L$, where:

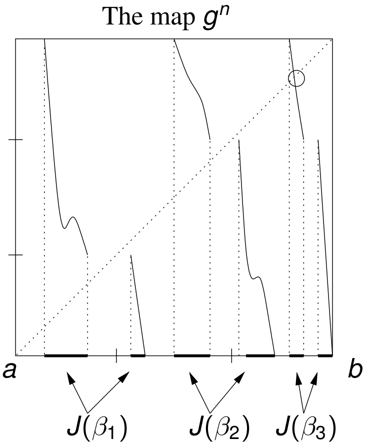
- K is the number of basic paths having some vertex in their interior.
- L is the maximum number of vertices contained in the interior of a basic path.

It is easy to see that

$$K \cdot L \geq M(S) := \frac{1}{2}|\operatorname{En}(S)| \cdot (|\operatorname{En}(S)| - 1) \cdot |V(S)|^2$$

When $n > K \cdot L$, there is a basic path (say, π_0) in the loop β such that the number of occurrences of π_0 in β is larger than the number of vertices in the interior of π_0. For instance, if π_0 has 2 vertices in its interior, then π_0 occurs at least 3 times in β.

Thus we can consider 3 (pairwise different) shifts of β starting at π_0 (say, β_1, β_2, β_3) and get the corresponding $J(\beta_1)$, $J(\beta_2)$ and $J(\beta_3)$:



So, we are left with the case:

$$1 < n < M(S) \text{ and } \beta \text{ negative}$$

Key tool to study these cases: A theorem of persistence of orbits (among “homotopically conjugate” graph maps) from

[AGGLMM] Ll. Alsedà, F. Gautero, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú,
Patterns and minimal dynamics for graph maps,
Proc. London Math. Soc. **91(2)** (2005), 9414–442.

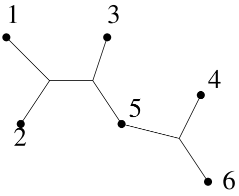
Key notions:

- Nielsen fixed point class
- Index of a Nielsen fixed point class

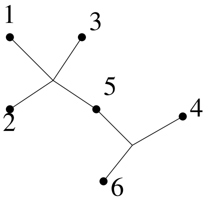
Definition

Let $f : G \longrightarrow G$ and $g : G' \longrightarrow G'$ be graph maps such that there exist two homotopy equivalences $r : G \longrightarrow G'$ and $s : G' \longrightarrow G$ satisfying $r \circ s \simeq \mathrm{Id}_{G'}$, $s \circ r \simeq \mathrm{Id}_G$ and $f \simeq s \circ g \circ r$. Then, there exists an index-preserving bijection that, for each $n \in \mathbb{N}$, sends essential fixed point classes of f^n to essential fixed point classes of g^n.

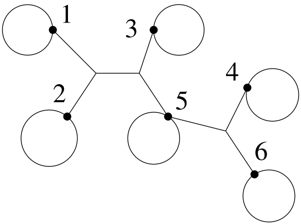
The trick to play



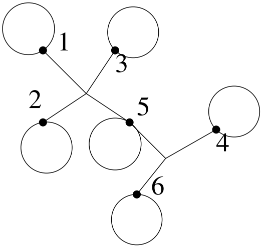
(T, A, f)



(S, P, g)



$(T^G, A, \overline{f})$



$(S^G, P, \overline{g})$

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THUS, BY THE THEOREM OF PERSISTENCE, THE SIGNIFICANT PERIODS OF A CANONICAL (SIMPLIFIED) MODEL ARE ALSO PERIODS OF THE ORIGINAL MODEL.