Introduction—Motivation

We want to show that the existence of attracting sets for quasiperiodically forced systems can be extended to appropriate skew-products on the cylinder, homotopic to the identity, in such a way that the general system will have (at least) one attracting set corresponding to every irrational rotation number $\omega$ in the rotation interval of the base map. This attracting set is a copy of the attracting set of the system quasiperiodically forced by a (rigid) rotation of angle $\omega$. This shows the co-existence of uncountably many attracting sets, one for each irrational in the rotation interval of the basis.

To fix ideas we will start by studying several examples of systems quasiperiodically forced by a (rigid) rotation (monotone and non-monotone). We also study these examples because this is the kind of models and dynamics that we want to extend simultaneously for uncountably many rotation numbers.

The Keller model

It is a skew product where the function in the second component has separated variables:

\[
\begin{align*}
\theta_{n+1} &= R(\theta_n) = \theta_n + \omega \pmod{1}, \\
x_{n+1} &= f(x_n)g(\theta_n)
\end{align*}
\]

where $x \in \mathbb{R}^+$, $\theta \in S^1$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and

1. $f : [0, \infty) \to [0, \infty)$ is $C^1$, bounded, strictly increasing, strictly concave and verifies $f(0) = 0$ (to fix ideas take $f(x) = \tanh(x)$ as in the [GOPY] model).

Thus, $x = 0$ will be invariant.

2. $g : S^1 \to [0, \infty)$ is bounded and continuous (to fix ideas take $g(\theta) = 2\sigma|\cos(2\pi\theta)|$ with $\sigma > 0$ in a similar way to the [GOPY] model – except for the absolute value).

Remark

In the pinched case any $T$-invariant set has to be 0 on a point and hence on a dense set because the circle $x \equiv 0$ is invariant and the $\theta$-projection of every invariant object must be invariant under $R$. There are big differences between the cases when $g$ takes the value 0 at some point: the pinched case and the case when $g$ is strictly positive.
A particular example (based in the [GOPY] model)

\[
\begin{align*}
\theta_{n+1} &= \theta_n + \omega \pmod{1}, \\
x_{n+1} &= 2\sigma \tanh(x_n)(\varepsilon + |\cos(2\pi \theta_n)|)
\end{align*}
\]

where \(x \in \mathbb{R}, \theta \in S^1\), \(\omega = \frac{\sqrt{5} + 1}{2}\). \(\sigma > 0\) and \(\varepsilon \geq 0\).

Remark

The attractor of the above system (if it exists) will be pinched if and only if \(\varepsilon = 0\).

Keller Theorem

There exists an upper semicontinuous map \(\phi : S^1 \rightarrow [0, \infty)\) whose graph is invariant under the Model (2). Moreover,

- The Lebesgue measure on the circle, lifted to the graph of \(\phi\) is a Sinai-Ruelle-Bowen measure,
- if \(\sigma \leq 1\) then \(\phi \equiv 0\),
- if \(\sigma > 1\) then \(\phi(\theta) > 0\) for almost every \(\theta\),
- if \(\sigma > 1\) and \(g(\theta_0) = 0\) for some \(\theta_0\) then the set \(\{\theta : \phi(\theta) > 0\}\) is meager and \(\phi\) is almost everywhere discontinuous,
- if \(\sigma > 1\) and \(g > 0\) then \(\phi\) is positive and continuous; if \(g\) is \(C^1\) then so is \(\phi\),
- if \(\sigma \neq 1\) then \(|x_n - \phi(\theta_n)|\) → 0 exponentially fast for almost every \(\theta\) and every \(x > 0\).

Other examples — The non-monotone case

We consider two basic situations. One with strict concavity and another one with non-strict concavity. In any case we take the golden mean as a rotation number \(\omega = \frac{\sqrt{5} + 1}{2}\) and \(g(\theta) = a\theta(1 - \theta)\) (thus we are in the pinched case).

\[
f(x) = 4x(1 - x) \quad a \in [1.8, 4]
\]

\[
f(x) = 1 - |2x - 1| \quad a \in [3.68, 4]
\]
The model we study

We denote the cylinder by $\mathbb{S}^1 \times K$ (where $K$ is either a closed interval of $\mathbb{R}$ containing zero, $\mathbb{R}^+$ or $\mathbb{R}$ itself). We consider the class of skew-products on $\mathbb{S}^1 \times K$ of the form:

\[ (\theta_{n+1}, x_{n+1}) = T(\theta_n, x_n) \]

where $T(\theta, x) = (f(\theta)p(x)q(\theta))$, $f$ is a continuous circle map of degree one with a lift $F$ such that $\text{Rot}(F)$ is non-empty (non-degenerate), $q$ is a continuous map from $\mathbb{S}^1$ to $K$ and $p$ is a continuous map from $K$ to itself.

A survey on rotation theory in the circle and water functions

We are only interested in continuous degree one circle maps. These are continuous maps such that $F(x + 1) = F(x) + 1$ for every $x \in \mathbb{R}$ and every lifting $F$. We denote by $\mathcal{L}_1$ the set of all liftings of continuous circle maps of degree one.

**Definition**

For each $F \in \mathcal{L}_1$ and $x \in \mathbb{R}$ we define the $F$-rotation number of $x$ as

\[ \rho_F(x) := \limsup_{n \to \infty} \frac{F^n(x) - x}{n}, \]

and the rotation set of $F$ as:

\[ \text{Rot}(F) := \{ \rho_F(x) : x \in \mathbb{R} \}. \]

From [Ito] we know that Rot($F$) is a closed interval of $\mathbb{R}$.

The notion of attracting set

Let $\mu$ be an ergodic measure of $f$ and let $\mathcal{U}$ be a measurable $f$-invariant set such that $\mu(\mathcal{U}) = 1$. Let $\varphi: \mathcal{U} \to K$ be a correspondence whose graph is $T$-invariant on $\mathcal{U}$ (i.e. $T(\text{graph}(\varphi)) = \text{graph}(\varphi)$). The closure of $\text{graph}(\varphi)$ will be called an attracting set with support $\mathcal{U}$ and generated by $\varphi$ whenever

\[ \lim_{n \to \infty} \| T^n(\theta, x) - T^n(\theta, z(x)) \| = 0 \]

for every $\theta \in \mathcal{U}$ and $x$ in a subset of $K$ of positive Lebesgue measure, and some $z(x) \in \varphi(\theta)$ (in particular, $\omega_T(\theta, x) \subset \omega_T(\theta, \varphi(\theta))$).

Upper and lower functions

Given $F \in \mathcal{L}_1$ we define the lower and upper liftings as follows:
Upper and lower functions

Given $F \in \mathcal{L}_1$ we define the lower and upper liftings as follows:

$$F_l(x) = \min \{ F(y) : y \geq x \}$$

$$F_u(x) = \max \{ F(y) : y \leq x \}$$

Properties of upper and lower functions

Clearly, $F_l$ and $F_u$ are non-decreasing functions from $\mathcal{L}_1$, $F_l \leq F \leq F_u$ and, if $F$ is non-decreasing then $F = F_l = F_u$.

Moreover, if $F, G \in \mathcal{L}_1$ are such that $F \leq G$, then $F_l \leq G_l$ and $F_u \leq G_u$.

If $F \in \mathcal{L}_1$ and it is non-decreasing,

$$\rho_F(x) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}$$

and it is independent on $x \in \mathbb{R}$. This number is denoted as $\rho(F)$ and called the rotation number of $F$.

Theorem

For every $F \in \mathcal{L}_1$, $\text{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

Water functions—a homotopy family from $F_l$ to $F_u$

Definition

Given $F \in \mathcal{L}_1$ and $0 \leq \alpha \leq \|F - F_l\|_\infty$, we define the water function of level $\alpha$ as

$$F_\alpha = (\min \{ F, F_l + \alpha \})_u$$
**Water functions** — a homotopy family from $F_l$ to $F_u$

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Given $F \in L_1$ and $0 \leq \alpha \leq \|F - F_l\|_{\infty}$, we define the 
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Observe that, $\min\{F, F_l + \alpha\} \in \mathcal{L}_1$
**Water functions**—a homotopy family from $F_l$ to $F_u$

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**Remarks**

- $F_\alpha$ is a non-decreasing functions from $\mathcal{L}_1$.
- $F_l = F_l$ and $F_\|F - F_l\|_{\infty} = F_u$.
- $F_\alpha \leq F_{\alpha'}$ whenever $\alpha \leq \alpha'$.
- For every $\alpha$, $F_\alpha$ coincides with $F$ in the complement of $\text{Const}(F_\alpha) \supset \text{Const}(F)$.

The map $\alpha \mapsto \rho(F_\alpha)$ from $[0, \|F - F_l\|_{\infty}]$ to $\text{Rot}(F)$ is continuous, onto and non-decreasing.

So, for every $\varrho \in \text{Rot}(F)$, there exists an $\alpha_\varrho \in [0, \|F - F_l\|_{\infty}]$ such that $\rho(F_{\alpha_\varrho}) = \varrho$.

**The main result — Theorem**

Consider a system of the form (3). To every $\varrho \in \text{Rot}(F) \setminus \mathbb{Q}$ we can associate a measurable $f$-invariant set $\mathcal{U}_\varrho \subset S^1$, a continuous non-decreasing circle map of degree one $h_\varrho$, and an $f$-ergodic measure $\mu_\varrho$ such that $\mu_\varrho(\mathcal{U}_\varrho) = 1$, $\rho_{\varrho}(e^{-1}(\text{Cl}(\mathcal{U}_\varrho))) = \varrho$, $h_\varrho|_{\mathcal{U}_\varrho} : \mathcal{U}_\varrho \rightarrow h_\varrho(\mathcal{U}_\varrho)$ is a homeomorphism and $h_\varrho(\mathcal{U}_\varrho)$ is a dense $\Phi_{\varrho}$-invariant set. Additionally, the sets $\text{Cl}(\mathcal{U}_\varrho)$ are pairwise disjoint.

Assume that, for every $\varrho \in \text{Rot}(F) \setminus \mathbb{Q}$, the system

$$
\begin{pmatrix}
\theta_{n+1} \\
x_{n+1}
\end{pmatrix} = S_\varrho
\begin{pmatrix}
\theta_n \\
x_n
\end{pmatrix}
\text{ where } S_\varrho
\begin{pmatrix}
\theta \\
x
\end{pmatrix} =
\begin{pmatrix}
\Phi_{\varrho}(\theta) \\
p(x)q(h_\varrho^{-1}(\theta))
\end{pmatrix},
$$

has an attracting set with support $h_\varrho(\mathcal{U}_\varrho)$ which is the closure of the graph of a correspondence $\varphi_\varrho : h_\varrho(\mathcal{U}_\varrho) \rightarrow K$. Then, the closure of the graph of $\varphi_\varrho \circ h_\varrho$ is an attracting set of $T$ with support $\mathcal{U}_\varrho$. Thus, whenever $\text{Rot}(F)$ is non-degenerate, $T$ has uncountably many attracting sets coexisting dynamically.
Idea of the proof

We start with system (3) and consider \( \varrho \in \text{Rot}(F) \setminus \mathbb{Q} \).

Then we take the water function \( F_{\alpha_{\varrho}} \) with \( \alpha_{\varrho} \in [0, \|F - F_i\|_{\infty}] \) such that \( \rho(F_{\alpha_{\varrho}}) = \varrho \).

We denote respectively by \( \mathcal{N}_{\varrho}, h_{\varrho} \) and \( \mu_{\varrho} \) the set \( \mathcal{N} \), the map \( h \) and the measure \( \mu \) given by the above Proposition for the map \( f_{\alpha_{\varrho}} \).

This already gives that \( \mathcal{N}_{\varrho} \) is measurable and \( f_{\alpha_{\varrho}} \)-invariant,

\[ \mu_{\varrho}(\mathcal{N}_{\varrho}) = 1, \quad h_{\varrho}|_{\mathcal{N}_{\varrho}} : \mathcal{N}_{\varrho} \rightarrow h_{\varrho}(\mathcal{N}_{\varrho}) \] is a homeomorphism and

\[ h_{\varrho}(\mathcal{N}_{\varrho}) \] is a dense \( \Phi_{\varrho} \)-invariant set.

Since \( f_{\alpha_{\varrho}} \) coincides with \( f \) in the complement of \( \text{Const}(f_{\alpha_{\varrho}}) \) and \( \text{Cl}(\mathcal{N}_{\varrho}) \) is disjoint from \( \text{Const}(f_{\alpha_{\varrho}}) \),

\[ f|_{\text{Cl}(\mathcal{N}_{\varrho})} = f_{\alpha_{\varrho}}|_{\text{Cl}(\mathcal{N}_{\varrho})}. \]

Hence, \( f(\mathcal{N}_{\varrho}) = \mathcal{N}_{\varrho} \) because \( \mathcal{N}_{\varrho} \) is \( f_{\alpha_{\varrho}} \)-invariant.

Idea of the proof (II)

Since \( F_{\alpha_{\varrho}} \) is non-decreasing,

\[ \rho_F(e^{-1}(\text{Cl}(\mathcal{N}_{\varrho}))) = \rho_{F_{\alpha_{\varrho}}}(e^{-1}(\text{Cl}(\mathcal{N}_{\varrho}))) = \varrho(F_{\alpha_{\varrho}}) = \varrho. \]

Consequently, if \( \varrho, \varrho' \in \text{Rot}(F) \) and \( \varrho \neq \varrho' \), then

\[ \text{Cl}(\mathcal{N}_{\varrho}) \cap \text{Cl}(\mathcal{N}_{\varrho'}) = \emptyset. \]

Also, it can be proved that \( \mu_{\varrho} \) is an ergodic measure of \( f \).

Now we assume that, for every \( \varrho \in \text{Rot}(F) \setminus \mathbb{Q} \), the system (4):

\[ \begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = S_{\varrho} \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} \]

\[ 
S_{\varrho}\left(\begin{pmatrix} \theta \\ x \end{pmatrix}\right) = \left(\Phi_{\varrho}(\theta), p(x)q(h_{\varrho}^{-1}(\theta))\right), \]

has an attracting set with support \( h_{\varrho}(\mathcal{N}_{\varrho}) \) which is the closure of the graph of a correspondence \( \varphi_{\varrho} : h_{\varrho}(\mathcal{N}_{\varrho}) \rightarrow \mathcal{K} \).

Idea of the proof (III)

We have to prove that the closure of the graph of \( \varphi_{\varrho} \circ h_{\varrho} \) is an attracting set of \( T \) with support \( \mathcal{N}_{\varrho} \).

Since \( h_{\varrho} \) is a semiconjugacy between \( f_{\alpha_{\varrho}} \) and \( \Phi_{\varrho} \), from (5) we get

\[ h_{\varrho}(f(\theta)) = h_{\varrho}(f_{\alpha_{\varrho}}(\theta)) = \Phi_{\varrho}(h_{\varrho}(\theta)) \]

for every \( \theta \in \mathcal{N}_{\varrho} \).

Set \( H := (h_{\varrho}, \text{Id}) \) which is a homeomorphism from \( \mathcal{N}_{\varrho} \times \mathcal{K} \) to \( h_{\varrho}(\mathcal{N}_{\varrho}) \times \mathcal{K} \). We have

\[ H(T(\theta, z)) = (h_{\varrho}(f(\theta)), p(z)q(\theta)) = \left(\Phi_{\varrho}(h_{\varrho}(\theta)), p(z)q(h_{\varrho}^{-1}(h_{\varrho}(\theta)))\right) \]

\[ = S_{\varrho}(H(\theta, z)), \]

for every \( \theta \in \mathcal{N}_{\varrho} \) and \( z \in \mathcal{K} \).

Idea of the proof (and IV)

That is, Systems (3) and (4) are conjugate by \( H \) on the set \( \mathcal{N}_{\varrho} \times \mathcal{K} \).

From this fact it is not difficult to prove the \( T \)- invariance of the graph of \( \varphi_{\varrho} \circ h_{\varrho} \). Also it follows that the closure of this graph is an attracting set of \( T \) with support \( \mathcal{N}_{\varrho} \).
Comments to the main theorem

Remark
It is precisely the need of the above conjugacy what forces us to replace the map \( q \) by \( q \circ h^{-1}_e \) on System (4).

Remark
The map \( q \circ h^{-1}_e \) is continuous in \( h^{-1}_e(\mathcal{U}_e) \) which is dense in \( S^1 \). Hence, if \( q \circ h^{-1}_e \) is discontinuous, it has only jump discontinuities in the complement of \( h^{-1}_e(\mathcal{U}_e) \). Therefore, since we want to reuse attractors from systems with a (rigid irrational) rotation in the base we are forced to consider System (4) on \( h^{-1}_e(\mathcal{U}_e) \times K \) and not on \( S^1 \times K \).

Attracting set with two minimal components—\( q \) negative

\[ f = \Phi_2^{\sqrt{\frac{5-1}{2}}} \]

When \( q \) is negative, the orbits keep alternating between \( \mathbb{R}^+ \) and \( \mathbb{R}^- \). Thus, typically, there will be an attracting set which is a 2-periodic orbit of function graphs.

\[ p(x) = \begin{cases} \tanh(x) & \text{if } x \geq 0 \\ \frac{\tanh(x-2) + \tanh(2)}{1 - \tanh(2)} & \text{if } x \leq 0 \end{cases} \]

\[ q(\theta) = \begin{cases} 7(\cos(2\pi\theta) - 2) & \text{if } \theta \in [0, \frac{1}{2}] \\ 5(\cos(2\pi\theta) - 4) & \text{otherwise} \end{cases} \]

On the number of pieces of an attracting set

Remark
The measure \( \mu_\theta \) lifted to the closure of the graph of \( \varphi_\theta \circ h_\theta \) is an invariant measure of \( T \). Moreover, this measure is ergodic if and only if the closure of the graph of \( \varphi_\theta \circ h_\theta \) is a minimal set of \( T \). In particular this is a criterion to decide the undecomposability of the attracting set into several smaller attracting sets.

When \( f \) is defined in the whole real line or in the interval \([−1, 1]\) so that \( \text{sign}(x) = \text{sign}(f(x)) \), there are two possibilities for such an attracting set: either its closure is a minimal attractor or it splits into two different minimal attractors and each of these attractors is the closure of the graph of a map from \( \mathcal{U} \) to the fibres.

2-periodic function graphs as attracting set—\( q \) positive

\[ f = \Phi_2^{\sqrt{\frac{5-1}{2}}} \]

\[ K = [-1, 1]. \]

When \( q \) is positive, System (3) can be split into two (one restricted to \( \mathbb{R}^+ \) and the other one restricted to \( \mathbb{R}^- \)). Consequently we get two attracting sets intersecting at \( x \equiv 0 \).

\[ p(x) = \begin{cases} x(1-x) & \text{if } x \geq 0 \\ \frac{1}{2}(x+1)(x+2) & \text{if } x \leq 0 \end{cases} \]

\[ q(\theta) = \begin{cases} 2.1 \cos(2\pi \theta) & \text{if } \theta \in [0, \frac{1}{2}] \\ 2.5 \sin(2\pi \theta) & \text{otherwise} \end{cases} \]