Patterns and minimal dynamics for graph maps

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Outline

Aims and summary

We study dynamical *persistence properties* of self-maps of (finite, connected) graphs, and explore dynamical consequences of the fact that a map has an orbit of a given type (pattern).

aims

For graph maps, generalise results known for interval maps (Sharkovskiĭ) and surface homeomorphisms. In particular, within a given homotopy (-equivalence) class of a graph map relative to one of its periodic orbits:

- define a notion of *pattern* of the orbit.
- find *canonical representatives*, which minimise topological entropy and periodic orbit structure within the given class.

This talk is essentially based on the paper

[AGGLMM] Ll. Alsedà, F. Gautero, J. Guaschi, J. Los, F. Mañosas, P. Mumbrú, *Patterns and minimal dynamics for graph maps*, Proc. London Math. Soc. **91** (2005), 414–442.

An introductory example: the interval case

The Sharkovskiĭ Ordering ${}_{\text{Sh}}\geq$:

$$3 \,{}_{\text{Sh}}\!> 5 \,{}_{\text{Sh}}\!> 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> 2\cdot 3 \,{}_{\text{Sh}}\!> 2\cdot 5 \,{}_{\text{Sh}}\!> 2\cdot 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!>$$
$$4\cdot 3 \,{}_{\text{Sh}}\!> 4\cdot 5 \,{}_{\text{Sh}}\!> 4\cdot 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!>$$
$$2^n\cdot 3 \,{}_{\text{Sh}}\!> 2^n\cdot 5 \,{}_{\text{Sh}}\!> 2^n\cdot 7 \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> 2^\infty \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!>$$
$$2^n \,{}_{\text{Sh}}\!> \cdots \,{}_{\text{Sh}}\!> 16 \,{}_{\text{Sh}}\!> 8 \,{}_{\text{Sh}}\!> 4 \,{}_{\text{Sh}}\!> 2 \,{}_{\text{Sh}}\!> 1.$$

is defined on the set

$$\mathbb{N}_{\text{Sh}} = \mathbb{N} \cup \{2^\infty\}$$

(we have to include the symbol 2^∞ to assure the existence of supremum for certain sets).

In the ordering ${}_{\text{Sh}}\!>$ the least element is 1 and the largest is 3. The supremum of the set $\{1, 2, 4, \dots, 2^n, \dots\}$ is 2^∞.

The Sharkovskiĭ Ordering formal definition

If $k = k' \cdot 2^p$ where p is non negative and k' is odd:

1. $k \,_{\mathrm{Sh}}\!> 2^\infty$ if $k' > 1$,
2. $2^\infty \,_{\mathrm{Sh}}\!> k$ if $k' = 1$,

and if $n = n' \cdot 2^q$ where q is non negative and n' is odd, then $n \,_{\mathrm{Sh}}\!> k$ if and only if one of the following next statements holds:

3. $k' > 1$, $n' > 1$ and $p > q$,
4. $k' > n' > 1$ and $p = q$,
5. $k' = 1$ and $n' > 1$,
6. $k' = 1$, $n' = 1$ and $p < q$.

Initial segments for the Sharkovskiĭ Ordering

For $s \in \mathbb{N}_{\text{Sh}}$, $S(s)$ denotes the set $\{k \in \mathbb{N} \colon s \,{}_{\text{Sh}}\!\geq k\}$. Examples of sets of the form $S(s)$ are:

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Note

$S(s)$ is finite if and only if $s \in S(2^\infty)$.

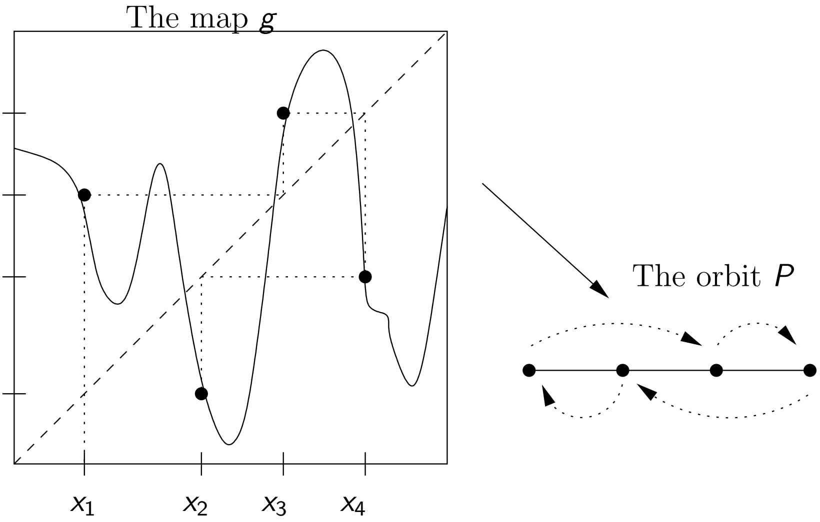
Sharkovskiĭ's Theorem

Theorem

For each continuous map g from a closed interval of the real line into itself, there exists $s \in \mathbb{N}_{\text{Sh}}$ such that $\operatorname{Per}(g) = S(s)$. Conversely, for each $s \in \mathbb{N}_{\text{Sh}}$ there exists a continuous map g from a closed interval of the real line into itself such that $\operatorname{Per}(g) = S(s)$.

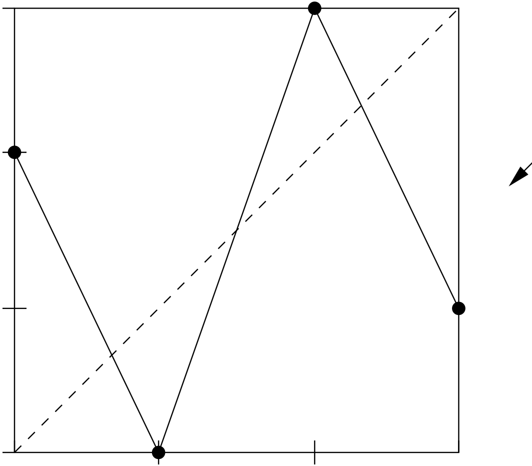
$\operatorname{Per}(g)$ denotes the set of (least) periods of all periodic points of g.

Idea of the proof of Sharkovskiĭ's Theorem



The minimal (connect–the–dots) map

The minimal map f_P



The pattern of P

$(1, 3, 4, 2)$

One has: $\mathrm{Per}(g) \supset \mathrm{Per}(f_P)$.

An introduction to the general notion of pattern

A summary of three known cases

PERIODIC ORBIT OF	PATTERN A	CANONICAL REPRESENTATIVES
interval map	permutation π induced by map on orbit	'Connect-the-dots' maps f_π
surface homeo.	braid type (isotopy class rel. orbit)	Nielsen-Thurston representatives
tree map	'relative positions' of the points of orbit	canonical models of trees

Basic properties of patterns

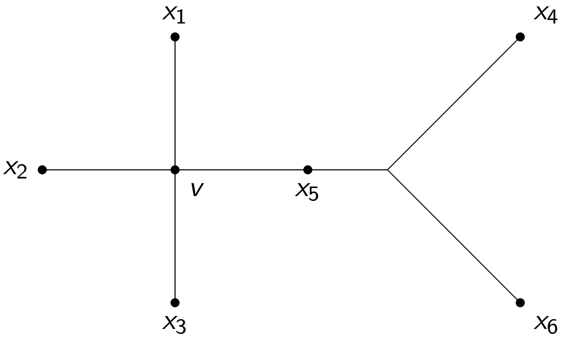
(A) f_π minimises topological entropy within the class of interval maps admitting a periodic orbit whose pattern is π.

(B) f_π admits a Markov partition which gives a good "coding" to describe the dynamics of the map f_π. The topological entropy of f_π may be calculated from this partition.

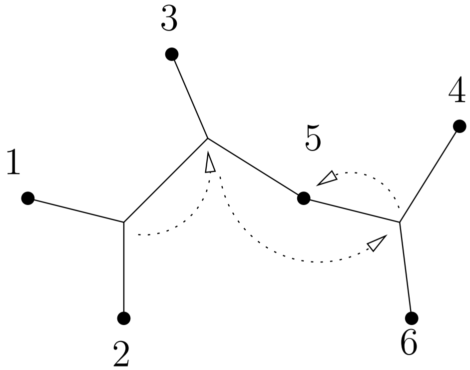
(C) f_π is essentially unique.

(D) the pattern of A forces a pattern ρ if and only if f_π has a periodic orbit whose pattern is ρ. We recall that a pattern A *forces* a pattern B if and only if each map exhibiting the pattern A also exhibits the pattern B. In this sense, the dynamics of f_π are minimal within the class of maps admitting a periodic orbit whose pattern is π_A.

The space cannot be fixed (the connect–the–dots map may not exist)!!



The correct space for the above model



The definition of a pattern

Let P (resp. Q) be a periodic orbit of a graph map $f\colon G \longrightarrow G$ (resp. $g\colon G' \longrightarrow G'$). The triples $(G, P, f), (G', Q, g)$ are said to *have the same pattern* if there exists a homotopy equivalence $r\colon G \longrightarrow G'$ such that:

1. $r\big|_P$ sends P bijectively onto Q.
2. the diagram:

$$\begin{array}{ccc} (G, P) & \xrightarrow{\ r\ } & (G', Q) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ (G, P) & \xrightarrow{\ r\ } & (G', Q) \end{array}$$

   commutes up to homotopy relative to P.

The resulting equivalence class, or *pattern*, of (G, P, f) is denoted by $[G, P, f]$.

Remarks to the definition of pattern

- This notion of pattern generalises the known ones in the case of interval maps and surface homeomorphisms (by taking r to be a homeomorphism).
- Our definition allows us to compare periodic orbits of maps of spaces having the same homotopy type, and not just self-maps of a space.
- We have an algebraic characterisation of pattern (conjugacy class of groupoid endomorphisms of fundamental groupoids — in $\mathrm{Aut}(\cdot)$).
- For trees, to have the same pattern is equivalent to to have the same period.
- In $\mathbb{S}^1$ all fixed points have the same pattern. However, already in two-foil, two fixed points may have different pattern.

The problem

To proceed as the known cases now we should be able to obtain canonical models (the equivalent of the “connect-the-dots” maps) relative to a pattern.

This is an open problem.

Graph maps

If G is a (finite, connected) graph then $\pi_1(G) \cong \mathbb{F}_n$, the free group of rank n.

A graph map $f\colon G \longrightarrow G$ induces an endomorphism $\Phi\colon \mathbb{F}_n \longrightarrow \mathbb{F}_n$, well defined up to inner automorphisms and conjugacy (choice of basepoint x, path from x to $f(x)$, identification of $\pi_1(G)$ with $\mathbb{F}_n$).

- f is called a *representative* for Φ.
- If further f sends vertexes to vertexes and edge-paths to edge-paths, it is called a *topological representative* for Φ.

Definition (Bestvina-Handel)

A topological representative $f\colon G \longrightarrow G$ for Φ is called *efficient* (or *train-track*) if it has no invariant forests, and if $\forall k \in \mathbb{N}$, the restriction of f^k to the interior of each edge is locally injective.

Remarks

- Φ admits efficient representatives if it is an irreducible free group automorphism (Bestvina-Handel, Los), or an irreducible free group endomorphism (Dicks-Ventura).
- An efficient representative minimises topological entropy within its homotopy equivalence class (Bestvina-Handel).

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Do efficient representatives minimise dynamics?
If yes with which "measuring device"?

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Answers

Do efficient representatives minimise dynamics?: yes
If yes with which "measuring device"?: patterns

Nielsen fixed point theory

Nielsen fixed point theory and the notion of index play an important role.

Let $f: G \longrightarrow G$ be a graph map.

Definition

- $x, y \in \mathrm{Fix}(f)$ belong to the same *Nielsen* or *fixed point class* for f if there exists an arc α from x to y such that $f(\alpha) \simeq \alpha$.
- If C is a Nielsen class of f then $\mathrm{ind}(C, f) \in \mathbb{Z}$ will denote its *index*.
- If $\mathrm{ind}(C, f) \neq 0$ then C will be called an *essential Nielsen class* of f.
- A periodic orbit P will be called *essential* if $\mathrm{ind}(C, f^{|P|}) \neq 0$, where C is a Nielsen class of $f^{|P|}$ containing any point of P.

[Jiang] B. Jiang, *Lectures on Nielsen fixed point theory*, American Mathematical Society, Providence, R.I., 1983. MR 84f:55002

Proposition

If x, y are periodic points of f of the same period k which belong to the same Nielsen class for f^k then the associated periodic orbits have the same pattern. The converse is false in general (Example: Two fixed points of the circle with different rotation number).

Non essential periodic orbits can be destroyed (think on fixed points).

We need to describe what happens with the pattern after such a destruction.

Index and efficient expanding maps

Definition

If f is an efficient graph map it will be called *expanding* if f expands each edge by some factor larger than one.

If f is an efficient, expanding map then each fixed point of f^n with $n \in \mathbb{N}$ is an isolated fixed point. Hence each fixed point class of f^n is finite, and the index of the class is just the sum of the indices for each fixed point in the class.

The notion of index in our context of graph maps has the following geometric formula due to [Jiang].

Let x be fixed under f^n, and let U_x be an open neighbourhood of x in G whose closure is homeomorphic to a tree (a valence(x)-star). Let E be the set of edges e of U_x that contain an interval I with endpoint x and such that $f^n(I) = e$ (that is, self-covered in an expanding way). Then $\operatorname{ind}(x, f^n)$ satisfies:

$$-1 \leq \operatorname{ind}(x, f^n) = \operatorname{Card}(E) - 1 \leq \operatorname{valence}(x) - 1.$$

A consequence of he above formula is:

Lemma

Let f be an efficient, expanding graph map, and let F be a fixed point class of f^n. If F has just one point which is not a vertex then $\operatorname{ind}(F, f^n) = \pm 1$. If the cardinal of F is greater than one then $\operatorname{ind}(x, f^n) = 1$ for all $x \in F \setminus V(G)$, $\operatorname{ind}(x, f^n) \geq 0$ for all $x \in F \cap V(G)$, and

$$\operatorname{ind}(F, f^n) \geq \operatorname{Card}(F) - \operatorname{Card}(F \cap V(G)).$$

Reductions

Recall the definition of a pattern

Let P (resp. Q) be a periodic orbit of a graph map $f\colon G \longrightarrow G$ (resp. $g\colon G' \longrightarrow G'$). The triples $(G, P, f), (G', Q, g)$ are said to have the *have the same pattern* if there exists a homotopy equivalence $r\colon G \longrightarrow G'$ such that:

1. $r\big|_P$ sends P bijectively onto Q.
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commutes up to homotopy relative to P.

In this definition we now replace 1 by the condition:

1' $r\big|_P\colon P \longrightarrow Q$ is onto but non injective,

Then we say that $[G', Q, g]$ is a *reduction* of $[G, P, f]$, and that $[G, P, f]$ is *reducible*.

A nother view of reductions

The following propositions characterise the notion of reducibility.

Proposition

Let $[G, P, f]$ be a pattern with $|P| = n$. Then $[G, P, f]$ is reducible if and only if there exists $m < n$ with $n = qm$, for some $q \in \mathbb{Z}^+ \setminus \{1\}$, such that for any $x \in P$ there exists a path γ from x to $f^m(x)$ satisfying:

$$[\gamma(f^m \circ \gamma) \dots (f^{(q-1)m} \circ \gamma)] = e_x,$$

where e_x denotes the homotopy class of the trivial loop based at x.

Proposition

If $[G', Q, g]$ is a reduction of $[G, P, f]$ and $x \in P$ then $\{f^{j \cdot |Q|}(x) \mid j \in \mathbb{Z}_+\}$ is contained in a Nielsen class of $f^{|P|}$.

Main Theorem: Preservation of patterns

Theorem

Let $f\colon G \longrightarrow G$ and $g\colon G' \longrightarrow G'$ be representatives of an endomorphism of a free group of finite rank. Then:

1. *there exists an index-preserving bijection κ that, for each $n \in \mathbb{N}$, sends essential fixed point classes of f^n to essential fixed point classes of g^n.*
2. *let P be an essential periodic orbit of f, let C be the fixed point class for $f^{|P|}$ of a point of P, and let Q be the g-orbit of a point of $\kappa(C)$. Then either $[G', Q, g] = [G, P, f]$, or $[G', Q, g]$ is a reduction of $[G, P, f]$.*

Minimal dynamics of efficient representatives

Theorem

Let $f\colon G \longrightarrow G$ be an efficient, expanding representative of an irreducible endomorphism Φ of a free group of rank n. Then there exists a cofinite subset $\mathcal{B}$ of the set of periodic orbits of f with the property that, for each representative $g\colon G' \longrightarrow G'$ of Φ, there exists a pattern-preserving injective function from $\mathcal{B}$ to the set of periodic orbits of g. Moreover, the number of periodic points of f whose orbit does not belong to $\mathcal{B}$ is at most $3\,\mathrm{Card}(V(G)) - 4\chi(G) \leq 10(n-1)$.

Remarks

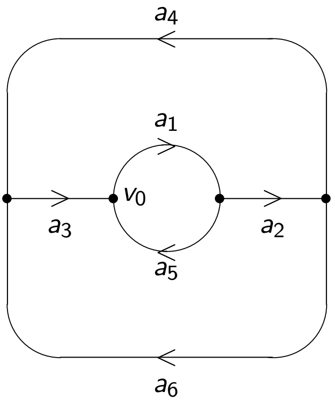
- Each point whose orbit P belongs to $\mathcal{B}$ is alone in its Nielsen class for all iterates of $f^{|P|}$.
- If $P \notin \mathcal{B}$ then either it is an inessential periodic orbit of vertexes, or else its pattern is reducible, and g exhibits the pattern $[G, P, f]$ or one of its reductions (we have examples of both phenomena).

Two efficient representatives have the same pattern

A direct consequence of the above theorem is that two efficient, expanding representatives of an irreducible endomorphism of a free group of rank n have (with at most $20(n-1)$ exceptions) the same number of periodic orbits of any pattern.

An example

Let G be the graph:



Let $f\colon G \longrightarrow G$ be defined by:

$$
\begin{aligned}
f(a_1) &= a_2,\\
f(a_2) &= a_6 a_3,\\
f(a_3) &= a_5 a_1,\\
f(a_4) &= a_1 a_2 a_6 a_3 a_1,\\
f(a_5) &= a_4 a_3 a_1,\\
f(a_6) &= a_1.
\end{aligned}
$$

Since f is a positive endomorphism, for all $n > 0$, there are no cancellations in the algebraic expression of f^n, and thus f^n restricted to any edge is locally injective. Since there are no invariant forests, f is efficient.

Consider the following generators of $\pi(G, \{v_0\})$:

$$\begin{aligned}
\alpha_1 &= a_1 a_2 a_6 a_3 a_1 a_5, \\
\alpha_2 &= a_1 a_2 a_4 a_6^{-1} a_2^{-1} a_1^{-1}, \\
\alpha_3 &= a_1 a_2 a_6 a_3,
\end{aligned}$$

and choose a_1 to be a path from v_0 to its image.

With this choice, the induced endomorphism $f^* : \pi(G, v_0) \longrightarrow \pi(G, v_0)$ is given by:

$$\begin{aligned} f^*([\alpha_1]) &= [\alpha_1][\alpha_2][\alpha_3], \\ f^*([\alpha_2]) &= [\alpha_3], \\ f^*([\alpha_3]) &= [\alpha_1]. \end{aligned}$$

Clearly f^* is an irreducible automorphism of $\mathbb{F}_3$. Thus f is an efficient representative of an irreducible automorphism of $\mathbb{F}_3$.

On the other hand, there exists a periodic orbit P of f of period 2 whose points, denoted respectively by p and q, lie in a_3 and a_5. Let ω be the oriented injective subpath of a_3 from p to v_0, and let π be the oriented injective subpath of $\bar{a}_5$ from v_0 to q. Direct computations show that $f(\omega\pi) = \bar{\pi} a_1 \bar{a}_1 \bar{\omega}$, and thus $[G, P, f]$ is reducible. The orbit $\{p, q\}$ is essential because $\operatorname{ind}(F, f^2) = 2$, where the fixed point class of p is denoted by F.

Another efficient representative of f^* may be obtained by considering the map $g : G' \longrightarrow G'$, where G' is the rose with three petals α, β and γ, given by:

$$\begin{aligned} g(\alpha) &= \alpha\beta\gamma, \\ g(\beta) &= \gamma, \\ g(\gamma) &= \alpha, \end{aligned}$$

which is also efficient.

Notice that this representative has an inessential periodic orbit of vertexes (in fact, a fixed point), while the preceding representative $f : G \longrightarrow G$ has no fixed points.

So we have an example of vanishing inessential fixed points in efficient models.

Since the orbit $\{p, q\}$ of f is essential, by the Main Theorem there exists a fixed point class C of g^2 that is associated with the class F. Since g has no periodic orbits of period 2, C must be the class of the fixed point.

We thus obtain an example of a reducible pattern in an efficient model that is reduced by a homotopy equivalence.