QUADRATIC SYSTEMS WITH A RATIONAL FIRST INTEGRAL OF DEGREE THREE: A COMPLETE CLASSIFICATION IN THE COEFFICIENT SPACE $\mathbb{R}^{12}$

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Abstract. A quadratic polynomial differential system can be identified with a single point of $\mathbb{R}^{12}$ through its coefficients. The phase portrait of the quadratic systems having a rational first integral of degree 3 have been studied using normal forms. Here using the algebraic invariant theory, we characterize all the non-degenerate quadratic polynomial differential systems of $\mathbb{R}^{12}$ having a rational first integral of degree 3. We show that there are only 31 different topological phase portraits in the Poincaré disc associated to this family of quadratic systems up to a reversal of the sense of their orbits, and we provide representatives of every class modulo an affine change of variables and a rescaling of the time variable. Moreover, each one of these 31 representatives is determined by a set of algebraic invariant conditions and we provide for it a first integral.

1. Introduction

Let $\mathbb{R}[x, y]$ be the ring of the polynomials in the variables x and y with coefficients in $\mathbb{R}$. We consider a system of polynomial differential equations or simply a *polynomial differential system* in $\mathbb{R}^2$ defined by

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$. We say that the maximum of the degrees of the polynomials P and Q is the *degree* of system (1). A *quadratic polynomial differential system* or simply a *quadratic system* is a polynomial differential system of degree 2. We say that the quadratic system (1) is *non–degenerate* if the polynomials P and Q are relatively prime or coprime.

Let U be an open and dense subset of $\mathbb{R}^2$, we say that a nonconstant function $\mathcal{H} : U \to \mathbb{R}$ is a *first integral* of system (1) on U if $\mathcal{H}(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of system (1). Obviously $\mathcal{H}$ is a first integral of system (1) if and only if

$$P(x, y)\frac{\partial \mathcal{H}}{\partial x}(x, y) + Q(x, y)\frac{\partial \mathcal{H}}{\partial y}(x, y) = 0, \tag{2}$$

for all $(x, y) \in U$. When a polynomial differential system (1) has a first integral we say that the system is *integrable*.

On the other hand given $f \in \mathbb{R}[x, y]$ we say that the curve $f(x, y) = 0$ is an *algebraic invariant curve* of system (1) if there exists $K \in \mathbb{R}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{3}$$

The name of invariant for such an algebraic curve $f(x, y) = 0$ is due to the fact that if a trajectory has a point on $f(x, y) = 0$, then the whole trajectory is contained in $f(x, y) = 0$.

The search of first integrals is a classic tool in order to describe the phase portraits of a planar differential system. As usual the *phase portrait* of a differential system is the decomposition of the domain of definition of the system as union of all its orbits or trajectories.

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