PHASE PORTRAITS OF THE FAMILY IV OF THE QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In the plane $\mathbb{R}^2$, the simplest non-linear differential systems are the quadratic polynomial differential systems. These type of differential systems have been studied intensively because of its non-linearity and their wide range of applications. These systems have been classified into ten classes. In this article, we characterize in the Poincaré disc all phase portraits topologically different from one of these classes.

1. Introduction and statement of the main results

A *quadratic polynomial differential system* usually only called *quadratic system* is a planar differential system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y whose maximum degree is two. As usual, the dot indicates the derivative with respect to the time t.

The interest in the study of quadratic systems began in the early 20th century. Coppel in [14] mentions that the first work on quadratic systems was published by Büchel [12] in 1904. Two studies on quadratic systems were published in 1966 by Coppel [14], and in 1982 by Chicone and Tian [13].

Quadratic systems have been studied intensively in recent decades and many interesting results have been published, see for example the books [11, 17, 26] and the references cited therein. But we are still far from a complete understanding of the dynamics of these systems.

In the paper [16] it was proved that an arbitrary quadratic system, after an affine change of variables and a change of scale of the time variable if necessary, becomes a quadratic system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y) = d + ax + by + \ell x^2 + mxy + ny^2, \tag{2}$$

where the differential equation $\dot{x} = P(x, y)$ is one of the following ten:

$$\begin{array}{llll}
\text{(I)} & \dot{x} = 1 + xy, & \text{(VI)} & \dot{x} = 1 + x^2, \\
\text{(II)} & \dot{x} = xy, & \text{(VII)} & \dot{x} = x^2, \\
\text{(III)} & \dot{x} = y + x^2, & \text{(VIII)} & \dot{x} = x, \\
\text{(IV)} & \dot{x} = y, & \text{(IX)} & \dot{x} = 1, \\
\text{(V)} & \dot{x} = -1 + x^2, & \text{(X)} & \dot{x} = 0.
\end{array}$$

A polynomial differential system in $\mathbb{R}^2$ can be extended analytically to the Poincaré disc, and in this way we can study the orbits of the polynomial differential system in a neighborhood of infinity. For more details, see subsection 2.2 of this paper as well as the Chapter 5 of [15]. Roughlly speaking, the *Poincaré disc* is

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