GLOBAL CONFIGURATIONS OF SINGULARITIES FOR QUADRATIC DIFFERENTIAL SYSTEMS WITH TOTAL FINITE MULTIPLICITY THREE AND AT MOST TWO REAL SINGULARITIES

Abstract. In this work we consider the problem of classifying all configurations of singularities, finite and infinite, of quadratic differential systems, with respect to the *geometric equivalence relation* defined in [2]. This relation is deeper than the *topological equivalence relation* which does not distinguish between a focus and a node or between a strong and a weak focus or between foci (or saddles) of different orders. Such distinctions are however important in the production of limit cycles close to the foci (or loops) in perturbations of the systems. The notion of *geometric equivalence relation* of configurations of singularities allows us to incorporate all these important geometric features which can be expressed in purely algebraic terms. This equivalence relation is also deeper than the *qualitative equivalence relation* introduced in [19]. The *geometric classification* of all configurations of singularities, finite and infinite, of quadratic systems was initiated in [3] where the classification was done for systems with total multiplicity m_f of finite singularities less than or equal to one. That work was continued in [4] where the geometric classification was done for the case $m_f = 2$. The case $m_f = 3$ has been split in two separate papers because of its length. The subclass of three real distinct singular points was done in [5] and we complete this case here.

In this article we obtain *geometric classification* of singularities, finite and infinite, for the remaining three subclasses of quadratic differential systems with $m_f = 3$ namely: (i) systems with a triple singularity (19 configurations); (ii) systems with one double and one simple real singularities (62 configurations) and (iii) systems with one real and two complex singularities (74 configurations). We also give here the global bifurcation diagrams of configurations of singularities, both finite and infinite, with respect to the *geometric equivalence relation*, for these subclasses of systems. The bifurcation set of this diagram is algebraic. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of invariant polynomials. This provides an algorithm for computing the geometric configuration of singularities for any quadratic system in this class.

1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p, q \in \mathbb{R}[x,y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. It is expected that we have a finite number of phase portraits in **QS**. We have phase portraits for several subclasses of **QS** but to obtain the complete topological classification of these systems, which occur rather often in applications, is a daunting task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time

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