FROM TOPOLOGICAL TO GEOMETRIC EQUIVALENCE IN THE CLASSIFICATION OF SINGULARITIES AT INFINITY FOR QUADRATIC VECTOR FIELDS

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Abstract. Planar quadratic differential systems occur in many areas of applied mathematics. Although more than one thousand papers were written on these systems, a complete understanding of this class is still missing. Classical problems, and in particular, Hilbert's 16th problem [19], are still open for this family. Even problems on quadratic systems not involving limit cycles remain unsolved, such as for example the problem of giving a complete classification of topologically distinct phase portraits without limit cycles. Although quadratic systems look simple, the set of all such systems is an object whose complete study is a rather daunting task. This is hardly a good enough reason to discourage further investigation because the quadratic family is too important for applications as well as for theoretical reasons. In recent years several studies of subfamilies of this class have been successfully completed and results were stated in invariant form. On the other hand the topological classification of singularities at infinity of the whole quadratic class was completely studied in [29]. An analogous work on finite singularities was achieved in [3]. In all these studies results were given in terms of polynomial invariants. Unlike so much work on quadratic systems done with respect to specific normal forms, all the above mentioned studies can be applied to any arbitrarily chosen normal form of any particular subfamily of systems.

In the topological classification of phase portraits no distinctions are made between a focus and a node and neither are they made between a strong and a weak focus or between foci of different orders. These distinction are however important in the production of limit cycles close to the foci in perturbations of the systems. The distinction between the one direction node and the two directions node, which plays a role in understanding the behavior of solution curves around the singularities at infinity, is also missing in the topological classification.

In this work we introduce the notion of *geometric equivalence relation* which incorporates these important purely algebraic features. The *geometric* equivalence relation is finer than the *topological* one and also finer than the *qualitative equivalence relation* introduced in [20]. We also list all possibilities we have for singularities finite and infinite taking into consideration these finer distinctions and introduce notations for each one of them. Our long term goal is to use this finer equivalence relation to classify the quadratic family according to their different *configurations of singularities*, finite and infinite.

In this work we accomplish a first step of this larger project. We give a complete global classification, using the *geometric equivalence* relation, of the whole quadratic class according to the singularities at infinity of the systems. Our classification Theorem is stated in terms of polynomial invariants and hence it can be applied to any family of quadratic systems with respect to any particular normal form. The theorem we give also contains the bifurcation diagram, done in the 12-parameter space, of the *geometric configurations* of singularities at infinity, and this bifurcation set is algebraic in the parameter space. To determine the bifurcation diagram of configurations of singularities at infinity for any family of quadratic systems, given in any normal form, becomes thus a simple task using computer algebra calculations.

1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p,\ q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$.

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