Preface

Our aim is to study *ordinary differential equations* or simply *differential systems* in two real variables

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{0.1}$$

where P and Q are C^r functions defined on an open subset U of $\mathbb{R}^2$, with $r = 1, 2, \ldots, \infty, \omega$. As usual C^ω stands for analyticity. We put special emphasis onto *polynomial differential systems*, i.e., on systems (0.1) where P and Q are polynomials.

Instead of talking about the differential system (0.1), we frequently talk about its associated *vector field*

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{0.2}$$

on $U \subset \mathbb{R}^2$. This will enable a coordinate-free approach, which is typical in the theory of dynamical systems. Another way expressing the vector field is by writing it as $X = (P, Q)$. In fact, we do not distinguish between the differential system (0.1) and its vector field (0.2).

Almost all the notions and results that we present for two-dimensional differential systems can be generalized to higher dimensions and manifolds; but our goal is not to present them in general, we want to develop all these notions and results in dimension 2. We would like this book to be a nice introduction to the qualitative theory of differential equations in the plane, providing simultaneously the major part of concepts and ideas for developing a similar theory on more general surfaces and in higher dimensions. Except in very limited cases we do not deal with bifurcations, but focus on the study of individual systems.

Our goal is certainly not to look for an analytic expression of the global solutions of (0.1). Not only would it be an impossible task for most differential systems, but even in the few cases where a precise analytic expression can be found it is not always clear what it really represents. Numerical analysis of a differential system (0.1) together with graphical representation are essential ingredients in the description of the phase portrait of a system (0.1) on U; that is, the description of U as union of all the orbits of the system. Of course, we do not limit our study to mere numerical integration. In fact in trying to do this one often encounters serious problems; calculations can take an enormous amount of time or even lead to erroneous results. Based however on a priori knowledge of some essential results on differential systems (0.1), these problems can often be avoided.

Qualitative techniques are very appropriate to get such an overall understanding of a differential system (0.1). A clear picture is achieved by drawing a phase portrait in which the relevant qualitative features are represented; it often suffices to draw the "extended separatrix skeleton." Of course, for practical reasons, the representation must not be too far from reality and has to respect some numerical accuracy. These are, in a nutshell, the main ingredients in our approach.

The basic results on differential systems and their qualitative theory are introduced in Chap. 1. There we present the fundamental theorems of existence, uniqueness, and continuity of the solutions of a differential system with respect the initial conditions, the notions of α- and ω-limit sets of an orbit, the Poincaré–Bendixson theorem characterizing these limit sets and the use of Lyapunov functions in studying stability and asymptotic stability. We analyze the local behavior of the orbits near singular points and periodic orbits. We introduce the notions of separatrix, separatrix skeleton, extended (and completed) separatrix skeleton, and canonical region that are basic ingredients for the characterization of a phase portrait.

The study of the singular points is the main objective of Chaps. 2, 3, 4, and 6, and partially of Chap. 5. In Chap. 2 we mainly study the elementary singular points, i.e., the hyperbolic and semi-hyperbolic singular points. We also provide the normal forms for such singularities providing complete proofs based on an appropriate two-dimensional approach and with full attention to the best regularity properties of the invariant curves. In Chap. 3, we provide the basic tool for studying all singularities of a differential system in the plane, this tool being based on convenient changes of variables called blow-ups. We use this technique for classifying the nilpotent singularities.

A serious problem consists in distinguishing between a focus and a center. This problem is unsolved in general, but in the case where the singular point is a linear center there are algorithms for solving it. In Chap. 4 we present the best of these algorithms currently available.

Polynomial differential systems are defined in the whole plane $\mathbb{R}^2$. These systems can be extended to infinity, compactifying $\mathbb{R}^2$ by adding a circle, and extending analytically the flow to this boundary. This is done by the so-called "Poincaré compactification," and also by the more general "Poincaré–Lyapunov compactification." In both cases we get an extended analytic differential system on the closed disk. In this way, we can study the behavior of the orbits near infinity. The singular points that are on the circle at infinity are called the infinite singular points of the initial polynomial differential system. Suitably gluing together two copies of the extended system, we get an analytic differential system on the two-dimensional sphere.

In Chap. 6 we associate an integer to every isolated singular point of a two-dimensional differential system, called its index. We prove the Poincaré–Hopf theorem for vector fields on the sphere that have finitely many singularities: the sum of the indices is 2. We also present the Poincaré formula for computing the index of an isolated singular point.

After singular points the main subjects of two-dimensional differential systems are limit cycles, i.e., periodic orbits that are isolated in the set of all periodic orbits of a differential system. In Chap. 7 we present the more basic results on limit cycles. In particular, we show that any topological configuration of limit cycles is realizable by a convenient polynomial differential system. We define the multiplicity of a limit cycle, and we study the bifurcations of limit cycles for rotated families of vector fields. We discuss structural stability, presenting a number of results and some open problems. We do not provide complete proofs but explain some steps in the exercises.

For a two-dimensional vector field the existence of a first integral completely determines its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a vector field on $\mathbb{R}^2$, how can one determine if this vector field has a first integral*? The easiest planar vector fields having a first integral are the Hamiltonian ones. The integrable planar vector fields that are not Hamiltonian are, in general, very difficult to detect. In Chap. 8 we study the existence of first integrals for planar polynomial vector fields through the Darbouxian theory of integrability. This kind of integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic curves that they have.

In Chap. 9 we present a computer program based on the tools introduced in the previous chapters. The program is an extension of previous work due to J. C. Artés and J. Llibre and strongly relies on ideas of F. Dumortier and the thesis of C. Herssens. Recently, P. De Maesschalck had made substantial adaptations. The program is called "Polynomial Planar Phase Portraits," abbreviated as P4 [9]. This program is designed to draw the phase portrait of any polynomial differential system on the compactified plane obtained by Poincaré or Poincaré–Lyapunov compactification; local phase portraits, e.g., near singularities in the finite plane or at infinity, can also be obtained. Of course, there are always some computational limitations that are described in Chaps. 9 and 10. This last chapter is dedicated to illustrating the use of the program P4.

Almost all chapters end with a series of appropriate exercises and some bibliographic comments.

The program P4 is freeware and the reader may download it at will from http://mat.uab.es/∼artes/p4/p4.htm at no cost. The program does not include either MAPLE or REDUCE, which are registered programs and must be acquired separately from P4. The authors have checked it to be bug free, but nevertheless the reader may eventually run into a problem that P4 (or the symbolic program) cannot deal with, not even by modifying the working parameters.

To end this preface we would like to thank Douglas Shafer from the University of North Carolina at Charlotte for improving the presentation, especially the use of the English language, in a previous version of the book.