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Dynamics of the Higgins–Selkov and Selkov systems

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ABSTRACT

We describe the global dynamics in the Poincaré disc of the Higgins–Selkov model

$x' = k_0 - k_1xy^2, \quad y' = -k_2y + k_1xy^2,$

where k_0, k_1, k_2 are positive parameters, and of the Selkov model

$x' = -x + ay + x^2y, \quad y' = b - ay - x^2y,$

where *a*, *b* are positive parameters. We determine the regions of initial conditions with biological meaning.

1. Introduction and statement of the results

The Higgins–Selkov model of glycolysis is

$$\dot{x} = k_0 - k_1xy^2, \quad \dot{y} = k_1xy^2 - k_2y, \tag{1}$$

where the unknowns x and y are concentrations which are non-negative and k_i for $i = 0, 1, 2$ are the reaction positive constants, see [7,13] for the biological details of this model.

We will describe the global dynamics of the differential system (1) in the Poincaré disc for all positive values of k_0, k_1 and k_2. For a definition of the Poincaré disc, and of its separatrices and canonical regions see Section A.2 of the Appendix. We denote by S (respectively R) the number of separatrices (respectively canonical regions) of a phase portrait in the Poincaré disc. Thus our first main result is:

Theorem 1. *The Higgins–Selkov system (1), after a rescaling of its variables, can be written as*

$$x' = 1 - xy^2, \quad y' = ay(xy - 1), \tag{2}$$

with $a > 0$. The global phase portraits of this system for $a \in \mathbb{R}$ is topologically equivalent to the one of

- *Fig. 1(A) for $a < 0$, with $S = 19$, $R = 6$;*
- *Fig. 1(B) for $a = 0$, with $S = \infty$;*
- *Fig. 1(C) for $a \in (0, 1]$, with $S = 17$, $R = 4$;*
- *Fig. 1(D) for $a \in (1, a^*]$ where $a^* \in (1.23, 1.24)$, with $S = 18$, $R = 5$;*
- *Fig. 1(E) for $a = a^*$, with $S = 16$, $R = 4$;*
- *Fig. 1(F) for $a \in (a^*, \infty)$, with $S = 17$, $R = 4$.*

For system (2) the bifurcation values of the parameter a are $a = 0$, $a = 1$ and $a = a^*$. We note that the Higgins–Selkov system (1) only reflects some biological meaning for some initial conditions when $a \in (0, a^*)$, otherwise the orbits that do not go to infinity have zero Lebesgue measure. In Fig. 1(C) and (D) the initial conditions with some biological meaning are in the shaded areas.

The proof of Theorem 1 modulo a conjecture is given in Section 2. In that section the conjecture is well stated. That conjecture is supported by numerical computations.

The Selkov model of glycolysis is given by the differential system

$$x' = -x + ay + x^2y, \quad y' = b - ay - x^2y, \tag{3}$$

with a and b positive parameters. The parameter b is named phosphofructokinase and the parameter a is called hexokinase which is the activant from all the glycolytic cycle. For a detailed derivation of system (3) see [13]. Our main aim is to study the global phase portraits on the Poincaré disc for the system (3) in function of its parameters. This is the content of the second main result in the paper.

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