1. Introduction

Let $H^\infty$ be the algebra of bounded analytic functions on the unit disc $D$ of the complex plane. A function in $H^\infty$ is called inner if it has radial limits of modulus one, almost everywhere on the unit circle. Given a sequence $\{z_n\}$ of points in $D$ satisfying the Blaschke condition $\sum_n (1-|z_n|) < +\infty$ and a real number $\gamma$, the Blaschke product

$$B(z) = e^{i\gamma z} \prod_{z_n \neq 0} \frac{z - z_n}{|z_n|} \frac{z_n}{1 - z_n \bar{z}}$$

for $z \in D$ is an inner function. Given a positive measure $\sigma$ on the unit circle, singular to Lebesgue measure, the singular function

$$S(\sigma)(z) = \exp \left( -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right)$$

for $z \in D$ is also inner. It is well known that any inner function can be factored into a Blaschke product and a singular function.

Let $I$ be an inner function and $\alpha \in D$. It is clear that

$$\tau_\alpha(I)(z) = \frac{I(z) - \alpha}{\bar{\alpha} I(z)}$$

for $z \in D$ is also inner. Actually, Frostman proved that for all $\alpha \in D$, except possibly for a set of logarithmic capacity zero, the function $\tau_\alpha(I)$ is a Blaschke product.

See [4, Chapter II] for the proofs of these results.

A Blaschke product $B$ is called indestructible if $\tau_\alpha(B)$ is a Blaschke product for all $\alpha \in D$, that is, if there is no exceptional set in Frostman's Theorem. As far as I know, the problem of characterizing the indestructible Blaschke products in terms of the distribution of their zeros remains open. In this paper we solve a conformal invariant version of that problem.

A positive measure $\mu$ on $D$ is a Carleson measure if there is a constant $C = C(\mu)$ such that $\mu(Q) \leq C \mu(Q)$, for every sector

$$Q = \{ z \in D : 1 - |z| \leq h, |\text{Arg} \ z - \theta| \leq h \}$$

(1.1)

where $\mu(Q) = h$.

A sequence $\{z_n\}$ of points in $D$ is called an interpolating sequence if, for every bounded sequence $\{w_n\}$ of complex numbers, there exists $f \in H^\infty$ such that $f(z_n) = w_n$. 

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for $n = 1, 2, \ldots$. Carleson proved that \{${z_n}$\} is an interpolating sequence if and only if
\[ \inf_{n \neq m} \frac{|(z_n - z_m)/(1 - \overline{z}_m z_n)|}{|z_n - z_m|^2} > 0 \quad \text{and} \quad \mu = \sum_n (1 - |z_n|^2) \delta_n \]
is a Carleson measure, where \(\delta_n\) is the Dirac measure at \(z_n\). (See [4, Chapter VII].)

An interpolating Blaschke product is a Blaschke product whose zero set is an interpolating sequence. It is known that a Blaschke product with zeros \{${z_n}$\} is a finite product of interpolating Blaschke products if and only if the measure \(\mu = \sum_n (1 - |z_n|^2) \delta_n\) is a Carleson measure [6]. This last condition is the conformal invariant version of the Blaschke condition; therefore the finite products of interpolating Blaschke products can be thought of, in terms of their zeros, as conformal invariant Blaschke products.

Our main result is a characterization of the Blaschke products \(B\) which are such that \(\tau_\alpha(B)\) is a finite product of interpolating Blaschke products for all \(\alpha \in \mathbb{D}\). Given a sector \(Q\) and \(N > 0\), we denote by \(NQ\) the dilatation of \(Q\) with factor \(N\), that is, the sector defined by the right-hand side of (1.1) with \(h\) replaced by \(Nh\).

**THEOREM.** Let \(B\) be a finite product of interpolating Blaschke products. Let \{${z_n}$\} be the sequence of zeros of \(B\), \(\delta_n\) the Dirac measure at \(z_n\) and \(\mu = \sum_n (1 - |z_n|^2) \delta_n\).

The following are equivalent.

(i) For all \(\alpha \in \mathbb{D}\), the function \(\tau_\alpha(B)\) is a finite product of interpolating Blaschke products.

(ii) For every \(M > 0\), there exist positive numbers \(\delta = \delta(M), \varepsilon = \varepsilon(M)\) such that if \(Q\) is a sector satisfying \(\ell(Q) < \delta\) and \(\mu(Q) > M\ell(Q)\), then there exists another sector \(Q'\) with \(\varepsilon Q < Q' \subset \varepsilon^{-1}Q\) such that
\[
\frac{\mu(Q)}{\ell(Q)} - \frac{\mu(Q')}{\ell(Q')} \geq \varepsilon.
\]

Condition (ii) is, in some sense, opposite to Bishop's condition characterizing the Blaschke products in the little Bloch space \(B_0\) (see [2]). Since Blaschke products in \(B_0\) are very far away from being interpolating, this should be not surprising. Actually, in the proof of (ii) \(\Rightarrow\) (i), we use some of Bishop's ideas.

We prove the Theorem in the next section. In Section 3, given a number \(m\) satisfying \(0 < m < 1\), we construct an interpolating Blaschke product \(B = B(m)\) such that \(\tau_\alpha(B)\) is not a finite product of interpolating Blaschke products, for all \(\alpha \in \mathbb{D}\) with \(|\alpha| > m\). So there is no analogue of Frostman's Theorem for the class of finite products of interpolating Blaschke products. We use this result in order to answer in the negative a question in [10] about the Nevanlinna–Pick interpolation problem. The last section contains some remarks.

I would like to thank Professor John Garnett for many helpful conversations.

2. Proof of the Theorem

Given a sector \(Q = \{z \in \mathbb{D} : 1 - |z| \leq h, \arg(z - \theta) \leq h\}\), define \(z_Q = (1 - h)e^{i\theta}\). For \(z \in \mathbb{D}\) and \(0 < \delta < 1\), let
\[
D_h(z, \delta) = \left\{ w \in \mathbb{D} : \rho(z, w) = \frac{|z - w|}{|1 - \overline{w}z|} < \delta \right\}
\]
be the pseudohyperbolic disc of centre \(z\) and radius \(\delta\). The following result follows easily from [5, Lemmas 1 and 3].
Lemma 1 [5]. Let $B$ be a Blaschke product with zeros $\{z_n\}$. Then $B$ is a finite product of interpolating Blaschke products if and only if there exists a number $m$ satisfying $0 < m < 1$ and a subsequence $\{c_n\}$ of $\{z_n\}$ such that the discs $D_N(c_n, m)$ are pairwise disjoint and

$$\inf \{ |B(z)| : z \notin \bigcup_n D_N(c_n, m) \} > 0.$$ 

Now let us go into the proof of the Theorem.

(i) $\Rightarrow$ (ii) Let $B$ be a finite product of interpolating Blaschke products and assume that (ii) fails. Then there exist $M > 0$, $\epsilon_j$ tending to zero, and sectors $Q_j$ such that

$$\mu(Q_j) > Ml(Q_j), \quad l(Q_j) \to 0,$$

and

$$\sup \left\{ \left( \frac{\mu(Q_j)}{l(Q_j)} - \frac{\mu(Q_j)}{l(Q_j)} \right) : \epsilon_j^2 Q_j \subset Q_j \subset \epsilon_j^{-2} Q_j \right\} \to 0. \quad (2.1)$$

Consider $\alpha_j = z_{Q_j}$. Using the inequality $\log x^{-1} \geq 2^{-1}(1 - x^2)$ for $0 < x < 1$, and the identity

$$1 - \left| \frac{z - w}{1 - \overline{w}z} \right|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\overline{w}|^2} \quad \text{for } z, w \in D,$$

one can get

$$\log |B(\alpha_j)|^{-1} \geq \frac{1}{2} \sum_{\epsilon_j \in \epsilon_j} \frac{(1 - |z_n|^2)(1 - |\alpha_j|^2)}{|1 - \overline{z_n}\alpha_j|^2} \geq \frac{1}{8(1 - |\alpha_j|^2)} \sum_{\epsilon_j \in \epsilon_j} (1 - |z_n|^2) = \frac{1}{8} \frac{\mu(Q_j)}{l(Q_j)} > \frac{1}{8} M.$$

Thus,

$$|B(\alpha_j)| \leq \exp(-\frac{1}{8} M) < 1.$$ 

We claim that it is sufficient to show that for each $m$ with $0 < m < 1$, one has

$$\sup \{|(1 - |z|)B'(z)| : z \in D_N(\alpha_j, m)\} \to 0. \quad (2.2)$$

Assume that (2.2) holds. Taking a subsequence if necessary, one can assume that

$$\lim_{j \to \infty} B(\alpha_j) = a \in D.$$ 

Now using (2.2), for each $0 < r < 1$ one has

$$\sup \left\{ \left| \frac{B(z) - a}{1 - \overline{a}B(z)} \right| : z \in D_N(\alpha_j, r) \right\} \to 0.$$ 

Applying Lemma 1, one gets that $\tau_a(B)$ is not a finite product of interpolating Blaschke products and this finishes the proof of (i) $\Rightarrow$ (ii). Thus, it suffices to show that (2.2) holds.

We shall omit the index $j$, writing $Q = Q_j, \alpha = \alpha_j, N = [\epsilon_j^{-1/2}]$ and $l(Q) \to 0, N \to \infty$ when $j \to \infty$. Consider the collection $\{Q^{(k)} : k = 1, \ldots, N^2\}$ of sectors with pairwise disjoint interiors lying inside $NQ$, with $l(Q^{(k)}) = N^{-1}l(Q)$ and $R = \bigcup_k Q^{(k)}$. If $l(Q)$ is sufficiently small, one has

$$\mu(NQ \setminus R) = 0. \quad (2.3)$$
Otherwise, there would exist \( z_n \in NQ \setminus R \), and taking
\[
T = \{ z \in D : 1 - |z| \leq 1 - |z_n|, |\text{Arg} z - \text{Arg} z_n| < 1 - |z_n| \}
\]
and \( T_1, T_2 \) the disjoint sectors inside \( T \) with \( l(T_i) = 2^{-i} l(T) \) for \( i = 1, 2 \), it would follow that
\[
\frac{\mu(T)}{l(T)} - \frac{\mu(T_1)}{l(T_1)} \geq 1 + \frac{\mu(T_1) + \mu(T_2)}{l(T)} - \frac{\mu(T_2)}{l(T_2)} = 1 + \frac{1}{2} \left[ \frac{\mu(T_2)}{l(T_2)} - \frac{\mu(T_1)}{l(T_1)} \right]
\]
and this would contradict (2.1). So (2.3) holds.

Applying (2.1), one gets
\[
\sup \left\{ \left| \frac{\mu(Q^{(i)})}{l(Q^{(i)})} - \frac{\mu(Q^{(k)})}{l(Q^{(k)})} \right| : i, k = 1, \ldots, N^2 \right\} \to 0 \quad \text{as} \quad l(Q) \to 0. \tag{2.4}
\]

Fix \( m \) such that \( 0 < m < 1 \). One can check that \( \rho(D_H(\alpha, m), D \setminus NQ) \to 0 \) as \( l(Q) \to 0 \), \( N \to \infty \). Also
\[
\sup \left\{ \frac{l(Q^{(k)})}{1 - |z|} : z \in D_H(\alpha, m) \right\} \leq \frac{N^{-1} l(Q)}{2^{-1} (1 - m) l(Q)} = 2N^{-1} (1 - m)^{-1}. \tag{2.5}
\]

Then (2.3) shows that
\[
\rho(D_H(\alpha, m), \{z_n\}) \to 1 \tag{2.6}
\]
as \( l(Q) \to 0, N \to \infty \). Applying Lemma 1, there exists a constant \( C > 0 \) such that
\[
\inf \{|B(z)| : z \in D_H(\alpha, m)\} \geq C > 0. \tag{2.7}
\]

Now, let us prove (2.2). Fixing \( z \in D_H(\alpha, m) \), one has
\[
(1 - |z|)|B'(z)| \leq (1 - |z|) \left| \frac{B'(z)}{B(z)} \right| \leq \left| \sum \frac{(1 - |z|)(1 - |z_n|^2)}{(z - z_n)(1 - \overline{z}_n z)} \right| \leq A + B,
\]
where
\[
A = \left| \sum_{z_n \in NQ} \frac{(1 - |z|)(1 - |z_n|^2)}{(z - z_n)(1 - \overline{z}_n z)} \right|, \quad B = \left| \sum_{z_n \in D \setminus NQ} \frac{(1 - |z|)(1 - |z_n|^2)}{(z - z_n)(1 - \overline{z}_n z)} \right|.
\]

Consider \( \|\mu\|_e = \sup \{\mu(Q) l(Q)^{-1} : Q \text{ is a sector of the form (1.1)}\} \). Applying (2.6) and the fact that \( \mu \) is a Carleson measure, one gets
\[
B \leq 2 \sum_{z_n \in NQ} \frac{(1 - |z|)(1 - |z_n|^2)}{|1 - \overline{z}_n z|^2} \leq 2 \sum_{k = \text{log}_Q(N)}^{\infty} (1 - |z|) \sum_{z_n \in Q \setminus \overline{Q} \setminus NQ} \frac{1 - |z_n|^2}{|1 - \overline{z}_n z|^2}
\]
\[
\leq 2 \sum_{k = \text{log}_Q(N)}^{\infty} \frac{\mu(2^{k+1}Q)}{2^{2k} (1 - |z|)} \leq \frac{4 \|\mu\|_e l(Q)}{N(1 - |z|)} \leq \frac{8 \|\mu\|_e}{N(1 - m)} \to 0 \quad \text{as} \quad l(Q) \to 0, N \to \infty.
\tag{2.8}
\]

On the other hand, (2.3) gives
\[
A = \left| \sum_{z_n \in \bigcup_{Q^{(k)}}} \frac{(1 - |z|)(1 - |z_n|^2)}{(z - z_n)(1 - \overline{z}_n z)} \right|.
\]
Take $\zeta_k = z_q^{(k)}$. Given $z_n \in Q^{(k)}$, a computation and (2.5) show that
\[
\left| \frac{1}{(z-z_n)(1-\overline{z}_n z)} - \frac{1}{(z-\zeta_k)(1-\overline{\zeta}_k z)} \right| \leq \frac{4N^{-1}l(Q)}{|1-z_n z|^2(1-|z|)} \leq \frac{8N^{-1}}{|1-z_n z|^2(1-m)}.
\]
Then, applying (2.7),
\[
\left| \sum_{k} \sum_{z_n \in Q^{(k)}} \left( \frac{1}{(z-z_n)(1-\overline{z}_n z)} - \frac{1}{(z-\zeta_k)(1-\overline{\zeta}_k z)} \right)(1-|z|)(1-|z_n|^2) \right|
\leq 8N^{-1}(1-m)^{-1} \sum_{k} \sum_{z_n \in Q^{(k)}} \frac{(1-|z|)(1-|z_n|^2)}{|1-\overline{z}_n z|^2}
\leq 16N^{-1}(1-m)^{-1} \log |B(z)|^{-1}
\leq 16N^{-1}(1-m)^{-1} \log (C^{-1}).
\]
Therefore
\[
A \leq 16N^{-1}(1-m)^{-1} \log (C^{-1}) + \left| \sum_{k} \sum_{z_n \in Q^{(k)}} \frac{(1-|z|)(1-|z_n|^2)}{(z-\zeta_k)(1-\overline{\zeta}_k z)} \right|
\]
\[
= 16N^{-1}(1-m)^{-1} \log (C^{-1}) + \sum_{k} \frac{(1-|z|)\mu(Q^{(k)})}{(z-\zeta_k)(1-\overline{\zeta}_k z)}.
\]
(2.9)
Applying (2.1) one gets
\[
\sup \left\{ \frac{\mu(Q^{(k)}) - \mu(Q^{(1)})}{l(Q^{(1)})} : k = 1, \ldots, N^2 \right\} \longrightarrow 0 \quad \text{as} \quad l(Q) \longrightarrow 0, N \longrightarrow \infty.
\]
Then, since $l(Q^{(1)}) \leq 2^{-1}||\zeta_k - \zeta_{k-1}||$ for $k = 1, \ldots, N^2$, one has
\[
\left| \sum_{k} \frac{(1-|z|)\mu(Q^{(k)}) - \mu(Q^{(1)})}{(z-\zeta_k)(1-\overline{\zeta}_k z)} \right|
\leq 2^{-1} \sup \left\{ \frac{\mu(Q^{(k)}) - \mu(Q^{(1)})}{l(Q^{(1)})} : k = 1, \ldots, N^2 \right\} \sum_{k} \frac{(1-|z|^2)||\zeta_k - \zeta_{k-1}||}{|1-\overline{\zeta}_k z|^2}
\leq \pi \sup \left\{ \frac{\mu(Q^{(k)}) - \mu(Q^{(1)})}{l(Q^{(1)})} : k = 1, \ldots, N^2 \right\} \longrightarrow 0 \quad \text{as} \quad l(Q) \longrightarrow 0, N \longrightarrow \infty,
\]
(2.10)
because the last sum is a Riemann sum of the Poisson kernel. Also
\[
\left| \sum_{k} \frac{(1-|z|)\mu(Q^{(1)})}{(z-\zeta_k)(1-\overline{\zeta}_k z)} \right| = \frac{\mu(Q^{(1)})}{l(Q^{(1)})} \sum_{k} \frac{l(Q^{(1)})(1-|z|)}{(z-\zeta_k)(1-\overline{\zeta}_k z)}
\]
is a Riemann sum of the integral
\[
\frac{\mu(Q^{(1)})}{l(Q^{(1)})} \int_{\Gamma} \frac{(1-|z|)d\zeta}{(z-\zeta)(1-\overline{\zeta})\zeta},
\]
where $\Gamma = \{\zeta \in D : |\zeta| = 1 - N^{-1}l(Q), \zeta \in NQ\}$. Since
\[
\left| \int_{|\zeta| = 1 - N^{-1}l(Q) \setminus \Gamma} \frac{(1-|z|)d\zeta}{(z-\zeta)(1-\overline{\zeta})\zeta} \right| \leq 2 \int_{|\zeta| = 1 - N^{-1}l(Q) \setminus \Gamma} \frac{(1-|z|)|\zeta|}{|z-\zeta|^2}d|\zeta| \longrightarrow 0
\]
as \( l(Q) \to 0, N \to \infty \), and

\[
\frac{(1-|z|) d\zeta}{|z| -1-N^{-1}l(Q)} \left( z - \zeta \right) \left( 1 - \frac{1}{z} \zeta \right) \zeta = 0,
\]

one gets

\[
\left| \sum_k \frac{(1-|z|) \mu(Q^{(k)})}{(z - \zeta_k)(1 - \frac{1}{z} \zeta_k)} \right| \to 0 \quad \text{as} \quad l(Q) \to 0, N \to \infty.
\]  

(2.11)

Now, (2.9), (2.10) and (2.11) give that \( A \to 0 \) as \( l(Q) \to 0, N \to \infty \). This shows that (2.2) holds and finishes the proof that (i) \( \Rightarrow \) (ii). (Recently, in a private communication, K. Oyma showed me a different proof of this implication, where he studies \(|B(z)|\) using harmonic measure techniques.)

(ii) \( \Rightarrow \) (i) If (i) fails, there exists \( \alpha \in D \) with \( \alpha \neq 0 \), such that \( \tau_\alpha(B) \) is not a finite product of interpolating Blaschke products. By Lemma 1, there exist \( \alpha_j \in D \) with \( |\alpha_j| \to 1 \), and \( m_j \) satisfying \( 0 < m_j \to 1 \), such that

\[
\sup \{|\log |B(\zeta)|^{-1} - \log |\alpha_j|^{-1}| : \zeta \in D_\mu(\alpha_j, m_j)\} \to 0 \quad \text{as} \quad j \to \infty.
\]  

(2.12)

We shall show that for each \( 0 < t < 1 \) one has

\[
\sup \left\{ \left| \mu(\zeta) - \log |\alpha_j|^{-1} \right| : \zeta \in D_\mu(\alpha_j, t) \right\} \to 0 \quad \text{as} \quad j \to \infty,
\]  

(2.13)

where \( Q_\zeta = \{ z \in D : 1-|z| \leq 1-|\zeta|, |\arg z - \arg \zeta| \leq 1-|\zeta| \} \). Since (2.13) contradicts (ii), this will finish the proof of the theorem.

Fix \( t \) with \( 0 < t < 1 \), \( \zeta \in D_\mu(\alpha_j, t) \) and \( Q = Q_\zeta \). Take \( s_j \) with \( 0 \leq s_j \to 1 \) such that \( (1-s_j)(1-m_j)^{-1} \to \infty \), and \( e_j \) with \( 0 < e_j \to 0 \) such that \( e_j(1-s_j)^{-1} \to \infty \). Consider

\[
R = R_j(\zeta) = \{ z \in Q : 1-|z| \leq (1-s_j)(1-|\zeta|) \},
\]

\[
L = L_j(\zeta) = (1-e_j)Q \cap \{ z : 1-|z| = (1-s_j)(1-|\zeta|) \}.
\]

From (2.12) it follows that \( B \) has no zeros in \( D_\mu(\alpha_j, m_j) \). The choice of the constants and a computation with the pseudohyperbolic distance, gives that for \( j \) sufficiently large, \( Q \setminus R \subset D_\mu(\alpha_j, m_j) \). Thus

\[
\mu(Q \setminus R) = 0,
\]  

(2.14)

\[
\inf \{ \rho(z, \{z_n\}) : z \in L = L_j(\zeta) \} \to 1 \quad \text{as} \quad j \to \infty.
\]  

(2.15)

Now using (2.15) and the facts that \( e_j(1-s_j)^{-1} \to \infty \) and \( \mu \) is a Carleson measure, one can see, as in (2.8), that

\[
\inf \{|B_{D \setminus R}(z)| : z \in L \} \to 1 \quad \text{as} \quad j \to \infty,
\]

where \( B_{D \setminus R} \) is the Blaschke product with zeros \( \{z_n : z_n \in D \setminus R\} \). Then, using (2.15) and the fact that \( (1-x^2)^{-1} \log x^2 \to 1 \) as \( x \to 1 \), one gets

\[
\sup \left\{ \left| 2 \log |B(z)|^{-1} - \int_R P_\rho(w) d\mu(w) \right| : z \in L \right\} \to 0 \quad \text{as} \quad j \to \infty,
\]

where \( P_\rho(w) = (1-|z|^2) 1-|\bar{w}z|^2 \). Since \( L \subset D_\mu(\alpha_j, m_j) \) for \( j \) sufficiently large, (2.12) shows that

\[
\sup \left\{ \left| 2 \log |\alpha_j|^{-1} - \int_R P_\rho(w) d\mu(w) \right| : z \in L \right\} \to 0 \quad \text{as} \quad j \to \infty.
\]
Now, parametrizing $L$ by $z = re^{i\theta}$, where $1 - r = (1 - s_j)(1 - |\zeta|)$, and integrating, one gets
\[
2|L| \log |x|^{-1} - \int R \int L P_{re^{i\theta}}(w) d\theta d\mu(w) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \tag{2.16}
\]
where $|L|$ is the Euclidian length of $L$. If $w$ is a zero of $B$ satisfying $w \in (1 - \varepsilon^j/2) Q$, one can check that
\[
\int L P_{re^{i\theta}}(w) d\theta = \int L \frac{1 - r^2}{|re^{i\theta} - w|^2} d\theta \longrightarrow 2\pi \quad \text{as } r \longrightarrow 1, \tag{2.17}
\]
because (2.14) and (2.15) show that the zeros of $B$ in $Q$ are much closer to the circle than the points of $L$ are. Also, since $\mu$ is a Carleson measure,
\[
\int \mathbb{R} \mathbb{I}(1 - \varepsilon^j/2) Q P_{re^{i\theta}}(w) d\theta \mu(w) \leq 2\pi \mu(\mathbb{R} \setminus (1 - \varepsilon^j/2) Q) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{2.18}
\]
Now, introducing (2.17) and (2.18) in (2.16), one gets
\[
2|L| \log |x|^{-1} - 2\pi \mu(\mathbb{R}) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.
\]
By (2.14), $\mu(\mathbb{R}) = \mu(Q)$ and $|L| = (1 - \varepsilon^j) l(Q)$. Therefore
\[
\left| \log |x|^{-1} - \frac{\pi \mu(Q)}{l(Q)} \right| \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.
\]
This proves (2.13) and finishes the proof of the theorem.

3. An example

Let $B$ be a Blaschke product. It follows from Lemma 1, that the set
\[
\{ \alpha \in D: \tau_\alpha(B) \text{ is not a finite product of interpolating Blaschke products} \}
\]
is closed. Let us remark that the exceptional set appearing in Frostman’s Theorem is not, in general, closed. In fact, it can even be dense on the unit disc (see [8, p. 714]).

In this section we shall show that there is no analogue of Frostman’s Theorem for the finite products of interpolating Blaschke products.

**PROPOSITION.** For each $m$ with $0 < m < 1$, there exists an interpolating Blaschke product $B = B_m$, such that $\tau_\alpha(B)$ is not a finite product of interpolating Blaschke products, for all $\alpha \in D$ and $|\alpha| \geq m$.

For the proof of the Proposition, we need the following results. Let $f \in H^\infty(D)$ and $e^{i\theta} \in \partial D$, then the radial cluster set of $f$ at $e^{i\theta}$ is the set of complex numbers $w$ such that there exists $r_k e^{i\theta}$ with $1 > r_k \to 1$, such that $f(r_k e^{i\theta}) \to w$.

**LEMMA 2.** There exists an interpolating Blaschke product whose radial cluster set at the point 1 contains the unit circle.

**LEMMA 3.** Let $f \in H^\infty(D)$, $z_k \in D$ and $t_k$ satisfying $0 < t_k < 1$, $t_k \to 1$ be such that
\[
\sup \{ |f(z) - \alpha| : z \in D_H(z_k, t_k) \} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\]
Then there exist $l_k$ with $0 < l_k < 1$, $l_k \longrightarrow 1$, such that
\[
\sup \{ |f(z) - f(z_k)| : z \in D_H(z_k, l_k) \} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\]
Proof of Lemma 2. Let \( \{e^{i\theta_k}\} \) be a dense sequence in the unit circle. We shall choose the zeros \( \{z_n\} \) of the Blaschke product by induction and we shall denote by \( B_N \) the Blaschke product with zeros \( z_1, \ldots, z_N \).

Choose \( r_1 \) and \( z_1 \) with \( 0 < r_1 < 1 \) and \( \rho(z_1, r_1) = 2^{-1} \) such that
\[
|B_1(r_1) - e^{i\theta_1}| = 2^{-1}.
\]

Assume that we have defined \( r_1, \ldots, r_k \) with \( 1 - r_j^2 \leq 2^{-3j}(1 - r_j^2) \) and \( z_1, \ldots, z_k \) with \( \rho(z_j, r_j) = 1 - 2^{-j} \) for \( j = 1, \ldots, k \), such that
\[
|B_k(r_k) - e^{i\theta_k}| \leq 2^{-k+1}.
\]

Then, choose \( r_{k+1} < 1 \) with \( 1 - r_{k+1}^2 < 2^{-3k}(1 - r_k^2) \) such that
\[
1 - |B_k(r_{k+1})| \leq 2^{-k-1}
\]
and \( z_{k+1} \) with \( \rho(z_{k+1}, r_{k+1}) = 1 - 2^{-k-1} \) such that
\[
\left| \frac{-z_{k+1}}{1 - z_{k+1}r_{k+1}} - \frac{e^{i\theta_{k+1}} B_k(r_{k+1})}{|B_k(r_{k+1})|} \right| = 2^{-k-1}.
\]

Now,
\[
|B_{k+1}(r_{k+1}) - e^{i\theta_{k+1}}| \leq 1 - |B_k(r_{k+1})| + \left| \frac{-z_{k+1}}{1 - z_{k+1}r_{k+1}} - \frac{e^{i\theta_{k+1}} B_k(r_{k+1})}{|B_k(r_{k+1})|} \right| \leq 2^{-k}.
\]

Since \( \rho(z_j, r_j) = 1 - 2^{-j} \) and \( 1 - r_{k+1}^2 \leq 2^{-3k}(1 - r_k^2) \), using the inequality
\[
\frac{|z| - |w|}{1 - |w||z|} \leq \rho(z, w) \leq \frac{|z| + |w|}{1 + |w||z|}
\]
for \( z, w \in D \) (see [4, p. 4]), one gets
\[
1 - |z_{k+1}|^2 \leq 1 - \left( \frac{r_{k+1} - (1 - 2^{-k-1})(1 - 2^{-k})}{1 - (1 - 2^{-k-1}) r_{k+1}} \right)^2 = \left( 1 - 2^{-k} \right) \frac{2^{2k} \left( 1 - 2^{-k} \right)^2}{1 - 2^{-k} r_{k+1}} \leq 2^{-3k} \frac{1 - r_k^2}{2^{-2k}} = 2^{-k} \left( 1 - r_k^2 \right) 2^{-2k} \leq 2^{-k} \left( 1 - \left( \frac{r_k + (1 - 2^{-k}) r_k}{1 + (1 - 2^{-k}) r_k} \right)^2 \right) \leq 2^{-k+2} (1 - |z_k|^2).
\]

This shows that \( \{z_n\} \) is an interpolating sequence. Let \( B \) be the Blaschke product with zeros \( \{z_n\} \). One has
\[
|B(r_k) - e^{i\theta_k}| \leq |B_k(r_k) - e^{i\theta_k}| + \left| \frac{B(r_k)}{B_k(r_k)} - 1 \right| \leq 2^{-k+1} + \sum_{j=k+1}^{\infty} \left| \frac{-z_j}{1 - z_j r_k} - \frac{z_j}{1 - z_j r_k} \right| \leq 2^{-k+1} + 2 \sum_{j=k+1}^{\infty} \frac{|z_j|}{1 - r_k} \leq 2^{-k+1} + 4 \frac{|z_{k+1}|}{1 - r_k} \leq 2^{-k+1} + 8 \cdot 2^{-3k}.
\]

Since \( \{e^{i\theta_k}\} \) is dense in the unit circle, the radial cluster set of \( B \) at the point 1 contains the unit circle.
Proof of Lemma 3. Assume that the conclusion fails. Then taking a subsequence if necessary, there exist \( l \) with \( 0 < l < 1 \) and points \( c_k \in D_{K}(z_k, l) \) such that 
\[
|f(c_k) - f(z_k)| \geq \delta > 0.
\]
Since 
\[
\delta \leq |f(c_k) - f(z_k)| \leq \int_{z_k}^{c_k} |f'(w)| \, dw,
\]
there exist \( \zeta_k \in D_{K}(z_k, l) \) and a constant \( c = c(l) > 0 \), such that 
\[
(1 - |\zeta_k|)|f'(\zeta_k)| \geq c\delta.
\]
Applying Bloch's Theorem [3, p. 295] to the function 
\[
f(\zeta_k) - f(z_k) \quad \text{for} \quad z \in D,
\]
one gets 
\[
f(D_{K}(\zeta_k, 2^{-1})) \supset D(f(\zeta_k), (72)^{-1}2^{-1}(1 - |\zeta_k|^2)|f'(\zeta_k)|) \supset D(f(\zeta_k), 144^{-1}c\delta)
\]
and this contradicts the hypothesis of Lemma 3.

Proof of the Proposition. Let \( B_1 \) be an interpolating Blaschke product satisfying the conditions of Lemma 2. Choose \( r_k \) with \( 0 < r_k < 1, r_k \to 1 \), such that 
\[
\{B_1(r_k) : k = 1, 2, \ldots\}
\]
is dense on the unit circle and 
\[
\frac{1 - r_{k+1}}{1 - r_k} \leq 2^{-2k} \quad \text{for} \quad k = 1, 2, \ldots. \tag{3.1}
\]
Take \( \alpha_k \) with \( m < \alpha_k < 1 - k^{-1} \), such that \( \{\alpha_k B_1(r_k) : k = 1, 2, \ldots\} \) is dense in 
\[
\{z : m < |z| < 1\}.
\]
We shall construct an interpolating Blaschke product \( B_2 \) such that \( B_1 B_2 \) is interpolating and for all \( k = 1, 2, \ldots \), the function 
\[
\frac{B_1 B_2(z) - \alpha_k B_1(r_k)}{1 - \alpha_k B_1(r_k) B_1 B_2(z)} \quad \text{for} \quad z \in D
\]
is not a finite product of interpolating Blaschke products. Then, the observation of the beginning of this section will give the proof of the Proposition.

Consider \( Q_k = \{z \in D : 1 - |z| \leq 1 - r_k, |\text{Arg} z| \leq 1 - r_k\} \), then 
\[
k\tilde{Q}_k \cap \{|z| = 1\} = \{e^{i\theta} : -a_k \leq \theta \leq a_k\},
\]
where \( a_k = k(1 - r_k) \). Define \( t_k \) by \( 1 - t_k = k^{-1}(1 - r_k) \) and \( s_k \) with \( 0 < s_k < \pi \) by 
\[
|\exp(is_k) - 1| = 2\pi(1-t_k)(\log|x_k^{-1}|)^{-1}, \quad \text{and put}
\]
\[
z_n^{(k)} = t_k \exp(is_k n) \quad \text{for} \quad n = -[a_k s_k^{-1}], \ldots, 0, \ldots, [a_k s_k^{-1}].
\]
Thus, 
\[
|z_n^{(k)} - z_{n+1}^{(k)}| = t_k 2\pi(1-t_k)(\log|x_k^{-1}|)^{-1} = t_k(1 - |z_n^{(k)}|) 2\pi(\log|x_k^{-1}|)^{-1}. \tag{3.2}
\]
If \( Q \) is a sector in the unit disc, one has 
\[
\sum_{n : z_n^{(k)} \in Q} (1 - |z_n^{(k)}|) \leq 2(2\pi)^{-1}\log|x_k^{-1}|l(Q) \leq 2(2\pi)^{-1}\log m^{-1}l(Q). \tag{3.3}
\]
Let $I_k$ be the Blaschke product with zeros $\{z_n^{(k)}: n = -[a_k s_k^{-1}], \ldots, [a_k s_k^{-1}]\}$. Let $Z(B)$ denote the zero set of the Blaschke product $B$. One can choose $r_k$ in such a way that $\rho(Z(I_k), Z(B_i)) \geq 2^{-1}$.

Let $a_k \sim b_k$ mean that $|a_k - b_k| \to 0$, as $k \to \infty$. Using (3.2), for each $M < 1$, one has

$$\sup \{|\log |I_k(z)|^{-1} - \log |\alpha_k|^{-1}|: z \in D_H(r_k, M)\} \sim \sup \left\{ \frac{1}{2} \sum_n \left( \frac{(1 - |z_n^{(k)}|^2)}{|1 - \overline{z_n^{(k)}} z|^2} - \log |\alpha_k|^{-1} \right): z \in D_H(r_k, M) \right\}$$

$$\sim \sup \left\{ (2\pi)^{-1}(\log |\alpha_k|^{-1}) \sum_n \frac{|z_n^{(k)} - z_n^{(0)}|}{|1 - \overline{z_n^{(k)}} z|^2} - \log |\alpha_k|^{-1} \right\}: z \in D_H(r_k, M) \right\} \to 0$$

as $k \to \infty$, because the last sum is a Riemann sum of the integral of the Poisson kernel at the point $z$ along the arc $\{t_k e^{i\theta}: -a_k \leq \theta \leq a_k\}$. Since the points $\{z_n^{(k)}\}$ are symmetric with respect to the real axis, one has $I_k(r_k) > 0$. So (3.4) and Lemma 3 give that

$$\sup \{|I_k(z) - \alpha_k|: z \in D_H(r_k, M)\} \to 0 \quad \text{as} \quad k \to \infty$$

for each $M$ with $0 < M < 1$. Consider $B_2 = \prod_k I_k$. Using (3.1) and (3.3) one can easily show that $B_2$ is an interpolating Blaschke product. Since $\rho(Z(B_2), Z(B_j)) \geq 2^{-1}$, it follows that $B_1 B_2$ is an interpolating Blaschke product. Also, using (3.1) and the symmetry of $\{z_n^{(k)}\}$, one can check that

$$\prod_{j \neq k} I_j(r_k) \to 1 \quad \text{as} \quad k \to \infty.$$

So, from (3.5) and Schwarz's Lemma, it follows that

$$\sup \{|B_2(z) - \alpha_k|: z \in D_H(r_k, M)\} \to 0 \quad \text{as} \quad k \to \infty$$

for each $M$ with $0 < M < 1$. Since $|B_1(r_k)| \to 1$ as $k \to \infty$, another application of Schwarz's Lemma gives that

$$\sup \{|B_1(z) - B_1(r_k)|: z \in D_H(r_k, M)\} \to 0 \quad \text{as} \quad K \to \infty$$

for each $0 < M < 1$. Now for fixed $a = \alpha_k B_1(r_k)$, (3.6) and (3.7) imply that there exists a subsequence $\{p_k\}$ of $\{r_k\}$ such that

$$\sup \left\{ \frac{B_1 B_2(z) - a}{|1 - \overline{a} B_1 B_2(z)|} : z \in D_H(p_k, M) \right\} \to 0 \quad \text{as} \quad k \to \infty$$

for each $M$ with $0 < M < 1$, and Lemma 1 shows that the function

$$\frac{B_1 B_2(z) - a}{1 - \overline{a} B_1 B_2(z)}$$

is not a finite product of interpolating Blaschke products. This finishes the proof of the Proposition.

Now, we use the Proposition in order to answer in the negative a question in [10, p. 515]. First, we recall some results.

Give two sequences of points $\{z_n\}, \{w_n\}$ in $D$, the Nevanlinna-Pick interpolation problem consists in finding analytic functions $f \in H^\infty$ satisfying

$$\|f\|_\infty = \sup \{|f(z)|: z \in D\} \leq 1 \quad \text{and} \quad f(z_n) = w_n \quad \text{for} \quad n = 1, 2, \ldots.$$
We shall denote it by

(\ast) \text{Find } f \in H^\infty, \|f\|_\infty \leq 1, f(z_n) = w_n, n = 1, 2, \ldots.

Pick and Nevanlinna found necessary and sufficient conditions in order that the problem (\ast) has a solution. Let $G$ be the set of all solutions of the problem (\ast). Nevanlinna showed that if $G$ consists of more than one element, there is a parametrization of the form

$$G = \left\{ f \in H^\infty : f = \frac{p\phi + q}{r\phi + s}, \phi \in H^\infty, \|\phi\|_\infty \leq 1 \right\},$$

where $p, q, r, s$ are certain analytic functions in $D$, depending on \{z_n\}, \{w_n\} and satisfying $ps - qr = B$, the Blaschke product with zeros \{z_n\}.

Later, Nevanlinna showed that for each unimodular constant $e^{i\theta}$, the function

$$I_\theta = \frac{p e^{i\theta} + q}{r e^{i\theta} + s}$$

is inner. Therefore, if the problem (\ast) has more than one solution, then there are inner functions solving it. See [4, pp. 6, 165] for the proofs of these results.

Recently, A. Stray [9] has proved that, in fact, for all unimodular constants $e^{i\theta}$ except possibly for a set of zero logarithmic capacity, the function $I_\theta$ is a Blaschke product. Also [10, Theorem 3], if \{z_n\} is an interpolating sequence, then there exists a number $r > 0$ depending only on \{z_n\}, such that if

$$\inf\{\|f\|_\infty : f \in G\} = r,$$

then the function $I_\theta$ is a finite product of interpolating Blaschke products for all unimodular constants $e^{i\theta}$.

In [10, p. 515], the question is asked if the same result is valid with some numerical constant $r$ independent of \{z_n\}. We now answer this question in the negative.

For each $m$ with $0 < m < 1$, let $B = B_m$ be the interpolating Blaschke product given by the Proposition. Let \{z_n\} be the sequence of zeros of $B$. Now, choose $\alpha = \alpha_m \in D$ with $|\alpha| = m$, and consider the following Nevanlinna–Pick problem.

(\ast)_m \text{ Find } f \in H^\infty, \|f\|_\infty \leq 1, f(z_n) = -\alpha, n = 1, 2, \ldots.

Let $G_m$ be the set of all solutions of (\ast)_m. It is clear that

$$G_m = \left\{ \frac{B\phi - \alpha}{1 - \bar{\alpha} B\phi}, \phi \in H^\infty, \|\phi\|_\infty \leq 1 \right\},$$

$$\inf\{\|f\|_\infty : f \in G_m\} = m.$$

Now the Proposition gives that the function

$$I_\theta = \frac{B e^{i\theta} - \alpha}{1 - \bar{\alpha} B e^{i\theta}}$$

is not a finite product of interpolating Blaschke products, for all $\theta \in [0, 2\pi]$. Since one can choose $m$ with $0 < m < 1$ to be arbitrarily small, this shows that the constant $r$ cannot be chosen independently of \{z_n\}. 
4. Remarks

Let $I$ be an inner function. A computation shows that $I$ is a finite product of interpolating Blaschke products if and only if there exists $r$ with $0 < r < 1$ such that

$$
\sup_{|w| < 1} \int_0^{2\pi} \log \left| I \left( \frac{re^{i\theta} + w}{1 + \overline{w}re^{i\theta}} \right) \right|^{-1} d\theta < +\infty.
$$

This could be understood as the conformal invariant version of Frostman's condition characterizing Blaschke products among inner functions (see [4, p. 56]).

Using the techniques of [1], one can show that conditions (i) or (ii) in the Theorem are also equivalent to any of the following.

(iii) For each $m$ with $0 < m < 1$, there exists $r$ with $0 < r < 1$ such that

$$
\inf \left\{ \int_{D_H(z, r)} |B'(w)|^2 dm(w) : |B(z)| \leq m \right\} > 0.
$$

(iv) For each $m$ with $0 < m < 1$, there exists $r$ with $0 < r < 1$ such that

$$
\inf \{ \text{diameter} (B(D_H(z, r))) : |B(z)| \leq m \} > 0.
$$

(v) For each $m$ with $0 < m < 1$, there exists $r$ with $0 < r < 1$ such that

$$
\inf \left\{ \frac{1}{(1-|z|)^2} \int_{D_H(z, r)} |B(w) - B(z)|^2 dm(w) : |B(z)| \leq m \right\} > 0.
$$

H. Morse [7] constructed a destructible Blaschke product which becomes indestructible when a single point is deleted from its zero-set. So no asymptotic condition on the measure $\mu$ can characterize indestructible Blaschke products.

References


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