Sampling real polynomials

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Paley-Wiener spaces

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Theorem (Whittaker-Kotelnikov-Shannon)

If $f \in PW$ then $(f(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}$$

They are a model for bandlimited signals with finite energy.

Orthogonal bases

Remark

Actually $\{\frac{\sin \pi(z-n)}{\pi(z-n)}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of reproducing kernels in PW.

We may ask, what other sequences $\{\lambda_n\} \subset \mathbb{R}$ instead of the integers provide orthogonal basis of reproducing kernels in PW?

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A more interesting problem is to consider Riesz basis of reproducing kernels, i.e. sequences λ_n of points in $\mathbb{R}$ such that there is a bounded invertible map $T : PW \to PW$ and

$$k_{\lambda_n} = \left\{ \frac{\sin \pi(z-\lambda_n)}{\pi(z-\lambda_n)} \right\}$$

such that $\{T(k_{\lambda_n})\}$ is an orthonormal basis of PW.

Frames and Sampling sequences

An alternative to orthogonal bases are frames: Let $\Lambda \subset \mathbb{R}$, then

Definition

A sequence of functions $\{f_\lambda(z) = \frac{\sin \pi(z-\lambda)}{\pi(z-\lambda)}\}_{\lambda \in \Lambda}$ is a frame whenever for all functions f in PW,

$$\|f\|^2 \simeq \sum_{\lambda \in \Lambda} |\langle f_\lambda, f \rangle|^2 = \sum_{\lambda \in \Lambda} |f(\lambda)|^2.$$

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One says that Λ is a sampling sequence for PW.
Any function in the space $f \in PW$ can be reconstructed from its samples

$$f(z) = \sum_{\lambda\in\Lambda} f(\lambda)\widetilde{f}_\lambda(z).$$

Here $f_\lambda(z)$ is the dual frame. The reconstruction is not unique.

The density of a sampling sequence

There is a density that almost describes sampling sequences

Definition

The lower Beurling-Landau density of a sequence $\Lambda \subset \mathbb{R}$ is

$$D^-(\Lambda) = \lim_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{\#\{\Lambda \cap (x-r, x+r)\}}{2r}.$$

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Theorem (Beurling)

A separated sequence $\Lambda \subset \mathbb{R}$ is sampling for PW if $D^-(\Lambda) > 1$. Moreover if Λ is sampling then $D^-(\Lambda) \geq 1$.

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Remark

The case $D^-(\Lambda) = 1$ is more delicate. There is a complete characterization of sampling sequences with the critical density

A finite dimensional model

Marzinkiewicz-Zygmund original result

We may model bandlimited functions by polynomials.

Theorem (Marzinkiewicz-Zygmund)

Let $\Lambda_n = \{e^{\frac{2ij\pi}{n+1}}\}_{j=0}^{n}$ be the $(n+1)$-roots of unity and let $1 < p < \infty$. There are constants C_p, such that

$$\frac{C_p^{-1}}{n} \sum_{\lambda \in \Lambda_n} |q(\lambda)|^p \leq \int_0^{2\pi} |q(e^{it})|^p \, dt \leq \frac{C_p}{n} \sum_{\lambda \in \Lambda_n} |q(\lambda)|^p$$

for all polynomials $q \in \mathcal{P}_n$.

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for all polynomials $q \in \mathcal{P}_n$.

The relevant point is that C_p is independent of the degree n. It is interesting to know what other possible sequences Λ_n are possible.

Beurling-Landau necessary conditions

An asymptotic density

We assume that $p = 2$ for simplicity, and that Λ_n are a collection of points with the same property as in the theorem of M-Z. Then

Theorem (Beurling-Landau)

If Λ_n are a sequence of finite sets with the Marzinkiewicz-Zygmund property for $q = 2$, then for any subinterval $I \subset \mathbb{T}$ we have

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The normalized Lebesgue measure in $\mathbb{T}$ is the critical Nyquist density in this context.

Our setting

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Let $\mathcal{P}_k$ be the real polynomials of degree k.
Let dV be the normalized Lebesgue measure restricted to Ω.
We denote by $N_k = dim(\mathcal{P}_k)$.
We endow $\mathcal{P}_k$ with the norm given by $L^2(V)$.

$$\|p\|^2 = \int_\Omega |f(x)|^2 \, dV(x).$$

Marzinkiewicz-Zygmund sequences

Let $\Lambda = \{\Lambda_k\}_k \subset \Omega$ be a sequence of finite sets of points of Ω.

Definition

We say that Λ is a Marzinkiewicz-Zygmund sequence if there is a constant $C > 0$ such that

$$C^{-1} \sum_{\lambda \in \Lambda_k} \frac{|p(\lambda)|^2}{c_{\lambda,k}} \leq \int_\Omega |p|^2 \, dV \leq C \sum_{\lambda \in \Lambda_k} \frac{|p(\lambda)|^2}{c_{\lambda,k}}, \qquad \forall p \in \mathcal{P}_k,$$

with a natural normalization $c_{\lambda,k}$.

We are interested in the geometric distribution of points in Λ.

The “natural” normalization

The natural normalization is

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This can be computed as follows. Take $p_1, \ldots, p_{N_k}$ an orthonormal basis of $\mathcal{P}_k$ and construct:

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Moreover K_k is the reproducing kernel:

$$p(z) = \int_\Omega K_k(z, w) p(w)\, dV(w), \qquad \forall p \in \mathcal{P}_k$$

Frames and sampling sequences

The orthogonal projection $L^2(\Omega) \to \mathcal{P}_k$ is given by the integral kernel K_k. We denote by κ_λ the normalized reproducing kernel

$$\kappa_\lambda(z) = K_k(\lambda, z)/\sqrt{K_k(\lambda, \lambda)}.$$

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Then Λ is a sampling sequence if and only if the normalized reproducing kernels form a frame in $\mathcal{P}_k$, i.e.:

$$C^{-1} \int_\Omega |p|^2 \, dV \leq \sum_{\lambda \in \Lambda_k} |\langle p, \kappa_\lambda \rangle|^2 \leq C \int_\Omega |p|^2 \, dV \quad \forall p \in \mathcal{P}_k$$

Landau's strategy

This is a strategy that was applied in bandlimited functions and does *not* work in this context, but nevertheless the heuristics give some insight on the problem.

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- If κ_λ is well localized around λ, the number of points of Λ_k in a subdomain should be bigger than the local dimension of $\mathcal{P}_k$.
- The local dimension in a domain U can be interpreted as $\int_U K_k(z,z)dV$.
- Altogether the above points show that if Λ is sampling then asymptotically Λ_k has a density bigger than some critical Nyquist density.

The Nyquist density

We try to identify which is the critical density. We will use the following result:

Theorem (Berman, Boucksom, Witt-Nyström)

If μ is a Bernstein-Markov measure then

$$\frac{K_k(x,x)d\mu(x)}{N_k} \xrightarrow{*} \mu^{eq}.$$

The Bernstein-Markov condition is technical and it is satisfied when $\mu = \chi_\Omega dV$. The measure μ^{eq} is the equilibrium measure.

The equilibrium potential

Definition

Given a compact $K = \overline{\Omega} \subset \mathbb{R}^n$ and any $z \in \mathbb{C}^n$ one defines the Siciak-Zaharjuta equilibrium potential as

$$u_K(z) = \sup\left\{ \frac{\log |p(z)|}{\deg(p)} : \sup_K |p| \leq 1 \right\}.$$

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Then the equilibrium measure is defined as the Monge-Ampere of u_K

$$\mu^{eq} = (i\partial\bar{\partial} u_K)^n.$$

The equilibrium measure is a positive measure supported on K.

What does μ^{eq} look like?

The measure μ^{eq} is a well-known object in pluripotential theory. In the examples we mentioned before it is well understood.

Theorem (Bedford-Taylor)

If Ω is an open bounded convex set in $\mathbb{R}^n$ then

$$d\mu^{eq}(x) \simeq d(x, \partial\Omega)^{-1/2} dV(x).$$

Main result

Theorem

If Λ is a Marzinkiewicz-Zygmund sequence for the polynomials in a bounded smooth convex domain then

$$\liminf_{k\to\infty} \frac{1}{N_k} \sum_{\lambda\in\Lambda_k} \delta_\lambda \geq \mu^{eq}.$$

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In particular, given any ball B in Ω we have

$$\liminf_{k\to\infty} \frac{\#(\Lambda_k \cap B)}{N_k} \geq \mu^{eq}(B),$$

thus μ^{eq} is the Nyquist density.

The Kantorovich-Wasserstein distance

Given a compact metric space K we defines the K-W distance between two probability measures μ and ν supported in K as

$$KW(\mu,\nu) = \inf_{\rho} \iint_{K\times K} d(x,y)d\rho(x,y),$$

where ρ is an admissible probability measure, i.e. the marginals of ρ are μ and ν respectively.

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where ρ is an admissible probability measure, i.e. the marginals of ρ are μ and ν respectively. Alternatively:

$$KW(\mu,\nu) = \inf_{\rho} \iint_{K\times K} d(x,y)d|\rho|(x,y),$$

where ρ is an admissible complex measure, i.e. the marginals of ρ are μ and ν respectively

The complex transport plan

The K-W distance metrizes the weak-$*$ convergence. We want to prove that

$$KW(b_k, \sigma_k) \to 0,$$

where $b_k = K_k(x,x)dV(x)/N_k$ is the Bergman measure and

$$\sigma_k \leq \frac{1}{N_k} \sum_{\lambda \in \Lambda_k} \delta_\lambda$$

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The transport plan ρ_k that is convenient to estimate is:

$$\rho_k(x,y) = \frac{1}{N_k} \sum_{\lambda \in \Lambda_k} \delta_\lambda(y) \times g_\lambda(x) \frac{K_k(\lambda, x)}{\sqrt{K_k(\lambda,\lambda)}} \, dV(x),$$

where g_λ is the dual frame to $\left\{ \frac{K_k(\lambda,x)}{\sqrt{K_k(\lambda,\lambda)}} \right\}_{\lambda \in \Lambda_k}$

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The two marginals of ρ_k are

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and

$$KW(\nu_k, \sigma_k) \leq \frac{1}{N_k} \sum_{\lambda \in \Lambda_k} \int_\Omega d(\lambda, x) |g_\lambda(x)| \frac{|K_k(\lambda, x)|}{\sqrt{K_k(\lambda,\lambda)}}\, dV(x).$$

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and

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Thus

$$KW^2(\nu_k,\sigma_k) \lesssim \frac{1}{N_k} \iint d^2(x,y)|K_k(x,y)|^2\, dV(x)\, dV(y).$$

An off-diagonal estimate

Given a bounded function f on M we denote by T_f be the Toeplitz operator on $\mathcal{P}_k \cap L^2(\Omega)$ with symbol f, i.e. $T_f := \Pi_k \circ f \cdot$ where Π_k denotes the orthogonal projection from $L^2(\Omega)$ to $\mathcal{P}_k$.

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$$\operatorname{Tr} T_f^2 - \operatorname{Tr} T_{f^2} = \frac{1}{2} \int_{\Omega \times \Omega} (f(x) - f(y))^2 \, |K_k(x,y)|^2 \, dV(x) dV(y).$$

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Now, setting $f := x_i$ we observe than on $\mathcal{P}_{k-1}$, $T_f(p) = x_i p$. Therefore $T_{f^2} - T_f^2 = 0$ on $\mathcal{P}_{k-2}$. Therefore:

$$\operatorname{Tr} T_f^2 - \operatorname{Tr} T_{f^2} = O(k^{n-1})$$

and

$$KW^2(\nu_k, \sigma_k) \lesssim \frac{1}{k}.$$

Some extensions

There are many extensions of this result. Of special interest:
Let M be a compact smooth algebraic variety in $\mathbb{R}^m$.
We endow the space of polynomials $\mathcal{P}_k$ restricted to M with the L^2 norm with respect to the Lebesgue measure. We define M-Z sequences Λ as before.

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Theorem

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$$\liminf_{k\to\infty} \frac{1}{N_k} \sum_{\lambda\in\Lambda_k} \delta_\lambda \geq \mu^{eq}.$$

The equilibrium measure in this setting is comparable to the Lebesgue measure.

Bernstein inequality

A Bernstein type inequality of independent interest:

Theorem

Given $M \subset \mathbb{R}^m$ be a smooth compact manifold. TFAE:

- *There is $C > 0$ such that for all polynomials p:*

$$\int_M |\nabla_t p| dV_M \leq C \deg p \int_M |p| dV_M.$$

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- *There is a uniformly separated Λ such that*

$$\int_M |p|^2 dV_M \lesssim \frac{1}{k^n} \sum_{\lambda \in \Lambda_k} |p(\lambda)|^2 \lesssim \int_M |p|^2 dV_M, \ \forall p \in \mathcal{P}_k(\mathbb{R}^m).$$

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- *M is algebraic.*