Extreme values of the Riemann zeta function

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Riemann zeta function

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Riemann zeta function

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**Meromorphic continuation:**

\[ \zeta(s) = \frac{1}{s-1} + A(s), \quad \Re(s) > 1, \]

\( A(s) \) is an entire function.
Functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s).$$
Symmetry

**Functional equation:**

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**Reason:** **Poisson summation formula**

\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \]
Symmetry

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\[ \zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s). \]

Reason: Poisson summation formula

\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \]

\[ \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/x} \]
Now take Mellin's transform of $\theta(x) = \sum_{n \in \mathbb{N}} e^{-\pi n^2 x} M_{\theta}(s) = \int_{0}^{\infty} x^{s-1} \theta(x) \, dx$.
Symmetry

Now take **Mellin’s transform** of

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\[
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\]

\[
\mathcal{M}\theta(s) = \int_0^\infty x^{s-1} \theta(x) \, dx
\]
Trivial zeros:
\[ \zeta(s) = 0, \text{ for } s = -2, -4, \ldots \]
There are many zeroes on \( 1/2 + it, \ t \in \mathbb{R} \).
Riemann hypothesis

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There are many zeroes on \( 1/2 + it, \ t \in \mathbb{R} \).

**Riemann hypothesis:** All nontrivial zeroes are on the critical line.
Lindelöf hypothesis

For any $\epsilon > 0$

$$|\zeta\left(\frac{1}{2} + it\right)| = o(t^\epsilon), \quad t \to \infty.$$
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On the Riemann hypothesis,

$$|\zeta(1/2 + it)| \ll \exp\left(\frac{c \log t}{\log \log t}\right).$$
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Bourgain (2016, JAMS):

$$|\zeta\left(\frac{1}{2} + it\right)| = O(t^{13/84 + \epsilon}).$$
\[ \zeta\left(\frac{1}{2} + it\right) = \sum_{n=1}^{N} n^{-1/2-it} - \frac{N^{1/2-it}}{1/2-it} + O(N^{-1/2}), \quad |t| < N. \]
Approximate formulae

\[\zeta\left(\frac{1}{2} + it\right) = \sum_{n=1}^{N} n^{-1/2-it} - \frac{N^{1/2-it}}{1/2 - it} + O(N^{-1/2}), \quad |t| < N.\]

Should be a lot of cancelations!
Approximate formulae

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Should be a lot of cancelations!

Why for any large \( a \in \mathbb{N} \) and \( t < a^{1/\epsilon} \)

\[ a^{it} + (a + 1)^{it} + \ldots + (2a)^{it} = o(a^{1/2+\epsilon})? \]
Asymmetry

\[ \sum_{n=1}^{N} \frac{n}{2} - \frac{1}{2} \]

attains "small values" on \([N/2, N] \), but doesn't attain "large values"!
It seems that

\[ \sum_{n=1}^{N} n^{-1/2-it} \]

attains “small values” on \([N/2, N]\), but doesn’t attain “large values”!
Montgomery; Balasubramanian and Ramachandra, 1977: \( \exists \) large \( T \) with

\[
|\zeta(1/2 + iT)| \geq \exp \left( c \sqrt{\frac{\log T}{\log \log T}} \right).
\]

Soundararajan (2008): \( c = 1 + o(1) \).
Lower bounds

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**Theorem 1.** (B, Seip; ArXiv, 2015)

\( \exists \) large \( T \) with

\[
|\zeta(1/2 + iT)| \geq \exp \left( c \sqrt{\frac{\log T \log \log \log T}{\log \log T}} \right),
\]

where \( c = 1/\sqrt{2} + o(1) \).
Farmer–Gonek–Hughes (2007) have conjectured, by use of random matrix theory, that the right bound is

$$\exp \left( \left( \frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log T \log \log T} \right).$$
What is the truth?

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\]

Example:

\[
|\zeta(1/2 + i \times 3.9246764\ldots \times 10^{31})| \approx 16244.86526
\]

For this particular \( T \)

\[
\exp \left( \left( \frac{1}{\sqrt{2}} \right) \sqrt{\log T \log \log T} \right) \approx 264964.
\]
Sketch of the proof

1. Resonance method

Assume that a function $F > 0$ is "localized" on $[-T, T]$

$$\int_R \zeta \left(1/2 + it\right) F(t) \, dt = M_1$$

$$\int_R F(t) \, dt = M_2$$

Then for some $t \in [-T, T]$, $|\zeta \left(1/2 + it\right)| \gg M_1/M_2$. 

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Sketch of the proof

\[ F = \left| \sum_{m \in M'} r(m) m^2 \Phi\left( \log T \right) \right|, \]

where \( \Phi(t) = e^{-t^2/2} \).

So, \( \Phi \) and \( \hat{\Phi} \) are positive!

\[ M_1 = \sqrt{\frac{2}{\pi} T \log T} \sum_{m, n \in M'} r(m)r(n) \Phi(\log m \log n) + \text{small terms}. \]

\[ M_2 = \sqrt{\frac{2}{\pi} T \log T} \sum_{m, n \in M'} \sum_{k \leq T} r(m)r(n) \sqrt{k} \Phi(\log m \log n) + \text{small terms}. \]
2. Choice of $F$

$$F = \left| \sum_{m \in M'} r(m) m^i t \right|^2 \Phi \left( \frac{\log T}{T} t \right),$$

where $\Phi(t) = e^{-t^2/2}$. 
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$$M_1 = \frac{\sqrt{2\pi} T}{\log T} \sum_{m,n \in \mathcal{M}'} r(m)r(n) \Phi \left( \frac{T}{\log T} \log \frac{m}{n} \right) + \text{small terms.}$$

$$M_2 = \frac{\sqrt{2\pi} T}{\log T} \sum_{m,n \in \mathcal{M}'} \sum_{k \leq T} \frac{r(m)r(n)}{\sqrt{k}} \Phi \left( \frac{T}{\log T} \log \frac{km}{n} \right) + \text{small terms.}$$
Sketch of the proof

3. Optimization problem

\[ |\mathcal{M}| = N \approx T^{1/2}, \]

\[ \sum_{m \in \mathcal{M}} f(m)^2 = 1 \]

Maximize

\[ \sum_{m,n \in \mathcal{M}, m=kn} \frac{f(n)f(m)}{\sqrt{k}}. \]
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\[ \sum_{m,n \in \mathcal{M}, m=kn} \frac{f(n)f(m)}{\sqrt{k}}. \]

Answer:

\[ \exp \left( \sqrt{\frac{\log N \log \log \log \log N}{\log \log N}} \right). \]
How to choose $\mathcal{M}$ and $f$?

A multiplicative function $f$ is such that $f(mn) = f(n)f(m)$ for $(m, n) = 1$ and supported on square-free numbers, where $f(p) := \sqrt{\log N \log^2 N \log^3 N \log p \log p}$. $M$ is the set where the "mass" of $f$ lives.
How to choose $\mathcal{M}$ and $f$?

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$$f(p) := \sqrt{\frac{\log N \log_2 N}{\log_3 N}} \frac{1}{\sqrt{p \log p}},$$

$$p \leq \log N \exp((\log_2 N)^{1-o(1)}).$$
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Related question

What is the maximum of

$$\sum_{m,n \in \mathcal{M}} c_m c_n \frac{(m, n)}{\sqrt{mn}},$$

where

$$\sum_{m \in \mathcal{M}} c_n^2 = 1?$$
Related question

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Or

$$\frac{1}{|\mathcal{M}|} \sum_{m,n \in \mathcal{M}} \frac{(m, n)}{\sqrt{mn}}?$$

They are almost the same!

Reason: \((m, n)\) is a certain inner product.

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$$\frac{1}{|\mathcal{M}|} \sum_{m,n \in \mathcal{M}} \frac{(m, n)}{\sqrt{mn}} ?$$

They are almost the same! Reason: (m,n) is a certain inner product.
Example: If $\mathcal{M}$ are all divisors of $p_1 \ldots p_\ell$ then

$$\frac{1}{|\mathcal{M}|} \sum_{m,n\in \mathcal{M}} \frac{(m, n)}{\sqrt{mn}} = \prod_{j=1}^{\ell} (1 + p_j^{-1/2})$$
Related question

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How to prove upper bounds?
Example: If $\mathcal{M}$ are all divisors of $p_1 \ldots p_\ell$ then

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How to prove upper bounds?

- It is enough to consider square free numbers
- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.
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How to prove upper bounds?

- It is enough to consider square free numbers
- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.
- Divisor closed extremal sets $\mathcal{M}$ enjoy the following completeness property: If $n \in \mathcal{M}$, $p|n$, $p' < p$, then either $p'|n$ or $p'n/p \in \mathcal{M}$. 
Combining the last with Aistleitner–Berkes–Seip arguments we obtain.

\[ \sum_{k,\ell=1}^{N} \left( \frac{\binom{n}{k} \binom{n}{\ell}}{\sqrt{n^k n^\ell}} \right) \approx \exp \left( A \sqrt{\frac{\log N}{\log \log N}} \right) \]

where \( 1 < A < 7 \).
Combining the last with Aistleitner–Berkes–Seip arguments we obtain.

**Theorem 2.** (B, Seip, 2015)

\[
\frac{1}{N} \sup_{1 \leq n_1 \ldots < n_N} \sum_{k, \ell = 1}^{N} \frac{(n_k, n_\ell)}{\sqrt{n_k n_\ell}} \approx \exp \left( A \sqrt{\frac{\log N \log \log \log N}{\log \log N}} \right),
\]

where \(1 \leq A < 7\).

Other tools: Bohr correspondence, multiplicative functions, Chauchy–Shwarz inequality
Questions

Is there a better choice of resonator, allowing for improvements?

Could we apply resonators for the upper bounds?
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- Could we apply resonators for the upper bounds?
THANK YOU!