HOROSPHERES IN HYPERBOLIC GEOMETRY

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ABSTRACT. In this paper we investigate the role of horospheres in Integral Geometry and Differential Geometry. In particular we study envelopes of families of horocycles by means of "support maps". We define invariant "linear combination" of support maps or curves. Finally we obtain Gauss-Bonnet type formulas and Chern-Lashof type inequalities.

1. INTRODUCTION

Some parts of Integral Geometry and Differential Geometry in euclidean spaces rely on the space of hyperplanes. For instance kinematic formulas, support functions, height functions in relation with the Gauss map and the total (absolute) curvature. Here the space of oriented hyperplanes is a cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ equipped with an isotropic metric invariant with respect to euclidean motions. In fact this isotropic metric is just the pullback of the metric on $\mathbb{S}^{n-1}$ under the canonical projection.

In Hyperbolic Geometry this situation looks quite different. The space of geodesic hyperplanes is topologically a cylinder but with a non-degenerated Lorentz-metric invariant with respect to hyperbolic motions (de Sitter sphere). In some sense horospheres are closer to euclidean hyperplanes. The space of horospheres is a half-cone $\mathbb{S}^{n-1} \times \mathbb{R}^+$ equipped with an invariant isotropic metric, which is a warped product of the metric on $\mathbb{S}^{n-1}$ with $\mathbb{R}^+$.

In this paper we investigate the role of horospheres in Integral Geometry and Differential Geometry. After some prelimineries

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we study in section 3 envelopes of families of horocycles by means of "support maps". In section 4 we define invariant "linear combination" of support maps or curves. Finally in section **??** we obtain Gauss-Bonnet type formulas and Chern-Lashof type inequalities.

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2. Preliminaries

We use the Lorentz space model for the Hyperbolic Geometry. In detail, the model lives in Lorentz space $\mathbb{R}^{n+1}_1$ with its Lorentz product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}\,.$$

The n-dimensional hyperbolic space $\mathbb{H}^n$ is realized as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = -1 \ \wedge \ x_{n+1} > 0\}$$

i.e. the upper half of an two-sheeted hyperboloid with the light cone $\mathcal{C}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0\}$ as asymptotic cone. The group G of hyperbolic motions of $\mathbb{H}^n$ is given by the subgroup of the Lorentz group leaving invariant $\mathbb{H}^n$.

The space $\mathcal{H}$ of horospheres of $\mathbb{H}^n$ is realized as the upper half of the light cone, i.e.

$$\mathcal{H} = \mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0 \ \wedge \ x_{n+1} > 0\,.\}$$

Indeed, horospheres in $\mathbb{H}^n$ are exactly the non-void sections of $\mathbb{H}^n$ with hyperplanes which are parallel to hyperplanes tangent to the light cone $\mathcal{C}^n$. Given $\theta \in \mathcal{C}^n_+$, then the affine hyperplane $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$ is parallel to the tangent hyperplane $T_\theta\mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = 0\}$ of $\mathcal{C}^n_+$ at θ and intersects $\mathbb{H}^n$ in horosphere which we also denote by Θ. Given a horosphere Θ in $\mathbb{H}^n$, i.e. the intersection of $\mathbb{H}^n$ with an affine hyperplane Θ parallel to a tangent hyperplane of $\mathcal{C}^n_+$ along along a half light-ray. Then there exists exactly one θ in this half light-ray such that $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$. (In the following we shall

always denote horospheres in $\mathbb{H}^n$ (or the underlying affine hyperplane) by capital greek letters and the vectors representing them in $\mathcal{C}_+^n$ by the corresponding small greek letters.)
The correspondence between θ and the hyperplane Θ comes exactly from the polarity relation with respect to the quadric $\pm\mathbb{H}^n \subset \mathbb{R}_1^{n+1}$. The Lorentz product induces on $\mathcal{C}^n$ a degenerated product (isotropic metric).

The light-rays in the cone $\mathcal{C}_+^n$ are exactely the pencils of "parallel" horospheres. Two parallel horospheres Θ_1 and Θ_2 touch one another at a point at infinity, and they lie in constant hyperbolic distance to each other. A little computation in the model shows that this distance is given by $|\ln\lambda|$, where $\lambda \in \mathbb{R}^+$ is given by $\theta_2 = \lambda\theta_1$. Here we use the signed distance from Θ_1 to Θ_2 by

$$\mathrm{d}(\Theta_1, \Theta_2) = -\ln\lambda\,. \tag{1}$$

For fixed Θ_1, as $\lambda \to +\infty$ the horosphere Θ_2 shrinks to the common point at infinity whereas the signed distance $\mathrm{d}(\Theta_1, \Theta_2) \to -\infty$. On the other side, if $\lambda \to 0$, then Θ_2 expands over the whole H^n and $\mathrm{d}(\Theta_1, \Theta_2) \to +\infty$.

The space of horospheres $\mathcal{H} = \mathcal{C}_+^n \subset \mathbb{R}_1^{n+1}$ is endowed with a n-form ω which is invariant under the Lorentz group. In terms of the coordinates $(x_1, \ldots, x_{n+1}) \in \mathbb{R}_1^{n+1}$ this form is given by

$$\omega = \frac{1}{x_{n+1}} dx_1 \wedge \ldots \wedge dx_{n+1} = x_{n+1}^{n-2} dx_{n+1} dv \tag{2}$$

where dv is the spherical volume element at $x_{n+1}^{-1}(x_1, \ldots, x_n) \in \mathbb{S}^{n-1}$, cf. [San67], [San68].

Our bridge between the point space H^n and the space of horospheres $\mathcal{C}_+^n$ is the following.

Definition 2.1. Let M be a smooth regular hypersurface in $\mathbb{H}^n$ and $\nu(x)$, $x \in M$, a unit normal vector field along M. Then $\theta(x) = x + \nu(x) \in \mathcal{C}_+^n$ represents the horosphere $\Theta(x)$ which is tangent to M at x such that $\nu(x)$ points into its convex side. We call

$$\theta : M \longrightarrow \mathcal{C}_+^n \quad , \quad x \mapsto x + \nu(x) \tag{3}$$

the *"support map"* of M.

The support map θ of M is smooth, and in general tranverse to the generators of $\mathcal{C}_+^n$.

3. Envelopes of horocycles

3.1. **Support maps: from c to θ.** Let us start with a regular parametrized curve $c(s)$ in $\mathbb{H}^2$, $s \in I$, s an arc length parameter.

In order to describe the differential geometry of the curve, we use the Frenet theory. That means we have the positive oriented Frenet frame along c, build by the unit tangent vector $e_1(s) = c'(s)$ and the normal unit vector $e_2(s)$. The Frenet equations $\bigtriangledown_{e_1(s)} e_1(s) = \kappa_g(s) e_2(s)$, $\bigtriangledown_{e_1(s)} e_2(s) = -\kappa_g(s) e_1(s)$ then define the geodesic curvature κ_g of c ($\bigtriangledown$ denotes the co-variant derivative in $\mathbb{H}^2$).

In order to describe the support map, let $\nu(s)$ be a unit normal vector field along c. We consider the support map θ of c with respect to ν, i.e.

$$\theta : I \to \mathcal{C}_+^2 \quad \text{with} \quad \theta(s) = c(s) + \nu(s).$$

The horocycle $\Theta(s)$ is tangent to c at $c(s)$ and $\nu(s)$ points into its convex side.

Then (the primes denote derivations with respect to s)

$$\theta' = c' + \nu' = c' + \epsilon e_2' = (1 - \epsilon\kappa_g) e_1$$

with $\epsilon := \langle \nu, e_2 \rangle$. (Note: $\langle e_2, e_2 \rangle = 1 \Rightarrow \langle e_2, e_2' \rangle = 0$, $\langle c, e_2 \rangle = 0 \Rightarrow 0 = \langle c', e_2 \rangle + \langle c, e_2' \rangle = \langle c, e_2' \rangle$, hence $e_2' \in T_c\mathbb{H}^2$.)

this shows that the curve $\theta(s)$ is regular parametrized iff $\kappa_g \neq \epsilon 1$. The curve $\theta(s)$ is then space-like, and its arc length parameter σ is given by

$$d\sigma = |1 - \epsilon\kappa_g| ds. \tag{4}$$

3.2. **Envelopes: from θ to c.** Let us start with a regular parametrized curve $\theta(\sigma)$ in $\mathcal{C}_+^2$ which is locally a graph with respect to the generators of $\mathcal{C}_+^2$, $\sigma \in I$, σ an arc length parameter, i.e $\langle\dot\theta,\dot\theta\rangle = 1$. We look for the envelope curve $c(\sigma)$ of the family $\Theta(\sigma)$ of horocycles in $\mathbb{H}^2$, i.e.

$$\begin{aligned} \langle c,c\rangle &= -1, \\ \langle c,\theta\rangle &= -1, \\ \langle \dot c,\theta\rangle &= 0 \quad \text{(envelope condition)}. \end{aligned} \tag{5}$$

For the curve θ we have
$\langle\theta,\theta\rangle = 0 \Rightarrow \langle\dot\theta,\theta\rangle = 0 \ \Rightarrow 0 = \langle\ddot\theta,\theta\rangle + \langle\dot\theta,\dot\theta\rangle = \langle\ddot\theta,\theta\rangle + 1$
(the points denote derivations with respect to σ),
$\langle\dot\theta,\dot\theta\rangle = 1 \ \Rightarrow \langle\dot\theta,\ddot\theta\rangle = 0$, and
$\langle\dot\theta,\ddot\theta\rangle = 0 \Rightarrow \langle\ddot\theta,\ddot\theta\rangle + \langle\dot\theta,\dddot\theta\rangle = 0$.
From (5) we get
$\langle c,c\rangle = -1 \Rightarrow \langle\dot c,c\rangle = 0$, and
$\langle c,\theta\rangle = -1 \Rightarrow 0 = \langle\dot c,\theta\rangle + \langle c,\dot\theta\rangle = \langle c,\dot\theta\rangle$.
Now, we assume that $\theta,\dot\theta,\ddot\theta$ are linear independent, and we try $c = \alpha\theta + \beta\dot\theta + \gamma\ddot\theta$ with unknown functions α,β,γ . We take into account the above relations, i.e.
$0 = \langle c,\dot\theta\rangle = \alpha\langle\theta,\dot\theta\rangle + \beta\langle\dot\theta,\dot\theta\rangle + \gamma\langle\ddot\theta,\dot\theta\rangle = \beta$, and
$-1 = \langle c,\theta\rangle = \alpha\langle\theta,\theta\rangle + \beta\langle\dot\theta,\theta\rangle + \gamma\langle\ddot\theta,\theta\rangle = -\gamma$, and
$-1 = \langle c,c\rangle = \alpha^2\langle\theta,\theta\rangle + \beta^2\langle\dot\theta,\dot\theta\rangle + \gamma^2\langle\ddot\theta,\ddot\theta\rangle + 2\alpha\beta\langle\theta,\dot\theta\rangle + 2\alpha\gamma\langle\theta,\ddot\theta\rangle + 2\beta\gamma\langle\dot\theta,\ddot\theta\rangle = \gamma^2\langle\ddot\theta,\ddot\theta\rangle - 2\alpha\gamma$.
And we get

$$c = \frac{1}{2}\Big(1 + \langle\ddot\theta,\ddot\theta\rangle\Big)\,\theta + \ddot\theta\ . \tag{6}$$

Now, with the expression (6) for c we directly check
$\langle c,c\rangle = -1$, i.e. $c \subset \mathbb{H}^n$, $\langle c,\theta\rangle = -1$, i.e. $c \in \Theta$, and $\langle\dot c,\theta\rangle = 0$. And therefore, c is the envelope of Θ we looked for.

From (6) we get by differentiation

$$\dot c = \frac{1 - \langle\ddot\theta,\ddot\theta\rangle}{2}\,\dot\theta\ . \tag{7}$$

(To this, we try $\dddot\theta$ as a linear combination of the vectors $\theta,\dot\theta,\ddot\theta$. We take into account $\langle\ddot\theta,\theta\rangle = -1 \Rightarrow \langle\ddot\theta,\dot\theta\rangle + \langle\dddot\theta,\theta\rangle = 0$ and $\langle\ddot\theta,\dot\theta\rangle = 0 \Rightarrow \langle\ddot\theta,\ddot\theta\rangle + \langle\dddot\theta,\dot\theta\rangle = 0$, in order to get $\dddot\theta = -\langle\dddot\theta,\ddot\theta\rangle\theta -$

$\langle\ddot\theta,\ddot\theta\rangle\dot\theta$.)
Formula (7) shows that the envelope c is regular iff $\langle\ddot\theta,\ddot\theta\rangle\neq 1$.

Remark 3.1. The condition $\langle\ddot\theta,\ddot\theta\rangle\neq 1$ means, that the osculating plane of the curve θ in $\mathbb{R}^3_1$ is not tangent to the model $\mathbb{H}^2$. This property characterizes curves θ in $\mathcal{C}^2_+$ which envelope regular curves in $\mathbb{H}^2$.

Remark 3.2. The osculating plane of θ at a fixed parameter defines a family of horocycles with the following geometric meaning: If the osculating plane is space-like, then the envelope curve c of θ has an osculating circle at the point under consideration. We have $|\kappa_g|>1$ at this point. And the family of horocycles envelopes this osculating circle on their concave sides or convex sides respectively, if the plane of the family intersects $\mathbb{H}^2$ or avoids $\mathbb{H}^2$ respectively.
If the osculating plane is of mixed type, then the envelope curve c of θ has an osculating equidistant at the point under consideration. We have $|\kappa_g|<1$ at this point. And the family of horocycles envelopes this equidistant.

Finally we compute the geodesic curvature κ_g of the curve c in terms of θ:
From (6) we have $c=\alpha\theta+\gamma\ddot\theta$, and further

$$c'=\frac{dc}{ds}=\frac{d\sigma}{ds}\dot c=\frac{d\sigma}{ds}\left(\dot\alpha\theta+\alpha\dot\theta+\dot\gamma\ddot\theta+\gamma\dddot\theta\right)\ .$$

Because of $c'\sim\dot\theta$ and $|\dot\theta|=1$ we can compute

$$\begin{aligned}1=|c'|=|\langle c',\dot\theta\rangle|&=|1-\epsilon\kappa_g|\left|\left\langle\dot\alpha\theta+\alpha\dot\theta+\dot\gamma\ddot\theta+\gamma\dddot\theta,\dot\theta\right\rangle\right|=\\ &=|1-\epsilon\kappa_g|\left|\left(\dot\alpha\langle\theta,\dot\theta\rangle+\alpha\langle\dot\theta,\dot\theta\rangle+\dot\gamma\langle\ddot\theta,\dot\theta\rangle+\gamma\langle\dddot\theta,\dot\theta\rangle\right)\right|=\\ &=|1-\epsilon\kappa_g|\left|\left(\alpha-\gamma\langle\ddot\theta,\ddot\theta\rangle\right)\right|\ .\end{aligned}$$

Inserting the coefficients α,β from (6) we get

$$1=\frac{|1-\epsilon\kappa_g|}{2}\left|\left(1-\langle\ddot\theta,\ddot\theta\rangle\right)\right|\ . \tag{8}$$

3.3. **Further relations between c and θ.** We want to write the length $L(c)$ and the total curvature $TC(c)$ of the point curve c in terms of the support curve θ.

Proposition 3.1. *Let $c(s), s \in I$, be a regular curve in H^2 parametrized by arc length s and $\nu(s)$ a unit normal vector field along c. Let $\theta : I \to \mathcal{C}^2_+$, $\theta(s) = c(s) + \nu(s)$ denote the support map of c with respect to ν, and set $\epsilon = \langle \nu, e_2 \rangle$.*
If $\epsilon = +1$ and $\kappa_g > 1$, or $\epsilon = -1$ and $\kappa_g > -1$ respectively, then

$$L(c) = \frac{1}{2} \int_\theta \epsilon \left(\kappa_\theta^2 - 1 \right) \, d\sigma, \tag{9}$$

where κ_θ is the curvature of the curve θ as a curve in $\mathbb{R}^3_1$, and

$$TC(c) = \int_c \kappa_g \, ds = \frac{1}{2} \int_\theta \left(\kappa_\theta^2 + 1 \right) \, d\sigma. \tag{10}$$

Proof. The case $\epsilon = +1$ and $\kappa_g > 1$: From (4) and (8) we get

$$ds = \frac{1}{\kappa_g - 1} \, d\sigma = \frac{|1 - \langle \ddot{\theta}, \ddot{\theta} \rangle|}{2} \, d\sigma.$$

Locally c lies in the convex side of Θ, hence we have $\langle c, \ddot{\theta} \rangle > 0$. (This can be seen in the model: Take the intersection of $\mathcal{C}^2_+$ and the plane through θ in direction span$(\dot{\theta}, \ddot{\theta})$ which represents the horocycles tangent to the osculating circle of c, and take into account that c locally lies in the convex side of Θ.) Hence through (6) we have $1 - \langle \ddot{\theta}, \ddot{\theta} \rangle < 0$. Because σ is an arc length parameter on θ we have $\langle \ddot{\theta}, \ddot{\theta} \rangle = \kappa_\theta^2$. Altogether we get (9).
From (4) and (8), taking into account $1 - \langle \ddot{\theta}, \ddot{\theta} \rangle < 0$, we get

$$\kappa_\theta \, ds = \frac{\langle \ddot{\theta}, \ddot{\theta} \rangle + 1}{2} \, d\sigma,$$

hence we get (10).
In case $\epsilon = -1$ and $\kappa_g > -1$ the proof runs analogously. □

4. Linear combinations of support maps

In the cone $\mathcal{C}^n_+$ each generator is a half-ray, i.e. $\mathbb{R}^+$. Hence along each generator we have an addition and a multiplication, as far as well defined. This way we are able to define "linear combinations" of support maps. In the following we investigate the 2-dimensional situation.

4.1. **The "λ-multiple".**

Definition 4.1. Let $c(s), s \in I$, be a regular curve in $\mathbb{H}^2$ parametrized by arc length s and $\nu(s)$ a unit normal vector field along c. Let $\theta : I \longrightarrow \mathcal{C}^2_+$, $\theta(s) = c(s) + \nu(s)$ denote the support map of c with respect to ν.
For $\lambda \in \mathbb{R}_+$, we call the envelope of $\lambda\theta$ in $\mathbb{H}^n$, in case it is well defined, the *"λ-multiple λc of c"*.

We consider the case $\epsilon = +1$ and $\kappa_g > 1$. Then locally c lies in the convex side of each of its tangent horocycles which are supporting c, and we have $\langle\ddot{\theta},\ddot{\theta}\rangle > 1$.
We consider $\theta^* = \lambda\theta$ with $\lambda > 0$. Using (1) we set $t = d(\Theta,\Theta^*) = -\ln\lambda$.

a) The case $\lambda > 1$: The envelope c^* of θ^* is the inner parallel curve to c at distance t.
We compute

$$\langle\frac{d^2\theta^*}{(d\sigma^*)^2},\frac{d^2\theta^*}{(d\sigma^*)^2}\rangle = \frac{1}{\lambda^2}\langle\ddot{\theta},\ddot{\theta}\rangle\ .$$

Taking into account (7) we get: If $\lambda^2 < \langle\ddot{\theta},\ddot{\theta}\rangle = \kappa_\theta^2$, then the envelope c^* is regular. Singular points occur for $\lambda^2 = \langle\ddot{\theta},\ddot{\theta}\rangle$.
In our case we have $\langle\ddot{\theta},\ddot{\theta}\rangle > 1$. Therefore (8) implies

$$\kappa_g = \frac{\langle\ddot{\theta},\ddot{\theta}\rangle + 1}{\langle\ddot{\theta},\ddot{\theta}\rangle - 1}. \tag{11}$$

The geodesic curvature κ_g and the curvature radius ρ are related by

$$\kappa_g = \coth\rho = \frac{e^{2\rho}+1}{e^{2\rho}-1}\ , \tag{12}$$

hence

$$e^{2\rho} = \langle \ddot{\theta}, \ddot{\theta} \rangle \ . \tag{13}$$

This shows that singular points occur for $t = -\rho$, i.e. singular points occur when the inner parallel curve of c runs through focal points of c.

b) The case $\lambda < 1$: The envelope $c^* = c_t$ of θ^* is the outer parallel curve to c at distance t.
Because of $\lambda < 1$, formula (7) shows that c_t is regular for all $t > 0$.
We now compute the length of c_t: Using (9), $d\sigma^* = \lambda \, d\sigma$, we get

$$\begin{aligned} L(c_t) = L(c^*) &= \frac{1}{2} \int_{\theta^*} \langle \frac{d^2\theta^*}{(d\sigma^*)^2}, \frac{d^2\theta^*}{(d\sigma^*)^2} \rangle \, d\sigma^* - \frac{1}{2} \int_{\theta^*} d\sigma^* = \\ &= \frac{1}{2} \frac{1}{\lambda} \int_\theta \langle \ddot{\theta}, \ddot{\theta} \rangle \, d\sigma - \frac{1}{2} \lambda \int_\theta d\sigma = \\ &= \frac{1}{2} \left(\frac{1}{\lambda} - \lambda \right) \int_c \kappa_g \, ds + \frac{1}{2} \left(\frac{1}{\lambda} + \lambda \right) L(c) \ . \end{aligned}$$

And replacing $\lambda = e^{-t}$, we arrive at

$$L(c_t) = \sinh(t) \int_c \kappa_g \, ds + \cosh(t) L(c) \ . \tag{14}$$

This is a well-known Steiner formula in hyperbolic plane, cf. e.g. [San76].

4.2. **The "sum".**

Definition 4.2. Let c_1, c_2 be two regular curves in $\mathbb{H}^2$ and θ_1, θ_2 their support maps with respect to unit normal fields ν_1, ν_2 along c_1, c_2. Then we call the envelope of $\theta_1 + \theta_2$ in $\mathbb{H}^n$, in case it is well defined, the *"sum $c_1 + c_2$ of c_1 and c_2"*.

Suppose $\theta_2 = \lambda \theta_1$, parametrized by the arc length parameter σ_1 on θ_1. Then we have

$$\frac{d\theta_2}{d\sigma_1} = \frac{d\lambda}{d\sigma_1} \theta_1 + \lambda \frac{d\theta_1}{d\sigma_1} \ ,$$

$$\langle \frac{d\theta_2}{d\sigma_1}, \frac{d\theta_2}{d\sigma_1} \rangle = \lambda^2 \langle \frac{d\theta_1}{d\sigma_1}, \frac{d\theta_1}{d\sigma_1} \rangle = \lambda^2 \ ,$$

hence

$$d\sigma_2 = \lambda\, d\sigma_1 \ . \tag{15}$$

We consider, in case it is well defined, $\theta^* = \theta_1 + \theta_2 = (1+\lambda)\,\theta_1$. Then we have

$$\frac{d\theta^*}{d\sigma_1} = \frac{d\lambda}{d\sigma_1}\theta_1 + (1+\lambda)\frac{d\theta_1}{d\sigma_1}$$

and

$$\langle \frac{d\theta^*}{d\sigma_1}, \frac{d\theta^*}{d\sigma_1} \rangle = (1+\lambda)^2 \langle \frac{d\theta_1}{d\sigma_1}, \frac{d\theta_1}{d\sigma_1} \rangle = (1+\lambda)^2 \ .$$

Therefore we get

$$d\sigma^* = (1+\lambda)\, d\sigma_1 = \frac{1+\lambda}{\lambda}\, d\sigma_2 = d\sigma_1 + d\sigma_2 \ , \tag{16}$$

and we arrive at

Proposition 4.1. *For the lengths of the support images involved in the "sum" the following relation holds*

$$L(\theta^*) = L(\theta_1 + \theta_2) = L(\theta_1) + L(\theta_2) \ . \tag{17}$$

In order to compute the length L^* and the total curvature TC^* of the sum in terms of the summands, we need the following

Definition 4.3. Let c_1, c_2 be two regular curves in $\mathbb{H}^2$ and θ_1, θ_2 their support maps with respect to unit normal fields ν_1, ν_2 along c_1, c_2. Then $\theta_2 = \lambda\theta_1$, and the signed distance $\mathrm{d}(\Theta_1, \Theta_2)$ from Θ_1 to Θ_2 is given by $\mathrm{d}(\Theta_1, \Theta_2) = -\ln\lambda$, cf. (1). We call

$$w_{12} : \theta_1 \to \mathbb{R} \quad , \quad \sigma_1 \mapsto -\ln\lambda(\sigma_1) \tag{18}$$

the *"mixed width function of c_1 and c_2 with respect to c_1".*

The mixed width function describes the relative position of c_1 and c_2 to one another in terms of the distance between parallel tangent horocycles.

Remark 4.1. If θ_1 is the support map of a point $O \in \mathbb{H}^2$, then w_{12} coincides with the horocycle support function of c_2 based at the point O, cf. [Fil70], [San67], [San68].

Remark 4.2. If $c_1 = c_2 = c$ and c is h-convex, then the mixed width function w_{12} coincides with the width function with respect to horocycles considered in [**?**].

4.2.1. *The sum of "concave-sided" support maps.* We consider the following situation: Let c_1, c_2 be two regular curves in $\mathbb{H}^2$ with geodesic curvatures $(\kappa_g)_1, (\kappa_g)_2 > -1$. We take the support maps θ_1, θ_2 according to $\epsilon_1 = \epsilon_2 = -1$, that means locally the curves lie on the concave sides of their respective support horocycles.

Proposition 4.2. *Suppose the situation described above. Then, whenever well defined, the sum $\theta^* = \theta_1 + \theta_2$ envelopes a regular curve $c^* = c_1 + c_2$ in $\mathbb{H}^2$ with*
(i) $\kappa_g^* > -1$, and
(ii) c^* lies locally on the concave sides of its respective support horocycles.

Proof. The curves c_1, c_2 lie locally on the concave sides of their respective support horocycles, therefore the osculating planes of θ_1, θ_2 intersect $\mathbb{H}^2$ without being tangent (cf. Remark 3.1).
Now, we keep fixed an arbitrary parameter σ_1.
The osculating plane of θ_1 at σ_1 is given by

$$\theta_1(\sigma_1) + \mathrm{span}(\dot{\theta}_1(\sigma_1), \ddot{\theta}_1(\sigma_1)).$$

Let P_1 denote the parallel plane through $\theta^*(\sigma_1)$, i.e.

$$P_1 = \theta^*(\sigma_1) + \mathrm{span}(\dot{\theta}_1(\sigma_1), \ddot{\theta}_1(\sigma_1)).$$

The osculating plane of θ_1 at σ_1 intersects $\mathbb{H}^2$ without being tangent, and $\theta^* = \theta_1 + \theta_2$, therefore P_1 also intersects $\mathbb{H}^2$ without being tangent.
Now $\theta_2 = \lambda\theta_1$, hence

$$\dot{\theta}_2 = \dot{\lambda}\theta_1 + \lambda\dot{\theta}_1 \quad \text{and} \quad \ddot{\theta}_2 = \ddot{\lambda}\theta_1 + 2\dot{\lambda}\dot{\theta}_1 + \lambda\ddot{\theta}_1 \tag{19}$$

(where the dots denote derivatives with respect to σ_1). And the osculating plane of θ_2 at σ_1 is given by

$$\theta_2(\sigma_1) + \mathrm{span}(\dot{\theta}_2(\sigma_1), \ddot{\theta}_2(\sigma_1)).$$

Let P_2 denote the parallel plane through $\theta^*(\sigma_1)$, i.e.

$$P_2 = \theta^*(\sigma_1) + \mathrm{span}(\dot\theta_2(\sigma_1), \ddot\theta_2(\sigma_1)).$$

The osculating plane of θ_2 at σ_1 intersects $\mathbb{H}^2$ without being tangent, we have $\theta^* = \theta_1 + \theta_2$, therefore P_2 also intersects $\mathbb{H}^2$ without being tangent.
The osculating plane of θ^* at σ_1 is given by

$$P^* = \theta^*(\sigma_1) + \mathrm{span}(\dot\theta^*(\sigma_1), \ddot\theta^*(\sigma_1))$$

with

$$\dot\theta^* = \dot\theta_2 + \dot\theta_1 \quad \text{and} \quad \ddot\theta^* = \ddot\theta_2 + \ddot\theta_1. \tag{20}$$

Let T be the tangent plane of $\mathcal{C}^2_+$ along the generator $\mathbb{R}_+ \cdot \theta_1(\sigma_1)$, i.e.
$T = \theta^*(\sigma_1) + \mathrm{span}(\theta_1(\sigma_1), \dot\theta_1(\sigma_1))$. For $a \geq 0$ let T_a denote the plane parallel to T given by $T_a = T + a\,\ddot\theta_1(\sigma_1)$.
Then T_a intersects P_1 in the line

$$\ell_{1a} = \theta^*(\sigma_1) + a\,\ddot\theta_1(\sigma_1) + \mathbb{R} \cdot \dot\theta_1(\sigma_1)\,.$$

And by (19), T_a intersects P_2 in the line

$$\ell_{2a} = \theta^*(\sigma_1) + \frac{a}{\lambda(\sigma_1)}\,\ddot\theta_2(\sigma_1) + \mathbb{R} \cdot \dot\theta_2(\sigma_1)\,.$$

And by (20), T_a intersects P^* in the line

$$\ell^*_a = \theta^*(\sigma_1) + \frac{a}{1+\lambda(\sigma_1)}\left(\ddot\theta_2(\sigma_1) + \ddot\theta_1(\sigma_1)\right) + \mathbb{R}\cdot\left(\dot\theta_2(\sigma_1) + \dot\theta_1(\sigma_1)\right)\,.$$

Let g_a denote the line in T_a given by

$$g_a = \theta^*(\sigma_1) + a\,\ddot\theta_1(\sigma_1) + \mathbb{R} \cdot \theta_1(\sigma_1)\,.$$

Then g_a intersects ℓ_{1a} in the point

$$Q_{1a} = \theta^*(\sigma_1) + a\,\ddot\theta_1(\sigma_1)\,.$$

And g_a intersects ℓ_{2a} in the point

$$Q_{2a} = \theta^*(\sigma_1) + a\,\ddot\theta_1(\sigma_1) + \frac{a(\lambda\ddot\lambda - 2\dot\lambda^2)}{\lambda^2}\Big|_{\sigma_1}\,\theta_1(\sigma_1)\,.$$

And g_a intersects ℓ_a^* in the point

$$Q_a^* = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \frac{a((1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2)}{(1+\lambda)^2}|_{\sigma_1}\,\theta_1(\sigma_1)\,.$$

The proof now splits into two cases.
The first case, $((1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2)|_{\sigma_1} \geq 0$:
P_1 intersects $\mathbb{H}^2$ without being tangent. Therefore there exists an $a > 0$ such that ℓ_{1a} intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. The axis of the parabola is $\theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \mathbb{R}\cdot\theta_1(\sigma_1)$. Hence the half-ray $Q_{1a} + \mathbb{R}_+ \cdot \theta_1(\sigma_1) \subset T_a$ lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. In the first case Q_a^* lies on this half-ray. Hence Q_a^* lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. Hence ℓ_a^* intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the osculating plane P^* of θ^* at σ_1 intersects $\mathbb{H}^2$ without being tangent.
The second case, $((1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2)|_{\sigma_1} < 0$:
P_2 intersects $\mathbb{H}^2$ without being tangent. Therefore there exists an $a > 0$ such that ℓ_{2a} intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the half-ray $Q_{2a} + \mathbb{R}_+ \cdot \theta_1(\sigma_1) \subset T_a$ lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. Through the assumption in the second case we have

$$\frac{\lambda\ddot{\lambda} - 2\dot{\lambda}^2}{\lambda^2}|_{\sigma_1} \leq \frac{(1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2}{(1+\lambda)^2}|_{\sigma_1}\,.$$

Hence Q_a^* lies on this half-ray. Hence ℓ_a^* intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the osculating plane P^* of θ^* at σ_1 intersects $\mathbb{H}^2$ without being tangent.
Altogether, this shows that the osculating planes of θ^* intersect $\mathbb{H}^2$ without being tangent. Therefore c^* is regular at σ_1, θ^* supports c^* concave-sided, and moreover $\kappa_g^* > -1$. □

Proposition 4.3. *Suppose the situation described above. Then the length L^* and the total curvature TC^* of $c^* = c_1 + c_2$ write in terms of c_1, c_2 and their relative position to each other in $\mathbb{H}^2$*

as follows:

$$L^* = -\frac{1}{2}\left(W(c_1, c_1 + c_2) - L_1 - TC_1 - L_2 - TC_2\right) \tag{21}$$

$$TC^* = \frac{1}{2}\left(W(c_1, c_1 + c_2) + L_1 + TC_1 + L_2 + TC_2\right), \tag{22}$$

with

$$W(c_1, c_1 + c_2) = TC^* - L^* = \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + \kappa_{\theta_1}^2\right) d\sigma_1$$

and the mixed with function

$$w_{1*}(\sigma_1) = -\ln(1 + \lambda(\sigma_1)).$$

Proof. By the assumptions on c_1, c_2 and by Proposition 4.2 we have for all three curves c_1, c_2, c^* that $\epsilon_1 = \epsilon_2 = \epsilon^* = -1$ and $(\kappa_g)_1, (\kappa_g)_2, \kappa_g^* > -1$. Therfore (6) writes $d\sigma = (\kappa_g + 1)\, ds$, hence

$$L(\theta) = \int_\theta d\sigma = \int_c (\kappa_g + 1)\, ds = \int_c \kappa_g\, ds + L(c)\,. \tag{23}$$

This and (17) gives

$$TC^* + L^* = TC_1 + L_1 + TC_2 + L_2. \tag{24}$$

From (4) and (8) we get

$$ds = \frac{1 - \langle\ddot{\theta}, \ddot{\theta}\rangle}{2}\, d\sigma,$$

$$L(c) = -\frac{1}{2}\int_\theta \kappa_\theta^2\, d\sigma + \frac{1}{2}L(\theta).$$

This applied to c^* yields

$$L(c^*) = -\frac{1}{2}\int_{\theta^*} \kappa_{\theta^*}^2\, d\sigma^* + \frac{1}{2}L(\theta^*). \tag{25}$$

Now a straightforward but lenghty computation, not acted out here, starts at $\theta^* = (1+\lambda)\,\theta_1$ and reaches

$$\langle \frac{d^2\theta^*}{d\sigma^{*2}}, \frac{d^2\theta^*}{d\sigma^{*2}} \rangle = \frac{1}{(1+\lambda)^2}\left[\left(\frac{d}{d\sigma_1}\left(\ln(1+\lambda)\right)\right)^2 - \right.$$
$$\left. -2\frac{d^2}{d\sigma_1^2}\left(\ln(1+\lambda)\right) + \langle \ddot{\theta}_1, \ddot{\theta}_1 \rangle \right]. \quad (26)$$

Using the mixed with function of c_1 and c_* with respect to c_1, i.e. $w_{1*} = -\ln(1+\lambda)$, formula (26) gives

$$\int_{\theta^*} \kappa_{\theta^*}^2\, d\sigma^* = \int_{\theta^*} \langle \frac{d^2\theta^*}{d\sigma^{*2}}, \frac{d^2\theta^*}{d\sigma^{*2}} \rangle\, d\sigma^* =$$
$$= \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + \kappa_{\theta_1}^2\right)\, d\sigma_1\,. \quad (27)$$

(Note: $d\sigma^* = (1+\lambda)\, d\sigma_1$ cf. (16).)
Hence (25), (27) and (23) yield

$$L^* = -\frac{1}{2}\int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + \kappa_{\theta_1}^2\right)\, d\sigma_1 + \frac{1}{2}TC^* + \frac{1}{2}L^*. \quad (28)$$

Finally (24) and (28) give the result. $\square$

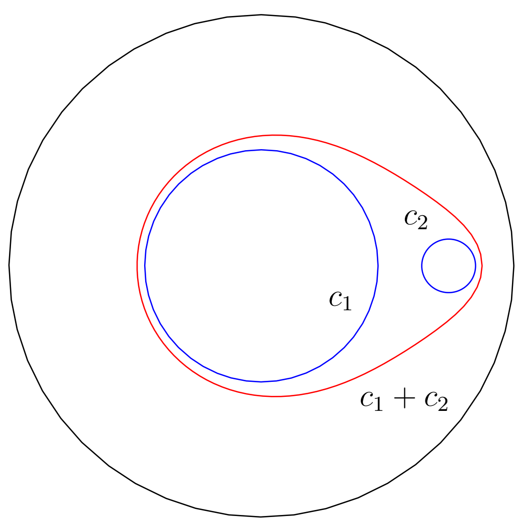

Figure 1: The sum $c_1 + c_2$ of two circles c_1, c_2 in the Poincaré disk, with radii $r_1 = 1, r_2 = 0.5$ and distance 2 between their centers

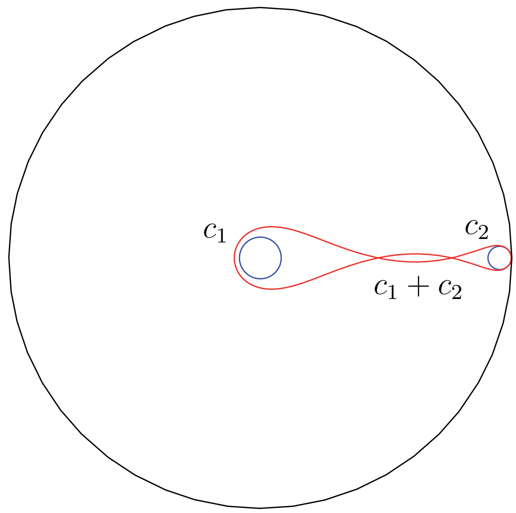

Figure 2: The sum $c_1 + c_2$ of two circles c_1, c_2 in the Poincaré disk, with radii $r_1 = 0.16, r_2 = 2$ and distance 5 between their centers

4.3. **The "rum".**

Definition 4.4. Let c_1, c_2 be two regular curves and θ_1, θ_2 their support maps with respect to unit normal fields ν_1, ν_2 along c_1, c_2. Then $\theta_2 = \lambda\theta_1$. Whenever well-defined, we call

$$c_1\#c_2 = c^* \quad \text{, given by} \quad \theta^* = \frac{\lambda}{1+\lambda}\,\theta_1 \tag{29}$$

the *"rum $c_1\#c_2$ of c_1 and c_2"*.

Geometrically, this definition is induced by the sum of the two parallel planes Θ_1 and Θ_2 in the vector space $\mathbb{R}^3_1$.

Lemma 4.1. *Let θ_1, θ_2 be support maps. Then $\theta^* = \theta_1\#\theta_2$ lies below θ_1 and θ_2 with respect to each of the generators of $\mathcal{C}^n_+$.*

Proof. This follows immediately from the geometric meaning of the definition of the rum. Alternatively:

We have $\theta_2 = \lambda\,\theta_1$ with $\lambda > 0$. Hence

$$\theta^* = \frac{\lambda}{1+\lambda}\theta_1 < \theta_1 \text{ , and}$$

$$\theta^* = \frac{\lambda}{1+\lambda}\theta_1 = \frac{1}{1+\lambda}\theta_2 < \theta_2 \ .$$

□

Proposition 4.4. ....

Proof. .... □

Now we bring orientations into game. We assume a given orientation on hyperbolic plane $\mathbb{H}^2$. For an oriented curve c in $\mathbb{H}^2$ we now fix $\nu = e_2$, i.e. $\epsilon = +1$, and we have the support map $\theta = c + e_2$. Horocycles Θ are orientated such that the convex region is on its left-hand side (i.e. we choose the positive orientation, i.e the counter-clockwise direction).

A view on oriented circles in $\mathbb{H}^2$:

An oriented circle c is given by its center $m \in \mathbb{H}^2$ and its radius $r \in \mathbb{R}$, thereby that the hyperbolic radius is $|r|$ and the orientation is counter-clockwise for $r > 0$ and clockwise for $r < 0$. Especially for $r = 0$ we get points.

If c is oriented counter-clockwise, then its θ supports convex-sided. If c is oriented clockwise, then its θ supports concave-sided.

If the circle is given by its support map θ, then θ is the intersection of $\mathcal{C}^2_+$ with a space-like plane $\langle n, x\rangle = -1$, n time-like and inside the half-cone $\mathcal{C}^2_+ \subset \mathcal{R}^3_1$. Its center is $m = n/|n| \in \mathbb{H}^2$ and its radius is $r = \ln|n|$. Moreover: $|n| > 1$ iff θ supports c convex-sided. $|n| < 1$ iff θ supports c concave-sided. $|n| = 1$ iff c is a point.

Proposition 4.5. *Let c_1, c_2 be circles or points in $\mathbb{H}^2$ with centers m_1, m_2 and signed radii r_1, r_2. The rum $c^* = c_1 \# c_2$ is a circle or a point which center m^* and signed radius r^* are given as follows:*

$$r^* = \frac{1}{2}\ln\left(e^{2r_1} + e^{2r_2} + 2e^{r_1+r_2}\cosh(\mathrm{d}(m_1, m_2))\right) \qquad (30)$$

where $\mathrm{d}(m_1, m_2)$ *is the hyperbolic distance between* m_1 *and* m_2*; and*

$$m^* = \frac{1}{|n_1 + n_2|}\,(n_1 + n_2) \tag{31}$$

with $n_1 = e^{r_1}\, m_1$ *and* $n_2 = e^{r_2}\, m_2$*. Moreover*

$$\frac{\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}^*))}{\cosh(\mathrm{d}(\mathrm{m}_2, \mathrm{m}^*))} = \frac{e^{r_1} + e^{r_2}\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}_2))}{e^{r_1}\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}_2)) + \mathrm{e}^{\mathrm{r}_2}}. \tag{32}$$

Proof. The support maps θ_1, θ_2 of c_1, c_2 are uniquely determined by their planes $\langle n_1, x\rangle = -1$, $\langle n_2, x\rangle = -1$ as described above. Then their centers and signed radii are given by $m_1 = n_1/|n_1|$, $m_2 = n_2/|n_1|$ and $r_1 = \ln|n_1|$, $r_2 = \ln|n_2|$ (cf. (13)). The support map θ^* is given by the plane $\langle n_1 + n_2, x\rangle = -1$. Then straightforward computations give the results. $\square$

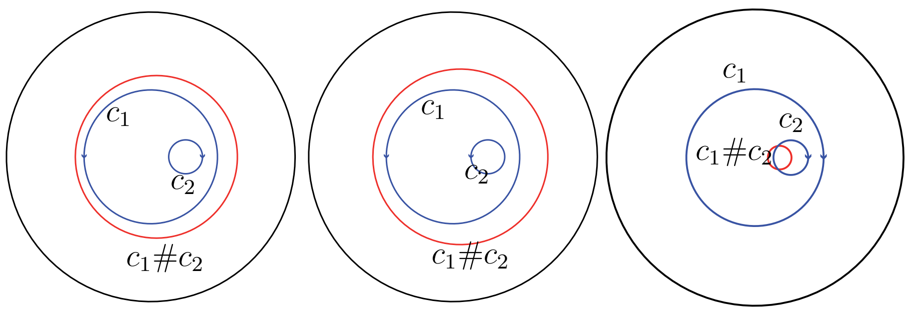

Figure 3: The rum $c_1\#c_2$ of two circles c_1, c_2 in the Poincaré disk, with signed radii $r_1 = 1, +1, -1, r_2 = -0.25, +0.25, -0.25$ and distance 0.5 between their centers

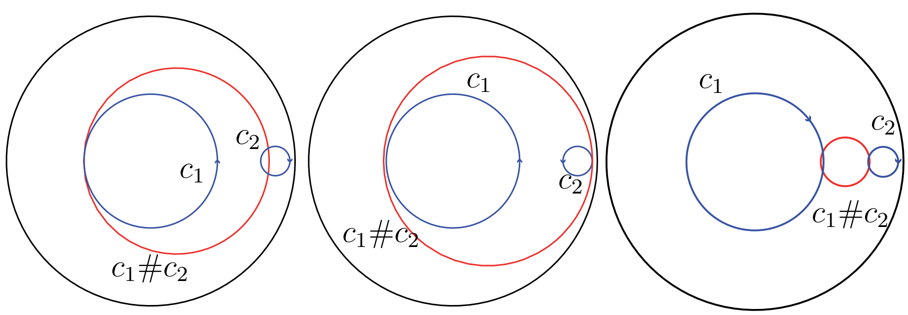

Figure 4: The rum $c_1\#c_2$ of two circles c_1, c_2 in the Poincaré disk, with signed radii $r_1 = +1, +1, -1, r_2 = -1, +1, -1$ and distance 3 between their centers

Proposition 4.6. *Let c_1, c_2 be counter-clockwise oriented circles or points in $\mathbb{H}^2$. Then the rum $c_1\#c_2$ of c_1 and c_2 is a circle containing both c_1 and c_2.*

Proof. c_1, c_2 are counter-clockwise oriented. Hence their support maps θ_1, θ_2 support convex-sided, and their planes do not intersect $\mathbb{H}^2$. Now θ^* lies below θ_1 and θ_2 with respect to each generator of $\mathcal{C}^2_+$ (cf. Lemma 4.1). Therefore the plane of θ^* does not intersect $\mathbb{H}^2$. Hence each θ^* supports $c_1\#c_2$ convex-sided and contains c_1 and c_2. $\square$

Proposition 4.7. *Let c_1, c_2 be counter-clockwise oriented smooth regular boundaries of h-convex bodies K_1, K_2 in $\mathbb{H}^2$. Then the rum $c^* = c_1\#c_2$ of c_1 and c_2 is the counter-clockwise oriented smooth regular boundary of an h-convex body K^*, also called rum $K^* = K_1\#K_2$ of K_1 and K_2. Moreover $K_1, K_2 \subset K^*$.*

Proof. The curves c_1, c_2 are oriented counter-clockwise and h-convex, hence their θ_1, θ_2 support convex-sided.
The second order situation of c_1 and c_2 at related points determines the second order situation of c^* at the envelope point. For more details at this place, one should especially take into account:
1) The support map of the osculating circle of a c in $\mathbb{H}^2$ is given by the intersection of the osculating plane of θ in $\mathbb{R}^3_1$ with $\mathcal{C}^2_+$.
2) The intersection of the osculating plane of θ with $\mathcal{C}^2_+$ is the osculating circle of the curve θ in $\mathbb{R}^3_1$ (use the Meusnier formula). And
3) The rum in $\mathcal{C}^2_+$ of the osculating circles of θ_1 and θ_2 in $\mathcal{C}^2_+$ is equal to the osculating circle of $\theta_1\#\theta_2$ (to this use 2) and (26)).
Now the second order situation of c_1, c_2 is given by their osculating circles $\mathrm{osc}_1, \mathrm{osc}_2$ in $\mathbb{H}^2$. Therefore the circle (cf. Proposition 4.6) $\mathrm{osc}_1\#\mathrm{osc}_2$ describes the second order situation of $c_1\#c_2$. Hence $\theta_1\#\theta_2$ supports $c_1\#c_2$ convex-sided, $c_1\#c_2$ is regular and

oriented counter-clockwise. And $\mathrm{osc}_1\#\mathrm{osc}_2$ is the osculating circle of $c_1\#c_2$. Therefore $c_1\#c_2$ is h-convex. Finally by Proposition 4.6, $K_1, K_2 \subset K^*$. $\square$

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