PERIMETER, DIAMETER AND AREA OF CONVEX SETS IN THE HYPERBOLIC PLANE.

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Abstract. In this paper we study the relation between the asymptotic values of the ratios area/length (F/L) and diameter/length (D/L) of a sequence of convex sets expanding over the whole hyperbolic plane. It is known (cf. [3] and [2]) that F/L goes to a value between 0 and 1 depending on the shape of the contour. Here, first of all it is seen that D/L has limit value between 0 and $1/2$ in strong contrast with the euclidean situation in which the lower bound is $1/\pi$ ($D/L = 1/\pi$ if and only if the convex set has constant width). Moreover, it is shown that, as the limit of D/L approaches to $1/2$, the possible limit values of F/L reduce. Examples of all possible limits F/L and D/L are given.

1. Introduction

In hyperbolic geometry, given a point p exterior to a line l there are infinitely many non secant lines. These lines lie between the two so called parallel lines to l. When the distance from p to l grows to infinity, the angle between the parallel lines goes to 0. This fact leads to the ambiguous idea that, in some sense, given a line l, the *probability* that a random line meets l is zero. In order to formalize this idea let us restrict our attention to the interiors of a sequence (K_n) of convex sets in the hyperbolic plane expanding to fill it. The probability for a random chord of K_n to meet r inside K_n *should* go to 0 as $n \to \infty$. It can be proved by using the Cauchy-Crofton formula, that this probability is $2\sigma_n/L_n$, where σ_n is the length of the chord $l \cap K_n$ and L_n denotes the length of ∂K_n. Because the length of the chord σ_n is less or equal than the diameter D_n of K_n, the study of $\lim \sigma_n/L_n$ is related to the knowledge of the asymptotic value of the ratio D_n/L_n where D_n.

The question of whether the asymptotic value of D/L is zero or not already appeared in [7]. In the present text we will see that there are many possible values for this limit and we will find them all. More precisely we will prove that for every $e \in [0, 1/2]$ there is a sequence (K_n) of hyperbolic convex sets such that $\lim D_n/L_n = e$. In fact it will be seen that, for convex sets with respect to equidistants intersecting infinity with angle θ, this limit can take values only below $(\sin\theta)/2$.

It must be noticed that in the euclidean case the situation is quite different: any convex set satisfies $1/\pi \leq D/L \leq 1/2$. The lower bound is reached only by constant width sets and the upper bound by the segments.

The paper is organized as follows. In sections 2 and 3 we introduce the basic concepts and notation. In section 4 we find lower and upper bounds for D/L in the λ-convex case, concluding that the asymptotic value of D/L for h-convex sequences is 0. Section 5 is devoted to the construction of examples showing that the preceding bounds are the best possible. In section 6 we recall the asymptotic behavior of the quotient F/L being F the area of the convex sets. We introduce, in section 7, the

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metric space of hyperbolic convex sets in order to treat an isoperimetric problem. Finally, in section 8, we give the relation between the asymptotic values of D/L and F/L. More precisely, we can state that

$$\lim_{n\to\infty} \frac{F_n}{L_n} \leq \sqrt{1 - \left(2 \lim_{n\to\infty} \frac{D_n}{L_n}\right)^2}.$$

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2. The hyperbolic plane

In this section we introduce the hyperbolic plane as well as some basic facts that will be used later on. The *hyperbolic plane*, $\mathbb{H}^2$, is the unique complete simply connected Riemannian manifold of dimension 2 with constant curvature -1. Its geometry corresponds to the one obtained from the absolute geometry given by the first four Euclid postulates and the Lobachevsky postulate: through every point P exterior to a line l pass more than one line not intersecting l. It is useful to have different models for this geometry, we shall describe their points, lines (geodesics) and rigid motions:

Half-plane model. It is the half-plane $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ with the metric $\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. The geodesics are half-circles centered in $\{y = 0\}$ and vertical half-lines. The rigid motions are composition of inversions with respect to these circles and symmetries with respect to these lines. This model is conformal since the metric is a multiple of the euclidean metric.

Disk model. It is the unit disk with the metric $\frac{4}{(1-x^2-y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. This model is also conformal. The geodesics are the a diameters of the disk and the arcs of circumference orthogonal to the border. The rigid motions are homographies of the complex plane fixing the disk.

Projective model. It is the unit disk with the metric $\frac{1}{1-r^2}(\frac{1}{1-r^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)$ where (r, θ) are the euclidean polar coordinates centered at the origin. The geodesics are chords of the disk. This fact makes this model become very useful when studying questions related to convex sets. The rigid motions are the projectivities fixing the disk.

In the following sections *polar coordinates* will be useful in the treatment of some problems. Whatever it is the model we work in, we can parametrize the points of the hyperbolic plane in the following way. Let O be a point called origin. We choose in O a direction $v \in T_O\mathbb{H}^2$. For each point P, let r be the length of the geodesic segment joining O and P, and let θ be the angle between this segment and v. Now, $\mathbb{H}^2 \setminus \{O\}$ is perfectly parametrized by the coordinates (r, θ). It can be easily checked out that in these coordinates the metric is written as follows.

$$g = \mathrm{d}r^2 + (\sinh r)^2\mathrm{d}\theta^2.$$

The volume element will be then $\sinh r \mathrm{d}r\mathrm{d}\theta$ and the area and perimeter of a circumference of radius r in $\mathbb{H}^2$ are

$$L = \int_0^{2\pi} \sinh r \mathrm{d}\theta = 2\pi \sinh r, \qquad F = \int_0^r \int_0^{2\pi} \sinh r \mathrm{d}\theta \mathrm{d}r = 2\pi(\cosh r - 1).$$

We shall need some formulas in *hyperbolic trigonometry*; proofs can be found in [5]. Let a, b and c be three sides of a geodesic triangle and let α, β and γ be their opposite angles. The following identities are then verified:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha,$$

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma},$$

$$\sinh a \cos \beta = \cosh b \sinh c - \sinh b \cosh c \cos \alpha.$$

The area of such triangle is

$$F = \pi - (\alpha + \beta + \gamma).$$

It is said that the area equals the *angular defect*.

To end this section we will just state some integral geometric formulas used later on. The *isoperimetric inequality*

$$L^2 - 4\pi F - F^2 \geq 0 \tag{1}$$

gives a relation between perimeter L and area F of an arbitrary compact domain in $\mathbb{H}^2$.

The *Cauchy-Crofton theorem* expresses the length of a curve in terms of the measure of lines (counted with its multiplicity) intersecting it. More precisely

$$2L = \int_{r \cap C \neq \emptyset} n(r) dr$$

where dr is a normalized isometry invariant density of geodesic lines and $n(r)$ the number of intersecting points of r and the curve C (cf. [6]).

3. Convexity and λ-convexity

In this section we introduce the concept of λ-convexity as well as some basic known facts about it. For a more detailed introduction see [3].

Definition 3.1. A subset $K \subset \mathbb{H}^2$ is said to be *convex* if for every pair of points in K, the geodesic segment joining them is also in K.

Notice that a set in $\mathbb{H}^2$ is convex if and only if in the projective model it looks like an euclidean convex set.

Definition 3.2. A closed convex set with nonempty interior is called a *convex domain.*

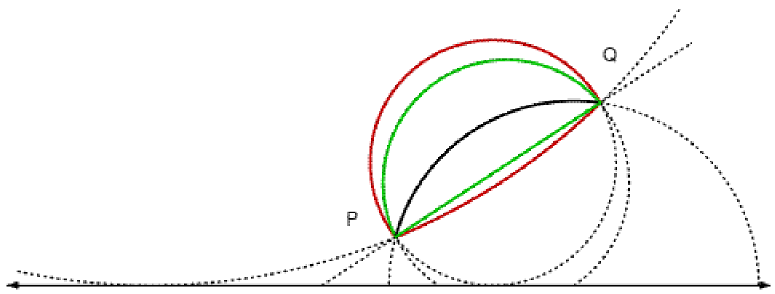
From now on all convex sets will be compact convex domains. If K is a convex set, then ∂K is C^2 except from, at most, a countable set of points. Moreover, ∂K must have finite length, which is called the *perimeter* L of K, and the *area* F of K must be finite too. The *diameter* is given by $D = \max\{d(p,q) \,|\, p, q \in \partial K\}$.

Definition 3.3. A sequence (K_n) of convex sets is said to *expand over the whole hyperbolic plane* if $K_n \subset K_{n+1}$ and $\forall p \in \mathbb{H}^2$ there is an n such that $p \in K_n$.

As in the euclidean case we have

Lemma 3.1. *A compact domain with piecewise C^2 boundary is convex if and only if its geodesic curvature does not change the sign and in the non C^2 points the interior angles are not greater than π.*

Given a geodesic line l in the euclidean plane, the set of equidistant points to l are two parallel lines symmetric with respect to l. In the hyperbolic plane this is no longer true; the set of equidistant points to l are two smooth curves called *equidistants.* If we consider the half-plane model, the equidistant curves to the hyperbolic line $x = 0$ are euclidean half-lines passing through $(0,0)$. Indeed any geodesic arc with center in $(0,0)$ going from $x = 0$ to $y = mx$ has the same length because they are (euclidean) homothetic and every homothety with center in the axis $y = 0$ is the composition of two inversions with respect circumferences centered in $y = 0$ which are hyperbolic isometries. In fact, if $m = \tan\theta$ the length of these



Special curves passing through two points P and Q

FIGURE 1

geodesic arcs is equal to $\log(\cot\theta/2)$. In this model (and in the disk model too) equidistant lines are, in general, arcs of euclidean circles meeting the infinity at two points. Every equidistant separates the plane in two regions such that only one of them is convex.

Definition 3.4. A λ-*geodesic* is an equidistant line meeting the infinity with an angle θ such that $|\cos\theta| = \lambda$.

0-geodesic lines are geodesics. It must be noticed that λ-geodesics have constant geodesic curvature $\pm\lambda$ at every point and the distance to equidistant geodesics is $\operatorname{arctanh}(\lambda)$.

Definition 3.5. An *horocycle* is a continuous curve orthogonal to a bundle of parallel lines.

In the half-plane and disk models, horocycles are euclidean circles tangent to the boundary. It can be easily seen that horocycles have constant geodesic curvature equal to ± 1, so when λ goes to 1 they can be considered a limit case of λ-geodesics. From now on geodesics and horocycles will be considered as particular cases of λ-geodesic lines.

As λ-geodesics are the unique solutions of the ordinary differential equation of order two $k_g = \pm\lambda$, given two points there are two and only two λ-geodesics passing through them. The length of these λ-geodesic segments will be called the λ-*geodesic distance* between these points. Similarly, given a direction at a point, there are two and only two λ-geodesic lines passing through it with the given direction. Later on, we will need the following result

Proposition 3.1. *Given a circle C and p an outer point to C, there are two and only two tangent λ-geodesics to C passing by p and leaving C in the convex side.*

Proof. Let us take a λ-geodesic tangent to C (leaving C in the convex side, cf. figure 2) . Carrying out a rotation, with respect to the center of C, we can make it pass through p. With a symmetry with respect to the line joining p and the center of C, we get the other λ-geodesic. □

Definition 3.6. A set $K \subset \mathbb{H}^2$ is said to be λ-*convex* if for every pair of points in K, the two λ-geodesic segments joining them are also in K. When $\lambda = 1$, K is said to be *convex with respect to horocycles* or h-*convex* for short.

See [1] for an equivalent definition of h-convexity based on supporting horocycles.

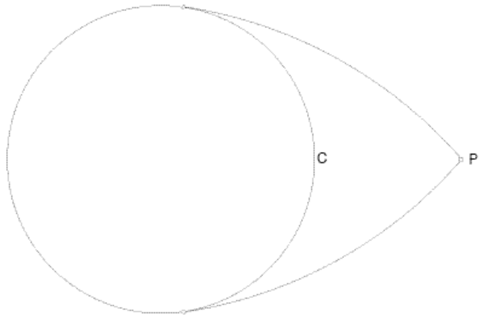


Figure 2

Definition 3.7. For $\lambda \leq 1$, a λ-*lens* is the convex domain bounded by two intersecting λ-geodesics (see figure 3). For $\lambda > 1$, a λ-lens is the convex domain bounded by two intersecting circles with curvature equal to λ. When the intersection points are at infinity we talk about *ideal* λ-*lens*.

If $\lambda_1 \leq \lambda_2$ then every λ_2-convex set is λ_1-convex. Indeed, the λ_1-geodesic segments joining two points lie between the λ_2-geodesic segments joining them. In particular every λ-convex set is convex.

Lemma 3.2. *Let K be a compact convex domain bounded by a piecewise C^2 curve. Then K is λ-convex if and only if its geodesic curvature satisfies $k_g \geq \lambda$ and in the angular points the interior angle is less or equal than π.*

Proof. Let $p \in \partial K$ such that $k_g(p) < \lambda$. Let (x, y) be geodesic normal coordinates such that p is given by $(0, 0)$ and $\partial/\partial x$ is tangent to ∂K in p. With respect to these coordinates, in a neighborhood of p, the boundary is the graph of

$$y = \frac{1}{2}k_g(p)x^2 + o(x^2)$$

and the λ-geodesic curves with direction $\partial/\partial x$ are the graph of

$$y = \pm\frac{1}{2}\lambda x^2 + o(x^2).$$

The fact that $k_g(p) < \lambda$ implies that one of these λ-geodesic is locally inside K. Since a compact domain cannot contain a whole λ-geodesic, we have contradiction with the λ-convexity of K.

Conversely, if K is not λ-convex there are two points x, $y \in \partial K$ such that the λ-geodesic between them is not contained in K. By lemma 3.1, K is convex so the geodesic segment r between x and y is in K. Let $0 \leq \mu < \lambda$ be the supremum of all nonnegative numbers such that the μ-geodesic between x and y is contained in K. If this μ-geodesic touches ∂K in a C^2 point we should have $k_g \leq \mu < \lambda$ in this point, a contradiction. The μ-geodesic cannot touch ∂K in an angular point and if it touches ∂K in a non-C^2 but non angular point we have that the lateral limits for k_g are not greater than μ, a contradiction. This implies, because μ is the supremum, that the μ geodesic is tangent at x or at y. But then $k_g \leq \mu < \lambda$ at x or at y, a contradiction. $\square$

4. Relation between diameter and perimeter of a convex set

In this section it will be shown that, for a sequence of λ-convex sets expanding over the whole hyperbolic plane, the asymptotic value of D/L is not greater than $\frac{1}{2}\sqrt{1-\lambda^2}$.

We shall need the following lemma , which was proved in [3].

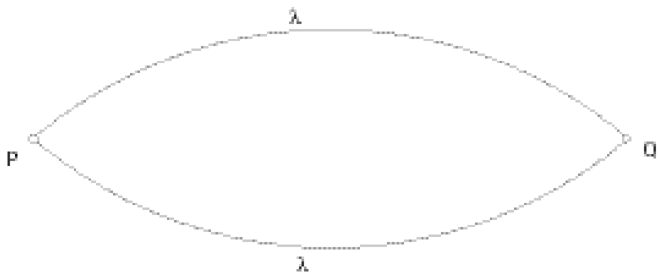


FIGURE 3

Lemma 4.1. *Given two points in the hyperbolic plane and* $0 \leq \lambda < 1$*, if* d *and* l *are, respectively, the geodesic and* λ*-geodesic distances between the points, then*

$$l = \frac{2}{\sqrt{1-\lambda^2}} \operatorname{arcsinh}\left(\sqrt{1-\lambda^2}\sinh\frac{d}{2}\right). \tag{2}$$

If $\lambda \to 1$ *then* $l \to 2\sinh\left(\frac{d}{2}\right)$.

Proposition 4.1. *Let* (K_n) *be a sequence of* λ*-convex sets expanding over the whole hyperbolic plane. If* D_n *are their diameters and* L_n *their perimeters, then*

$$0 \leq \liminf \frac{D_n}{L_n} \leq \limsup \frac{D_n}{L_n} \leq \frac{1}{2}\sqrt{1-\lambda^2}. \tag{3}$$

Proof. For each n let p_n and q_n be points in ∂K_n such that the chord $p_n q_n$ has length equal to D_n. Let A_n be the λ-lens with endpoints p_n and q_n. Since K_n is λ-convex, $A_n \subset K_n$ so the perimeter of A_n is less than L_n. So we get

$$\frac{4}{\sqrt{1-\lambda^2}} \operatorname{arcsinh}\left(\sqrt{1-\lambda^2}\sinh\frac{D_n}{2}\right) \leq L_n$$

and

$$\lim_{n\to\infty} \frac{D_n}{L_n} \leq \lim_{n\to\infty} \frac{D_n\sqrt{1-\lambda^2}}{4\operatorname{arcsinh}(\sqrt{1-\lambda^2}\sinh\frac{D_n}{2})} = \frac{1}{2}\sqrt{1-\lambda^2}.$$

□

Corollary 4.1. *If* (K_n) *is a sequence of* h*-convex sets expanding over the whole hyperbolic plane then*

$$\lim_{n\to\infty} \frac{D_n}{L_n} = 0.$$

5. A FAMILY OF EXAMPLES

In this section we are going to construct sequences of λ-convex sets showing that inequalities in (3) are the best possible.

Let C be a circumference with radius r centered at a point O. Let s be a geodesic segment with midpoint O and length $2R$ ($R > r$). We call $K_\lambda(R, r)$ the smallest λ-convex set containing C and s (see figure 4). Let us describe the boundary of $K_\lambda(R, r)$. When $r \geq \operatorname{arctanh}(\lambda)$ the boundary of $K_\lambda(R, r)$ is formed by the two λ-geodesic segments tangent to C leaving C in the convex side union with the arcs of C between the tangency points. By lemma 3.1, this curve bounds a λ-convex domain.

We shall see that, for suitable r and R, the quotient D/L can be as close as possible to any value between 0 and $\frac{1}{2}\sqrt{1-\lambda^2}$. Let P be one of the ends of s and Q be the tangency point with C of one of the λ-geodesic segments starting at P.

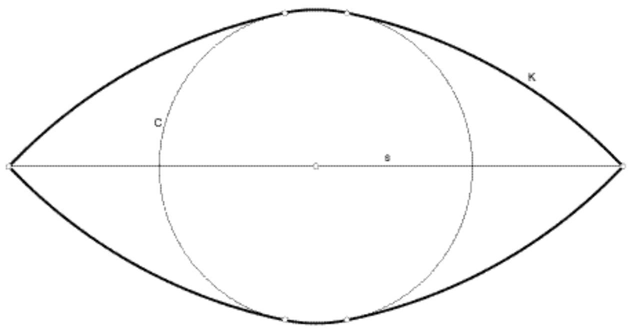


FIGURE 4

Let d and l be the geodesic and λ-geodesic distances between P and Q. Finally, let α be the angle POQ and β be the angle OQP.

We present some interesting formulas in the next lemma.

Lemma 5.1. *With the notation as above*

$$\tan\left(\frac{\pi}{2} - \beta\right) = \frac{\lambda}{\sqrt{\coth^2 \frac{d}{2} - \lambda^2}} \tag{4}$$

$$l = \frac{2}{\sqrt{1-\lambda^2}} \operatorname{arcsinh} \sqrt{\frac{1-\lambda^2}{2} \frac{\cosh R - \cosh r}{\cosh r - \lambda \sinh r}}. \tag{5}$$

Proof. The proof of the first formula can be found in [3]. From (4) it follows that

$$\cos\beta = \lambda \tanh \frac{d}{2} \tag{6}$$

using the first cosine law on the hyperbolic triangle OPQ

$$\cosh R = \cosh r \cosh d - \sinh r \sinh d \cos\beta.$$

Using (6) and isolating $\cosh d$ from the last equality we get

$$\cosh d = \frac{\cosh R - \lambda \sinh r}{\cosh r - \lambda \sinh r} \tag{7}$$

Substituting in (2) and bearing in mind that $\sinh \frac{d}{2} = \sqrt{\frac{\cosh d - 1}{2}}$ we get the equation we were looking for. □

Corollary 5.1. *If we take $R = \mathrm{e}^{2r}$ or $R = a\mathrm{e}^r$ with $a > 0$ then*

$$\begin{aligned} \lim_{r\to\infty} \frac{l}{R} &= \frac{1}{\sqrt{1-\lambda^2}} \\ \lim_{r\to\infty} \alpha &= 0 \end{aligned}$$

where, as above, l is the λ-geodesic distance between P and Q and α is the angle POQ.

Proof. Using the fact that $\log(x) \sim \operatorname{arcsinh}(x)$ when x goes to infinity and formula (5) we have

$$\lim_{r\to\infty} \frac{l}{R} = \lim_{r\to\infty} \frac{2}{R\sqrt{1-\lambda^2}} \log \sqrt{\frac{1-\lambda^2}{2} \frac{\cosh R - \cosh r}{\cosh r - \lambda \sinh r}},$$

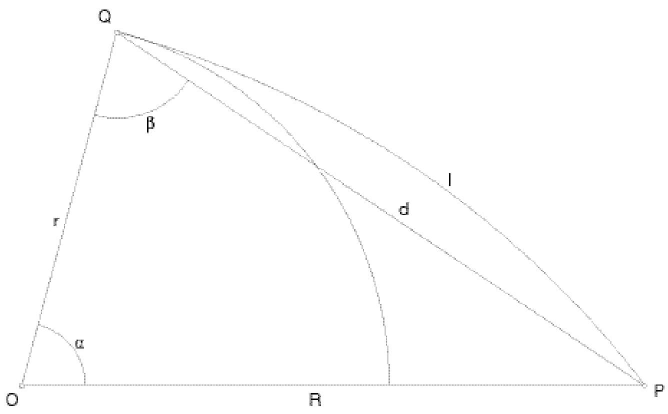


FIGURE 5

then

$$\lim_{r\to\infty}\frac{l}{R}=\frac{1}{\sqrt{1-\lambda^2}}\lim_{r\to\infty}\frac{\log\cosh R-\cosh r}{R},$$

and the last limit is 1.

It remains to prove that the angles α tend to 0. By the first cosinus law applied to the triangle OPQ

$$\cosh d=\cosh R\cosh r-\sinh R\sinh r\cos\alpha.$$

Isolating $\cos\alpha$ in the last expression and using (7) we easily get that

$$\lim_{r\to\infty}\cos\alpha=1.$$

□

Proposition 5.1. *For every n, let $r_n=n$, $R_n=\mathrm{e}^{2n}$ and $K_n=K_\lambda(R_n,r_n)$, the λ-convex set described above. If L_n and D_n are the perimeter and the diameter of K_n then*

$$\lim_{n\to\infty}\frac{L_n}{D_n}=\frac{2}{\sqrt{1-\lambda^2}}.$$

Moreover, if we take $r_n=n$ and $R_n=a\mathrm{e}^{n}$ with $a>0$, then

$$\lim_{n\to\infty}\frac{L_n}{D_n}=\frac{2}{\sqrt{1-\lambda^2}}+\frac{\pi}{2a}.$$

Proof. Using corollary 5.1

$$\lim_{n\to\infty}\frac{L_n}{D_n}=\lim_{n\to\infty}\frac{4(\sinh r_n(\frac{\pi}{2}-\alpha)+l_n)}{2R_n}=2\lim_{n\to\infty}\frac{\pi\sinh r}{2R_n}+\frac{l_n}{R_n}.$$

□

Then we have found, for every l between 0 and $\frac{1}{2}\sqrt{1-\lambda^2}$, a sequence of λ-convex sets such that $\lim\frac{D}{L}=l$. We summarize this result in the following theorem

Theorem 1. *Let $0\le\lambda\le1$, for every l in $[0,\frac{1}{2}\sqrt{1-\lambda^2}]$ there exists a sequence (K_n) of λ-convex sets expanding over the whole hyperbolic plane such that*

$$\lim_{n\to\infty}\frac{D_n}{L_n}=l$$

where D_n and L_n are, respectively, the diameter and the perimeter of K_n.

Note that, as it was said in proposition 4.1, $\frac{1}{2}\sqrt{1-\lambda^2}$ is the upper bound for $\lim D/L$.

6. Relation between area and perimeter of a convex set

In the euclidean plane, given a sequence (K_n) of convex sets expanding over the whole plane, if F_n and L_n are the area and the perimeter of K_n then the quotient F_n/L_n always goes to infinity. Indeed, it can be proved that $F/L \geq r_i/2$ where r_i is the radius of the greatest circumference contained in K (this easily follows from the expression $F = \frac{1}{2}\int pds$ where p is the distance to the origin of the circumference and the support lines of the convex).

In the hyperbolic plane, for any sequence (K_n) of convex sets expanding over the whole hyperbolic plane, we have that

$$\limsup \frac{F_n}{L_n} \leq 1$$

where F_n and L_n are the area and the perimeter of K_n. This is a consequence of the hyperbolic isoperimetric inequality (1). If K_n are supposed to be h-convex and bounded by piecewise C^2 curves it is known that

$$\lim \frac{F_n}{L_n} = 1.$$

In the general case, it was proved in [2] that for every nonnegative $l \leq 1$ there exists a sequence (K_n) of convex sets expanding over the whole hyperbolic plane such that

$$\lim_{n\to\infty} \frac{F_n}{L_n} = l$$

where F_n and L_n are, respectively, the area and the perimeter of K_n.

Let us recall how these examples were constructed. Let K_n be a regular polygon formed by $3\cdot 2^{n-1}$ isosceles triangles inscribed in a circle of radius R_n. If d_n is the length of the basis of one of this triangles and h_n its area, then $F_n/L_n = h_n/d_n$. If $\alpha_n = 2\pi/(3\cdot 2^{n-1})$ is the opposite angle to d_n then

$$d_n = 2\,\mathrm{arcsinh}\left(\sinh R_n \cdot \sin\left(\frac{\alpha_n}{2}\right)\right)$$

and

$$h_n = \pi - \left(\alpha_n + 2\arctan \frac{1}{\tan\frac{\alpha_n}{2}\cdot \cosh R_n}\right).$$

Taking $R_n = n$ we have that $\lim h_n/d_n = 0$. Taking $R_n = \log(4/\mu\alpha_n)$ with $\mu > 0$ we have that

$$\lim\left(\tan\frac{\alpha_n}{2}\cdot\cosh R_n\right) = \lim \frac{\alpha_n}{2}\frac{2}{\mu\alpha_n} = \frac{1}{\mu},$$

hence,

$$\lim h_n = \pi - 2\cdot \arctan\mu.$$

In an analogous way

$$\lim d_n = 2\,\mathrm{arcsinh}\,\frac{1}{\mu}.$$

So we have that

$$\lim\frac{F_n}{L_n} = \frac{\pi - 2\arctan\mu}{2\,\mathrm{arcsinh}\,\frac{1}{\mu}}$$

that takes, depending on the parameter μ, all values between 0 and 1.

It is interesting to calculate $\lim D_n/L_n$ being D_n the diameter of these polygons.

$$\lim\frac{D_n}{L_n} = \lim\frac{2R_n}{3\cdot 2^n\,\mathrm{arcsinh}(\sinh R_n\cdot\sin(\frac{\alpha_n}{2}))} = 0$$

Using the sequences constructed in section 5, we can show in an alternative way that $\lim F/L$ can take any value between 0 and 1. Indeed, let $K_n = K_\lambda(R_n, r_n)$ with $r_n = n$, $R_n = e^{2n}$ and $0 \leq \lambda \leq 1$. If F_n and L_n are the area and the perimeter of K_n then, by the Gauss-Bonnet formula

$$\lim \frac{F_n}{L_n} = \lim \frac{\int_{\partial K_n} k_g \, \mathrm{d}s \, + \, \beta_n}{L_n} = \lim \frac{\int_{\partial K_n} k_g \, \mathrm{d}s}{L_n} =$$
$$= \lim \frac{\lambda(L_n \, - 4\alpha_n \sinh r_n) \, + \, \coth r_n 4\alpha_n \sinh r_n}{L_n} =$$
$$= \lambda \, + \, \lim \frac{4\alpha_n \sinh r_n (\coth r_n - \lambda)}{D_n} \lim \frac{D_n}{L_n} = \lambda$$

where β_n are the interior angles in ∂K_n and $2\alpha_n$ is the angle described by one of the arcs of circle in ∂K_n.

It is interesting to remark that the sequence with $\lim F/L = 0$ is precisely the sequence with $\lim D/L = 1/2$. This seems not to be casual, D/L goes to $1/2$ because the convex sets are "very thin" so it is not surprising that F/L goes to 0.

It seems natural to ask if a sequence of convex sets expanding over the whole plane with $\lim D/L = 1/2$ must have $\lim F/L = 0$. In the next sections we will look for bounds $0 \leq f(l) \leq g(l) \leq 1$ such that if a sequence has $\lim D/L = l$ then

$$f(l) \leq \lim \frac{F_n}{L_n} \leq g(l). \tag{8}$$

Taking into account the sequences of polygons used above , $f(0)$ must be 0 and $g(0)$ must be 1.

A first step, in order to find g, is to find a bound for F, the area of a convex set with fixed perimeter and diameter.

7. Extremal values of the area for a given perimeter and diameter

It is known that, given positive L_0, the compact domain with perimeter L_0 and maximum area is a circle. This is a consequence of the hyperbolic isoperimetric inequality (1). If we restrict to compact domains with diameter greater or equal than a given value D_0 and fixed perimeter $L_0 < 2\pi \sinh D_0/2$, circles are not allowed. Then we consider the problem of finding which of these domains has maximum area. As a first step we have

Proposition 7.1. *If K is a compact domain with diameter greater or equal than D_0, fixed perimeter L_0 and maximum area, then K is convex.*

Proof. Indeed, if K is not convex there must exist two boundary points x and y such that the geodesic segment s joining them is in K^c . Let γ be the piece of ∂K between x and y. Performing a reflection with respect to s of γ we can construct a new domain with perimeter L_0 and more area than K. $\square$

Let $\mathcal{C}$ be the set of all compact convex domains in the hyperbolic plane. If $K \in \mathcal{C}$ we define its *hyperbolic parallel convex sets* at distance ϵ as follows

$$K^{\epsilon} = \{p \in \mathbb{H}^2 \,|\, d(p, K) \leq \epsilon\}$$
$$K^{-\epsilon} = \{p \in K \,|\, d(p, \partial K) \geq \epsilon\}.$$

In $\mathcal{C}$ we define the following distance:

$$d(K_1, K_2) = \min\{\epsilon > 0 \,|\, K_1^{\epsilon} \subset K_2 \, \text{and} \, K_2^{\epsilon} \subset K_1\}$$

Now, $\mathcal{C}$ is a metric space and we consider the induced metric topology. Distance d is the hyperbolic version of Hausdorff distance for convex sets in the euclidean plane (cf. for instance [4]). In fact, $\mathcal{C}$ can be seen in the projective model as the set of euclidean convex domains contained in the unit disk $\mathbb{D}$. If d_e is the euclidean distance and K_e^ϵ, $K_e^{-\epsilon}$ denote the *euclidean parallel convex sets* to K, we can define

$$d_e(K_1,K_2) = \inf\{\epsilon > 0 \,|\, (K_1)_e^\epsilon \subset K_2 \,\text{and}\, (K_2)_e^\epsilon \subset K_1\}$$

where K_1 and K_2 are convex domains contained in $\mathbb{D}$. If K is a convex subset of $\mathbb{D}$, the ball $\mathcal{B}(K,d) = \{K' \in \mathcal{C} \,|\, d(K,K') \leq d\}$ contains the ball $\mathcal{B}_e(K,d) = \{K' \in \mathcal{C} \,|\, d_e(K,K') \leq \epsilon\}$ where

$$\epsilon = inf\{d_e(\partial K, \partial K^d), d_e(\partial K, \partial K^{-d})\}$$

Indeed, if $K' \in \mathcal{B}_e(K,d)$ then $K' \in \mathcal{B}(K,d)$ since

$$K^{-d} \subset K_e^{-\epsilon} \subset K' \subset K_e^\epsilon \subset K^d.$$

Similarly, every *euclidean* ball contains a *hyperbolic* one. Therefore the topologies defined in $\mathcal{C}$ by d and by d_e are equivalent.

Let $\mathcal{B}$ be the ball in $\mathcal{C}$ with radius L_0 and center the convex set containing only the origin. We are interested in convex domains belonging to $\mathcal{B}$ because every convex domain with perimeter L_0 can be moved to be in $\mathcal{B}$.

We have (cf. [4]) the following

Theorem 2. **[Blaschke Selection Theorem]** *A bounded infinite family of euclidean convex sets has a sequence converging to some convex set.*

Corollary 7.1. *$\mathcal{B}$ is compact.*

Proof. In metric spaces a set A is compact if and only if every infinite subset of A contains an accumulation point. Since the *euclidean* and *hyperbolic* topologies are equivalent we can use theorem 2 to state that any infinite family in $\mathcal{B}$ accumulates to a convex set. Since $\mathcal{B}$ is closed we are done. □

Proposition 7.2. *The diameter, D, perimeter, L, and area, F, functions are continuous over $\mathcal{C}$, the set of all compact convex domains in the hyperbolic plane with the Hausdorff topology.*

Proof. Let $K \in \mathcal{C}$ and let K_n be a sequence of convex domains such that $d(K_n, K)$ tends to 0. By the definition of the distance between two convex sets

$$K^{-d_n} \subset K_n \subset K^{d_n}$$

where $d_n = d(K_n, K)$. Therefore

$$D(K) - 2d_n = D(K_{-d_n}) \leq D(K_n) \leq D(K_{d_n}) = D(K) + 2d_n$$

and $\lim D(K_n) = D(K)$. For the perimeter, using the hyperbolic Crofton formula we have

$$\begin{aligned} \lim L(K_{\pm d_n}) = \lim \int_R \chi\{r \in R \,|\, r \cap K_{\pm d_n} \neq \emptyset\} dr = \\ = \int_R \lim \chi\{r \in R \,|\, r \cap K_{\pm d_n} \neq \emptyset\} dr = \\ = \int_R \chi\{r \in R \,|\, r \cap K \neq \emptyset\} dr = L(K) \end{aligned}$$

where R is the set of lines in $\mathbb{H}^2$ and χ is the characteristic function. Therefore $\lim L(K_n) = L(K)$. The continuity for the area follows analogously. □

Let $\mathcal{S} = \{K \in \mathcal{B} \,|\, D(K) \geq D \quad L(K) = L_0\} = D^{-1}([D_0, L_0/2]) \cap L^{-1}(L_0) \cap \mathcal{B}$. $\mathcal{S}$ is a closed subset of $\mathcal{B}$ so it is compact. Then F must have a maximum and a minimum over $\mathcal{S}$.

Corollary 7.2. *In the set of hyperbolic convex domains with diameter bounded below by D_0 and fixed perimeter L_0 the area function attains its maximum value.*

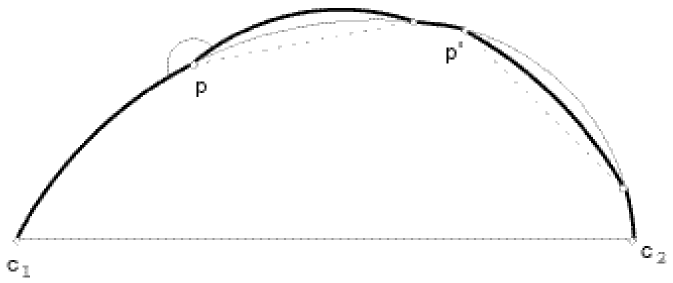


Figure 6

This proves the existence question. The uniqueness is discussed in the next theorem.

Theorem 3. *Given D_0 and $L_0 < 2\pi \sinh D_0/2$, the compact convex domain with diameter greater or equal than D_0 and perimeter L_0 that maximizes the area is a λ-lens with diameter exactly D_0.*

We need some previous results. Let K be a convex set in $\mathcal{B}$ maximizing the area. Let c_1 and c_2 be the endpoints of a diameter of K and $\gamma(s)$ be the curve ∂K parametrized by the arc.

First we see that C^1 points are locally equivalent.

Lemma 7.1. *Let $p = \gamma(s)$ and $p' = \gamma(s')$ be two points different from c_1 and c_2. If γ is C^1 in p and p' then there exists a rigid motion that moves a neighborhood of p in γ onto a neighborhood of p'.*

Proof. Let $\gamma(s) = p$ and $\gamma(s') = p'$. Let ϵ be small enough to make $c_1, c_2 \notin \gamma([s-\epsilon, s+\epsilon])$ and $c_1, c_2 \notin \gamma([s'-\epsilon, s'+\epsilon])$. For any $t \in (s, s+\epsilon]$ let $t' > s$ be the first one with $d(\gamma(s), \gamma(t)) = d(\gamma(s'), \gamma(t'))$. Swapping $\gamma([s,t])$ and $\gamma([s',t'])$ we obtain the border of a new domain K' with the same area as K, the same perimeter and, perhaps, a greater diameter. The angle between $\gamma'(s)$ and $\overline{\gamma(s)\gamma(t)}$ is equal to the angle between $\gamma'(s')$ and $\overline{\gamma(s')\gamma(t')}$. Indeed, if one of these angles is greater that the other, in $\partial K'$ there would be an interior angle greater than π; contradicting the fact that K' must be convex (cf. lemma 3.1)

So, in polar coordinates with center p and direction $\gamma'(s)$, the curve $\gamma([s-\epsilon, s+\epsilon])$ has the same expression as $\gamma([s'-\epsilon, s'+\epsilon])$, in polar coordinates with center p' and direction $\gamma'(s')$. If g is the motion that moves p on p' and $\gamma'(s)$ on $\gamma'(s')$, then g moves the neighborhood of p onto the neighborhood of p'. □

Lemma 7.2. *γ is of class C^1 except in c_1 and c_2.*

Proof. Since K is convex, ∂K must be C^1 except from, at most, in a countable set of points. Let $\gamma(s) = x \neq c_1, c_2$ be one of these points. Let (s_n) be a sequence such that $\lim s_n = s$ and $x_n = \gamma(s_n)$ are C^1 points. Let ϵ be such that $c_1, c_2 \notin U_n = \gamma([s_n - \epsilon, s_n + \epsilon])$ for any n. U_n are not geodesic segments so U_0 must contain three non aligned points $p_1 = \gamma(s_0 + t_1)$, $p_2 = \gamma(s_0 + t_2)$ and $p_3 = \gamma(s_0 + t_3)$. Let g_n be the motion that moves U_0 onto U_n. The group of rigid motions in $\mathbb{H}^2$ can be identified with the set of triangles congruent to $p_1p_2p_3$. Since

$$\lim g_n(\gamma(s_0 + t_i)) = \lim \gamma(s_n + t_i) = \gamma(s + t_i), \quad i = 1, 2, 3$$

the sequence (g_n) converges to a motion g and for every $|t| < \epsilon$

$$g(\gamma(s_0 + t)) = (\lim g_n)(\gamma(s_0 + t)) = \lim \gamma(s_n + t) = \gamma(s + t).$$

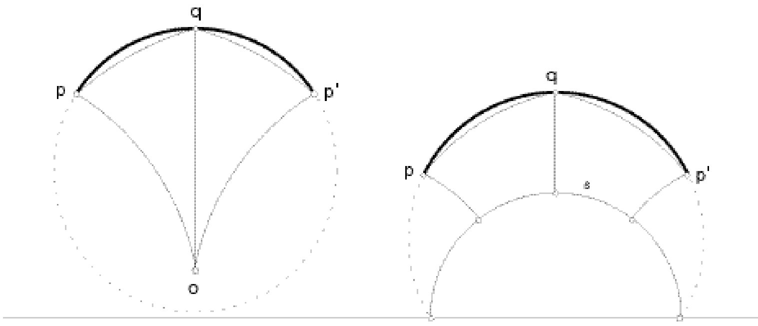


FIGURE 7

So, g moves a neighborhood of x_0 (in γ) onto a neighborhood of x. This contradicts the fact that, in x, γ is not C^1. □

Now we can afford the

Proof of theorem 3. Let $p = \gamma(s)$ and $p' = \gamma(s')$ be two border points of K such that neither c_1 nor c_2 are in $\gamma([s, s'])$. Let r and r' be the lines through p and p', respectively, orthogonal to γ in these points. Two cases are possible, r intersects r' or not.

If r and r' intersect in o, let $q = \gamma(t)$ be the point in $\gamma([s, s'])$ such that $d(p, q) = d(p', q)$ (see figure 7). The argument used in the proof of lemma 7.1 implies that the angles $(pq, \gamma'(s))$, $(\gamma'(t), pq)$, $(qp', \gamma'(t))$ and $(\gamma'(s'), qp')$ must be equal. Then the triangles opq and $op'q$ are isosceles with $d(o, p) = d(o, q) = d(o, p')$. This argument could be repeated starting with p and q or with q and p'. Repeating it indefinitely we get a dense subset $\Omega \subset \gamma([s, s'])$ at a constant distance from o. By continuity of the distance, $\gamma([s, s'])$ must be an arc of circle with center o.

If r and r' are nonsecant they have a common perpendicular line s. In an analogous way we can see that $\gamma([s, s'])$ must be a piece of an equidistant of s (see figure 7). Anyway, the curve between p and p' is C^2 and has constant curvature. Then, each piece of $\partial K/\{c_1, c_2\}$, must be C^2 with constant curvature. Lemma 7.1 implies that the curvature must be the same in the two pieces. □

For every pair (D_0, L_0) there exists a unique convex set with diameter D_0 and perimeter L_0 that maximizes the area and it is a λ-lens. It is important to remark that the value of this λ is a function on D_0 and L_0 and that

$$\begin{aligned} \lambda < 1 \quad &\text{if} \quad 2D_0 \le L_0 < 4\sinh\frac{D_0}{2} \\ \lambda \ge 1 \quad &\text{if} \quad 4\sinh\frac{D_0}{2} \le L_0 \le 2\pi\sinh\frac{D_0}{2}. \end{aligned}$$

Notice that in the first case the boundary lines are λ-geodesic segments and in the second case they are arcs of circumference.

Now we can treat the problem of finding the bounds in formula (8).

8. UPPER AND LOWER BOUNDS FOR $\lim F/L$ WITH RESPECT $\lim D/L$

As usual, let (K_n) be a sequence of convex sets expanding over the whole hyperbolic plane. Let F_n, L_n and D_n be, respectively, the area, perimeter and diameter of K_n. Let us suppose that $\lim D_n/L_n = l \neq 0$. The domain with diameter D_n and perimeter L_n of maximum area is a λ_n-lens with area $F(D_n, L_n)$ and interior

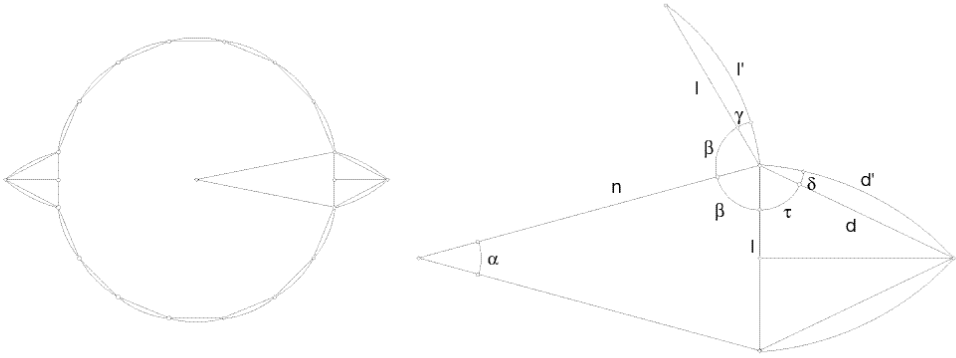


FIGURE 8

angles β_n. In this case

$$\lim \frac{F_n}{L_n} \leq \lim \frac{F(D_n, L_n)}{L_n} = \lim \frac{\lambda_n \cdot L_n - 2\beta_n}{L_n} = \lim \lambda_n.$$

Notice that we can suppose, for n big enough, $\lambda_n < 1$ and $\lim \lambda_n \neq 1$. Indeed, if $\liminf \lambda_n \geq 1$ then $L_n \geq 4 \sinh D_n/2$ for n arbitrarily big. Therefore $\lim D_n/L_n$ must be 0 and in this case all values of $\lim F_n/L_n$ between 0 and 1 can be attained (see section 6).

Using (2)

$$\begin{aligned} \lim \frac{L_n}{D_n} &= \lim \frac{4}{\sqrt{1-\lambda_n^2} D_n} \cdot \operatorname{arcsinh}\left(\sqrt{1-\lambda_n^2} \sinh \frac{D_n}{2}\right) \\ &= \lim \frac{4}{\sqrt{1-\lambda_n^2}} \frac{\log(\sinh \frac{D_n}{2})}{D_n} = \lim \frac{2}{\sqrt{1-\lambda_n^2}}. \end{aligned}$$

So $\lim \lambda_n$ exists and its value is $\sqrt{1-(2l)^2}$. We can state

Theorem 4. *Let (K_n) be a sequence of convex sets expanding over the whole hyperbolic plane. Let F_n, L_n and D_n denote their area, perimeter and diameter. If $\lim D_n/L_n = l$ then*

$$\lim \frac{F_n}{L_n} \leq \sqrt{1-(2l)^2}. \tag{9}$$

Sequences constructed in section 5 show that (9) could not be better. Taking $K_n = K_\lambda(\mathrm{e}^{2n}, n)$ we know that

$$\lim \left(\frac{F_n}{L_n}\right)^2 + \lim \left(\frac{2D_n}{L_n}\right)^2 = \lambda^2 + (1-\lambda^2) = 1.$$

The following proposition shows that the only lower bound for the $\lim F_n/L_n$ is $f(l) \equiv 0$.

Proposition 8.1. *For every $0 \leq l \leq 1/2$ and every $0 \leq \lambda \leq \sqrt{1-(2l)^2}$ there exists a sequence of convex sets expanding over the whole hyperbolic plane with $\lim D_n/L_n = l$ and $\lim F_n/L_n = \lambda$.*

Proof. Let K_n be the regular polygon with $3 \cdot 2^{n-1}$ sides inscribed in a circle of radius n. Let K_n' be the polygon K_n with two isosceles triangles of height $k \cdot 2^n n$ $(k > 0)$ attached in two opposite sides of K_n. Now, let K_n^λ be the domain bounded by the exterior λ-geodesic segments corresponding to each side of K_n' (see figure 8). Let F_n, L_n and D_n be the area, perimeter and diameter of K_n^λ respectively. Let l_n be the length of each side of K_n. Let d_n be the length of the equal sides in

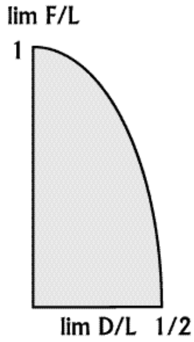


FIGURE 9. Possible values for $\lim F_n/L_n$ with respect to $\lim D_n/L_n$

the attached triangles. We denote l'_n and d'_n the lengths of the λ-geodesic segments corresponding to l_n and d_n respectively. Let γ_n and δ_n be the angles between l_n and l'_n and between d_n and d'_n respectively. Let β_n be the half part of the interior angles in ∂K_n and let τ_n be the value of the two equal angles in each attached triangle.

The domains K_n^λ are convex for n big enough. Indeed, β_n and τ_n go to 0 and, taking (4) into account, γ_n and δ_n go to $\arccos\lambda < \pi/2$. Therefore, for n big enough, the interior angles of ∂K_n^λ are not greater than π and the K_n^λ are convex domains.

Using hyperbolic trigonometry,

$$l_n = 2\,\mathrm{arcsinh}\left(\sinh n \sin(\frac{\pi}{3\cdot 2^{n-1}})\right) \sim 2n\log(e/2)$$

and

$$d_n = \mathrm{arccosh}\left(\cosh l_n \cosh(k2^n n)\right) \sim k2^n n$$

when n goes to infinity. Using (2) we have

$$l'_n \sim \frac{2n\log(e/2)}{\sqrt{1-\lambda^2}} \qquad d'_n \sim \frac{k2^n n}{\sqrt{1-\lambda^2}}.$$

Since $D_n - 2k2^n n < 2n$, $D_n \sim 2k2^n n$. Therefore

$$\lim_{n\to\infty}\frac{L_n}{D_n} = \lim_{n\to\infty}\frac{(3\cdot 2^{n-1}-2)l'_n + 4d'_n}{2k2^n n} = \frac{1}{\sqrt{1-\lambda^2}}\left(\frac{3\log(e/2)}{2k}+2\right)$$

which takes, depending on k all the values between $2/\sqrt{1-\lambda^2}$ and infinity.

Finally, using the Gauss-Bonnet formula

$$\begin{aligned}\lim_{n\to\infty}\frac{F_n}{L_n} &= \lim_{n\to\infty}\frac{\lambda L_n + \sum \text{angles} - 2\pi}{L_n}\\ &= \lambda + \lim_{n\to\infty}\frac{(3\cdot 2^{n-1}-n)(\pi - 2(\gamma_n+\beta_n))}{L_n} = \lambda.\end{aligned}$$

Notice that this computations give only the cases $0 < l < 1/2$ and $\lambda < \sqrt{1-(2l)^2}$. We can use the examples given in section 5 for $l = 1/2$ and $\lambda = \sqrt{1-(2l)^2}$ and those in section 6 for $l = 0$ $\square$

So, we have seen that the upper bound $g(l)$ for $\lim F/L$ with respect $l = \lim D/L$ is the function $\sqrt{1-(2l)^2}$. This value and all the lower ones are attained for every l between 0 and 1/2 (see figure 9).

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