PERIMETER, DIAMETER AND AREA OF CONVEX SETS IN THE HYPERBOLIC PLANE.

EDUARDO GALLEGO AND GIL SOLANES

Abstract. In this paper we study the relation between the asymptotic values of the ratios area/length \( F/L \) and diameter/length \( D/L \) of a sequence of convex sets expanding over the whole hyperbolic plane. It is known (cf. [3] and [2]) that \( F/L \) goes to a value between 0 and 1 depending on the shape of the contour. Here, first of all it is seen that \( D/L \) has limit value between 0 and 1/2 in strong contrast with the euclidean situation in which the lower bound is 1/\( \pi \) (\( D/L = 1/\pi \) if and only if the convex set has constant width). Moreover, it is shown that, as the limit of \( D/L \) approaches to 1/2, the possible limit values of \( F/L \) reduce. Examples of all possible limits \( F/L \) and \( D/L \) are given.

1. Introduction

In hyperbolic geometry, given a point \( p \) exterior to a line \( l \) there are infinitely many non secant lines. These lines lie between the two so called parallel lines to \( l \). When the distance from \( p \) to \( l \) grows to infinity, the angle between the parallel lines goes to 0. This fact leads to the ambiguous idea that, in some sense, given a line \( l \), the probability that a random line meets \( l \) is zero. In order to formalize this idea let us restrict our attention to the interiors of a sequence \( (K_n) \) of convex sets in the hyperbolic plane expanding to fill it. The probability for a random chord of \( K_n \) to meet \( r \) inside \( K_n \) should go to 0 as \( n \to \infty \). It can be proved by using the Cauchy-Crofton formula, that this probability is \( 2\sigma_n/L_n \), where \( \sigma_n \) is the length of the chord \( l \cap K_n \) and \( L_n \) denotes the length of \( \partial K_n \). Because the length of the chord \( \sigma_n \) is less or equal than the diameter \( D_n \) of \( K_n \), the study of \( \lim \sigma_n/L_n \) is related to the knowledge of the asymptotic value of the ratio \( D_n/L_n \) where \( D_n \).

The question of whether the asymptotic value of \( D/L \) is zero or not already appeared in [7]. In the present text we will see that there are many possible values for this limit and we will find them all. More precisely we will prove that for every \( e \in [0,1/2] \) there is a sequence \( (K_n) \) of hyperbolic convex sets such that \( \lim D_n/L_n = e \). In fact it will be seen that, for convex sets with respect to equidistants intersecting infinity with angle \( \theta \), this limit can take values only below \( (\sin \theta)/2 \).

It must be noticed that in the euclidean case the situation is quite different: any convex set satisfies \( 1/\pi \leq D/L \leq 1/2 \). The lower bound is reached only by constant width sets and the upper bound by the segments.

The paper is organized as follows. In sections 2 and 3 we introduce the basic concepts and notation. In section 4 we find lower and upper bounds for \( D/L \) in the \( \lambda \)-convex case, concluding that the asymptotic value of \( D/L \) for \( h \)-convex sequences is 0. Section 5 is devoted to the construction of examples showing that the preceding bounds are the best possible. In section 6 we recall the asymptotic behavior of the quotient \( F/L \) being \( F \) the area of the convex sets. We introduce, in section 7, the

1991 Mathematics Subject Classification. Primary 52A55; Secondary 52A10.

Key words and phrases. Hyperbolic plane, diameter, perimeter, area, \( \lambda \)-geodesic, \( \lambda \)-convex set.

Work partially supported by DGYCIT grant number PB96 – 1178.
metric space of hyperbolic convex sets in order to treat an isoperimetric problem. Finally, in section 8, we give the relation between the asymptotic values of \( D/L \) and \( F/L \). More precisely, we can state that

\[
\lim_{n \to \infty} \frac{F_n}{L_n} \leq \sqrt{1 - \left(2 \lim_{n \to \infty} \frac{D_n}{L_n}\right)^2}.
\]

We wish to thank professor A. Reventós for many helpful conversations during the preparation of this work.

2. The hyperbolic plane

In this section we introduce the hyperbolic plane as well as some basic facts that will be used later on. The hyperbolic plane, \( \mathbb{H}^2 \), is the unique complete simply connected Riemannian manifold of dimension 2 with constant curvature \(-1\). Its geometry corresponds to the one obtained from the absolute geometry given by the first four Euclid postulates and the Lobachevsky postulate: through every point \( P \) exterior to a line \( l \) pass more than one line not intersecting \( l \). It is useful to have different models for this geometry, we shall describe their points, lines (geodesics) and rigid motions:

- **Half-plane model.** It is the half-plane \( \{(x, y) \in \mathbb{R}^2| y > 0\} \) with the metric \( \frac{1}{y^2}(dx^2 + dy^2) \). The geodesics are half-circles centered in \( \{y = 0\} \) and vertical half-lines. The rigid motions are composition of inversions with respect to these circles and symmetries with respect to these lines. This model is conformal since the metric is a multiple of the euclidean metric.

- **Disk model.** It is the unit disk with the metric \( \frac{4}{(1 - x^2 - y^2)^2}(dx^2 + dy^2) \). This model is also conformal. The geodesics are the a diameters of the disk and the arcs of circumference orthogonal to the border. The rigid motions are homographies of the complex plane fixing the disk.

- **Projective model.** It is the unit disk with the metric \( \frac{1}{1 - r^2}(\frac{1}{r^2}dr^2 + r^2d\theta^2) \) where \((r, \theta)\) are the euclidean polar coordinates centered at the origin. The geodesics are chords of the disk. This fact makes this model become very useful when studying questions related to convex sets. The rigid motions are the projectivities fixing the disk.

In the following sections **polar coordinates** will be useful in the treatment of some problems. Whatever it is the model we work in, we can parametrize the points of the hyperbolic plane in the following way. Let \( O \) be a point called origin. We choose in \( O \) a direction \( v \in T_O\mathbb{H}^2 \). For each point \( P \), let \( r \) be the length of the geodesic segment joining \( O \) and \( P \), and let \( \theta \) be the angle between this segment and \( v \). Now, \( \mathbb{H}^2 \setminus \{O\} \) is perfectly parametrized by the coordinates \((r, \theta)\). It can be easily checked out that in these coordinates the metric is written as follows.

\[
g = dr^2 + (\sinh r)^2d\theta^2.
\]

The volume element will be then \( \sinh r dr d\theta \) and the area and perimeter of a circumference of radius \( r \) in \( \mathbb{H}^2 \) are

\[
L = \int_0^{2\pi} \sinh r d\theta = 2\pi \sinh r, \quad F = \int_0^r \int_0^{2\pi} \sinh r d\theta dr = 2\pi(\cosh r - 1).
\]

We shall need some formulas in **hyperbolic trigonometry**, proofs can be found in [5]. Let \( a, b \) and \( c \) be three sides of a geodesic triangle and let \( \alpha, \beta \) and \( \gamma \) be their opposite angles. The following identities are then verified:

\[
\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha,
\]
\[
\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma},
\]
\[
\sinh a \cos \beta = \cosh b \sinh c - \sinh b \cosh c \cos \alpha.
\]

The area of such triangle is
\[
F = \pi - (\alpha + \beta + \gamma).
\]
It is said that the area equals the \textit{angular defect}.

To end this section we will just state some integral geometric formulas used later on. The \textit{isoperimetric inequality}
\[
L^2 - 4\pi F - F^2 \geq 0
\]
gives a relation between perimeter \(L\) and area \(F\) of an arbitrary compact domain in \(\mathbb{H}^2\).

The \textit{Cauchy-Crofton theorem} expresses the length of a curve in terms of the measure of lines (counted with its multiplicity) intersecting it. More precisely
\[
2L = \int_{r \cap C \neq \emptyset} n(r) dr
\]
where \(dr\) is a normalized isometry invariant density of geodesic lines and \(n(r)\) the number of intersecting points of \(r\) and the curve \(C\) (cf. [6]).

3. \textbf{Convexity and }\lambda\textbf{-convexity}

In this section we introduce the concept of \(\lambda\)-convexity as well as some basic known facts about it. For a more detailed introduction see [3].

\textbf{Definition 3.1.} A subset \(K \subset \mathbb{H}^2\) is said to be \textit{convex} if for every pair of points in \(K\), the geodesic segment joining them is also in \(K\).

Notice that a set in \(\mathbb{H}^2\) is convex if and only if in the projective model it looks like an euclidean convex set.

\textbf{Definition 3.2.} A closed convex set with nonempty interior is called a \textit{convex domain}.

From now on all convex sets will be compact convex domains. If \(K\) is a convex set, then \(\partial K\) is \(C^2\) except from, at most, a countable set of points. Moreover, \(\partial K\) must have finite length, which is called the \textit{perimeter} \(L\) of \(K\), and the \textit{area} \(F\) of \(K\) must be finite too. The \textit{diameter} is given by \(D = \max\{d(p, q) \mid p, q \in \partial K\}\).

\textbf{Definition 3.3.} A sequence \((K_n)\) of convex sets is said to \textit{expand over the whole hyperbolic plane} if \(K_n \subset K_{n+1}\) and \(\forall p \in \mathbb{H}^2\) there is an \(n\) such that \(p \in K_n\).

As in the euclidean case we have

\textbf{Lemma 3.1.} A \textit{compact domain} with piecewise \(C^2\) boundary is convex if and only if its geodesic curvature does not change the sign and in the non \(C^2\) points the interior angles are not greater than \(\pi\).

Given a geodesic line \(l\) in the euclidean plane, the set of equidistant points to \(l\) are two parallel lines symmetric with respect to \(l\). In the hyperbolic plane this is no longer true; the set of equidistant points to \(l\) are two smooth curves called \textit{equidistants}. If we consider the half-plane model, the equidistant curves to the hyperbolic line \(x = 0\) are euclidean half-lines passing through \((0, 0)\). Indeed any geodesic arc with center in \((0, 0)\) going from \(x = 0\) to \(y = mx\) has the same length because they are (euclidean) homothetic and every homothety with center in the axis \(y = 0\) is the composition of two inversions with respect circumferences centered in \(y = 0\) which are hyperbolic isometries. In fact, if \(m = \tan \theta\) the length of these
geodesic arcs is equal to \( \log(\cot \theta/2) \). In this model (and in the disk model too) equidistant lines are, in general, arcs of euclidean circles meeting the infinity at two points. Every equidistant separates the plane in two regions such that only one of them is convex.

**Definition 3.4.** A \( \lambda \)-geodesic is an equidistant line meeting the infinity with an angle \( \theta \) such that \( |\cos \theta| = \lambda \).

0-geodesic lines are geodesics. It must be noticed that \( \lambda \)-geodesics have constant geodesic curvature \( \pm \lambda \) at every point and the distance to equidistant geodesics is \( \text{arctanh} (\lambda) \).

**Definition 3.5.** An horocycle is a continuous curve orthogonal to a bundle of parallel lines.

In the half-plane and disk models, horocycles are euclidean circles tangent to the boundary. It can be easily seen that horocycles have constant geodesic curvature equal to \( \pm 1 \), so when \( \lambda \) goes to 1 they can be considered a limit case of \( \lambda \)-geodesics. From now on geodesics and horocycles will be considered as particular cases of \( \lambda \)-geodesic lines.

As \( \lambda \)-geodesics are the unique solutions of the ordinary differential equation of order two \( k_\lambda = \pm \lambda \), given two points there are two and only two \( \lambda \)-geodesics passing through them. The length of these \( \lambda \)-geodesic segments will be called the \( \lambda \)-geodesic distance between these points. Similarly, given a direction at a point, there are two and only two \( \lambda \)-geodesic lines passing through it with the given direction. Later on, we will need the following result

**Proposition 3.1.** Given a circle \( C \) and \( p \) an outer point to \( C \), there are two and only two tangent \( \lambda \)-geodesics to \( C \) passing by \( p \) and leaving \( C \) in the convex side.

**Proof.** Let us take a \( \lambda \)-geodesic tangent to \( C \) (leaving \( C \) in the convex side, cf. figure 2). Carrying out a rotation, with respect to the center of \( C \), we can make it pass through \( p \). With a symmetry with respect to the line joining \( p \) and the center of \( C \), we get the other \( \lambda \)-geodesic.

**Definition 3.6.** A set \( K \subset \mathbb{H}^2 \) is said to be \( \lambda \)-convex if for every pair of points in \( K \), the two \( \lambda \)-geodesic segments joining them are also in \( K \). When \( \lambda = 1 \), \( K \) is said to be convex with respect to horocycles or \( h \)-convex for short.

See [1] for an equivalent definition of \( h \)-convexity based on supporting horocycles.
Definition 3.7. For $\lambda \leq 1$, a $\lambda$-lens is the convex domain bounded by two intersecting $\lambda$-geodesics (see figure 3). For $\lambda > 1$, a $\lambda$-lens is the convex domain bounded by two intersecting circles with curvature equal to $\lambda$. When the intersection points are at infinity we talk about ideal $\lambda$-lens.

If $\lambda_1 \leq \lambda_2$ then every $\lambda_2$-convex set is $\lambda_1$-convex. Indeed, the $\lambda_1$-geodesic segments joining two points lie between the $\lambda_2$-geodesic segments joining them. In particular every $\lambda$-convex set is convex.

Lemma 3.2. Let $K$ be a compact convex domain bounded by a piecewise $C^2$ curve. Then $K$ is $\lambda$-convex if and only if its geodesic curvature satisfies $k_\gamma \geq \lambda$ and in the angular points the interior angle is less or equal than $\pi$.

Proof. Let $p \in \partial K$ such that $k_\gamma(p) < \lambda$. Let $(x, y)$ be geodesic normal coordinates such that $p$ is given by $(0, 0)$ and $\partial/\partial x$ is tangent to $\partial K$ in $p$. With respect to these coordinates, in a neighborhood of $p$, the boundary is the graph of

$$y = \frac{1}{2} k_\gamma(p) x^2 + o(x^2)$$

and the $\lambda$-geodesic curves with direction $\partial/\partial x$ are the graph of

$$y = \pm \frac{1}{2} \lambda x^2 + o(x^2).$$

The fact that $k_\gamma(p) < \lambda$ implies that one of these $\lambda$-geodesic is locally inside $K$. Since a compact domain cannot contain a whole $\lambda$-geodesic, we have contradiction with the $\lambda$-convexity of $K$.

Conversely, if $K$ is not $\lambda$-convex there are two points $x, y \in \partial K$ such that the $\lambda$-geodesic between them is not contained in $K$. By lemma 3.1, $K$ is convex so the geodesic segment $\gamma$ between $x$ and $y$ is in $K$. Let $0 \leq \mu < \lambda$ be the supremum of all nonnegative numbers such that the $\mu$-geodesic between $x$ and $y$ is contained in $K$. If this $\mu$-geodesic touches $\partial K$ in a $C^2$ point we should have $k_\gamma \leq \mu < \lambda$ in this point, a contradiction. The $\mu$-geodesic cannot touch $\partial K$ in an angular point and if it touches $\partial K$ in a non-$C^2$ point we have that the lateral limits for $k_\gamma$ are not greater than $\mu$, a contradiction. This implies, because $\mu$ is the supremum, that the $\mu$ geodesic is tangent at $x$ or at $y$. But then $k_\gamma \leq \mu < \lambda$ at $x$ or at $y$, a contradiction.

4. Relation between diameter and perimeter of a convex set

In this section it will be shown that, for a sequence of $\lambda$-convex sets expanding over the whole hyperbolic plane, the asymptotic value of $D/L$ is not greater than $\frac{1}{2} \sqrt{1 - \lambda^2}$.

We shall need the following lemma, which was proved in [3].
Lemma 4.1. Given two points in the hyperbolic plane and $0 \leq \lambda < 1$, if $d$ and $l$ are, respectively, the geodesic and $\lambda$-geodesic distances between the points, then

$$l = \frac{2}{\sqrt{1 - \lambda^2}} \arcsinh \left( \sqrt{1 - \lambda^2} \sinh \frac{d}{2} \right).$$

If $\lambda \to 1$ then $l \to 2 \sinh \left( \frac{d}{2} \right)$.

Proposition 4.1. Let $(K_n)$ be a sequence of $\lambda$-convex sets expanding over the whole hyperbolic plane. If $D_n$ are their diameters and $L_n$ their perimeters, then

$$0 \leq \liminf \frac{D_n}{L_n} \leq \limsup \frac{D_n}{L_n} \leq \frac{1}{2} \sqrt{1 - \lambda^2}.$$  

Proof. For each $n$ let $p_n$ and $q_n$ be points in $\partial K_n$ such that the chord $p_nq_n$ has length equal to $D_n$. Let $A_n$ be the $\lambda$-lens with endpoints $p_n$ and $q_n$. Since $K_n$ is $\lambda$-convex, $A_n \subset K_n$ so the perimeter of $A_n$ is less than $L_n$. So we get

$$\frac{4}{\sqrt{1 - \lambda^2}} \arcsinh \left( \sqrt{1 - \lambda^2} \sinh \frac{D_n}{2} \right) \leq L_n$$

and

$$\lim_{n \to \infty} \frac{D_n}{L_n} \leq \lim_{n \to \infty} \frac{D_n \sqrt{1 - \lambda^2}}{4 \arcsinh(\sqrt{1 - \lambda^2} \sinh \frac{D_n}{2})} = \frac{1}{2} \sqrt{1 - \lambda^2}.$$

Corollary 4.1. If $(K_n)$ is a sequence of $h$-convex sets expanding over the whole hyperbolic plane then

$$\lim_{n \to \infty} \frac{D_n}{L_n} = 0.$$

5. A family of examples

In this section we are going to construct sequences of $\lambda$-convex sets showing that inequalities in (3) are the best possible.

Let $C$ be a circumference with radius $r$ centered at a point $O$. Let $s$ be a geodesic segment with midpoint $O$ and length $2R$ ($R > r$). We call $K_\lambda(R, r)$ the smallest $\lambda$-convex set containing $C$ and $s$ (see figure 4). Let us describe the boundary of $K_\lambda(R, r)$. When $r \geq \arctanh(\lambda)$ the boundary of $K_\lambda(R, r)$ is formed by the two $\lambda$-geodesic segments tangent to $C$ leaving $C$ in the convex side union with the arcs of $C$ between the tangency points. By lemma 3.1, this curve bounds a $\lambda$-convex domain.

We shall see that, for suitable $r$ and $R$, the quotient $D/L$ can be as close as possible to any value between 0 and $\frac{1}{2} \sqrt{1 - \lambda^2}$. Let $P$ be one of the ends of $s$ and $Q$ be the tangency point with $C$ of one of the $\lambda$-geodesic segments starting at $P$. 
Let $d$ and $l$ be the geodesic and $\lambda$-geodesic distances between $P$ and $Q$. Finally, let $\alpha$ be the angle $POQ$ and $\beta$ be the angle $OQP$.

We present some interesting formulas in the next lemma.

**Lemma 5.1.** With the notation as above

\[ \tan \left( \frac{\pi}{2} - \beta \right) = \frac{\lambda}{\sqrt{\cosh^2 \frac{d}{2} - \lambda^2}} \]  

(4)

\[ l = \frac{2}{\sqrt{1 - \lambda^2}} \text{arcsinh} \left( \sqrt{\frac{1 - \lambda^2}{2}} \frac{\cosh R - \cosh r}{\cosh r - \lambda \sinh r} \right) \]  

(5)

*Proof.* The proof of the first formula can be found in [3]. From (4) it follows that

\[ \cos \beta = \lambda \tan \frac{d}{2} \]  

(6)

using the first cosine law on the hyperbolic triangle $OPQ$

\[ \cosh R = \cosh r \cosh d - \sinh r \sinh d \cos \beta. \]

Using (6) and isolating $\cosh d$ from the last equality we get

\[ \cosh d = \frac{\cosh R - \lambda \sinh r}{\cosh r - \lambda \sinh r} \]  

(7)

Substituting in (2) and bearing in mind that $\sinh \frac{d}{2} = \sqrt{\frac{\cosh^2 d - 1}{2}}$ we get the equation we were looking for. \hfill \Box

**Corollary 5.1.** If we take $R = e^{2r}$ or $R = ae^r$ with $a > 0$ then

\[ \lim_{r \to \infty} \frac{l}{R} = \frac{1}{\sqrt{1 - \lambda^2}} \]

\[ \lim_{r \to \infty} \alpha = 0 \]

where, as above, $l$ is the $\lambda$-geodesic distance between $P$ and $Q$ and $\alpha$ is the angle $POQ$.

*Proof.* Using the fact that $\log(x) \sim \arcsinh(x)$ when $x$ goes to infinity and formula (5) we have

\[ \lim_{r \to \infty} \frac{l}{R} = \lim_{r \to \infty} \frac{2}{R \sqrt{1 - \lambda^2}} \log \sqrt{\frac{1 - \lambda^2}{2}} \frac{\cosh R - \cosh r}{\cosh r - \lambda \sinh r}, \]
Figure 5

then

\[ \lim_{r \to \infty} \frac{l}{R} = \frac{1}{\sqrt{1-\lambda^2}} \lim_{r \to \infty} \frac{\log \cosh R - \cosh r}{R}, \]

and the last limit is 1.

It remains to prove that the angles \( \alpha \) tend to 0. By the first cosinus law applied to the triangle \( OPQ \)

\[ \cosh d = \cosh R \cosh r - \sinh R \sinh r \cos \alpha. \]

Isolating \( \cos \alpha \) in the last expression and using (7) we easily get that

\[ \lim_{r \to \infty} \cos \alpha = 1. \]

Proposition 5.1. For every \( n \), let \( r_n = n, R_n = e^{2n} \) and \( K_n = K_\lambda(R_n, r_n) \), the \( \lambda \)-convex set described above. If \( L_n \) and \( D_n \) are the perimeter and the diameter of \( K_n \) then

\[ \lim_{n \to \infty} \frac{L_n}{D_n} = \frac{2}{\sqrt{1-\lambda^2}}. \]

Moreover, if we take \( r_n = n \) and \( R_n = ae^n \) with \( a > 0 \), then

\[ \lim_{n \to \infty} \frac{L_n}{D_n} = \frac{2}{\sqrt{1-\lambda^2}} + \frac{\pi}{2a}. \]

Proof. Using corollary 5.1

\[ \lim_{n \to \infty} \frac{L_n}{D_n} = \lim_{n \to \infty} \frac{4(\sinh r_n(\frac{e}{2} - \alpha) + l_n)}{2R_n} = 2 \lim_{n \to \infty} \frac{\pi \sinh r_n}{2R_n} + \frac{l_n}{R_n}. \]

Then we have found, for every \( l \) between 0 and \( \frac{1}{2} \sqrt{1-\lambda^2} \), a sequence of \( \lambda \)-convex sets such that \( \lim \frac{D_n}{L_n} = l \). We summarize this result in the following theorem

Theorem 1. Let \( 0 \leq \lambda \leq 1 \), for every \( l \) in \( [0, \frac{1}{2} \sqrt{1-\lambda^2}] \) there exists a sequence \((K_n)\) of \( \lambda \)-convex sets expanding over the whole hyperbolic plane such that

\[ \lim_{n \to \infty} \frac{D_n}{L_n} = l \]

where \( D_n \) and \( L_n \) are, respectively, the diameter and the perimeter of \( K_n \).

Note that, as it was said in proposition 4.1, \( \frac{1}{2} \sqrt{1-\lambda^2} \) is the upper bound for \( \lim D/L \).
6. Relation between area and perimeter of a convex set

In the euclidean plane, given a sequence \((K_n)\) of convex sets expanding over the whole plane, if \(F_n\) and \(L_n\) are the area and the perimeter of \(K_n\) then the quotient \(F_n/L_n\) always goes to infinity. Indeed, it can be proved that \(F/L \geq r_i/2\) where \(r_i\) is the radius of the greatest circumference contained in \(K\) (this easily follows from the expression \(F = \frac{1}{2} \int pds\) where \(p\) is the distance to the origin of the circumference and the support lines of the convex).

In the hyperbolic plane, for any sequence \((K_n)\) of convex sets expanding over the whole hyperbolic plane, we have that

\[
\lim \sup \frac{F_n}{L_n} \leq 1
\]

where \(F_n\) and \(L_n\) are the area and the perimeter of \(K_n\). This is a consequence of the hyperbolic isoperimetric inequality (1). If \(K_n\) are supposed to be \(h\)-convex and bounded by piecewise \(C^2\) curves it is known that

\[
\lim \frac{F_n}{L_n} = 1.
\]

In the general case, it was proved in [2] that for every nonnegative \(l \leq 1\) there exists a sequence \((K_n)\) of convex sets expanding over the whole hyperbolic plane such that

\[
\lim \frac{F_n}{L_n} = l
\]

where \(F_n\) and \(L_n\) are, respectively, the area and the perimeter of \(K_n\).

Let us recall how these examples were constructed. Let \(K_n\) be a regular polygon formed by \(3 \cdot 2^{n-1}\) isosceles triangles inscribed in a circle of radius \(R_n\). If \(d_n\) is the length of the basis of one of this triangles and \(h_n\) its area, then \(F_n/L_n = h_n/d_n\). If \(\alpha_n = 2\pi/(3 \cdot 2^{n-1})\) is the opposite angle to \(d_n\) then

\[
d_n = 2 \text{arcsinh} \left( \sinh R_n \cdot \sin \left( \frac{\alpha_n}{2} \right) \right)
\]

and

\[
h_n = \pi - \left( \alpha_n + 2 \arctan \frac{1}{\tan \frac{\alpha_n}{2} \cdot \cosh R_n} \right).
\]

Taking \(R_n = n\) we have that \(\lim h_n/d_n = 0\). Taking \(R_n = \log(4/\mu \alpha_n)\) with \(\mu > 0\) we have that

\[
\lim \left( \tan \frac{\alpha_n}{2} \cdot \cosh R_n \right) = \lim \frac{\alpha_n}{2} \frac{2}{\mu \alpha_n} = \frac{1}{\mu},
\]

hence,

\[
\lim h_n = \pi - 2 \cdot \arctan \mu.
\]

In an analogous way

\[
\lim d_n = 2 \text{arcsinh} \frac{1}{\mu}.
\]

So we have that

\[
\lim \frac{F_n}{L_n} = \frac{\pi - 2 \arctan \mu}{2 \text{arcsinh} \frac{1}{\mu}}
\]

that takes, depending on the parameter \(\mu\), all values between 0 and 1.

It is interesting to calculate \(\lim D_n/L_n\) being \(D_n\) the diameter of these polygons.

\[
\lim \frac{D_n}{L_n} = \lim \frac{2R_n}{3 \cdot 2^n \text{arcsinh} (\sinh R_n \cdot \sin (\frac{\alpha_n}{2}))} = 0
\]
Using the sequences constructed in section 5, we can show in an alternative way that \( \lim F/L \) can take any value between 0 and 1. Indeed, let \( K_n = K(\lambda, r_n) \) with \( r_n = n, R_n = e^{2n} \) and \( 0 \leq \lambda \leq 1 \). If \( F_n \) and \( L_n \) are the area and the perimeter of \( K_n \) then, by the Gauss-Bonnet formula

\[
\lim \frac{F_n}{L_n} = \lim \frac{\int_{\partial K_n} k_g \, ds + \beta_n}{L_n} = \lim \frac{\int_{\partial K_n} k_g \, ds}{L_n} = \\
\lim \frac{\lambda(L_n - 4\alpha_n \sinh r_n) + \coth r_n 4\alpha_n \sinh r_n}{L_n} = \\
\lambda + \lim \frac{4\alpha_n \sinh r_n (\coth r_n - \lambda)}{D_n} \lim \frac{D_n}{L_n} = \lambda
\]

where \( \beta_n \) are the interior angles in \( \partial K_n \) and \( 2\alpha_n \) is the angle described by one of the arcs of circle in \( \partial K_n \).

It is interesting to remark that the sequence with \( \lim F/L = 0 \) is precisely the sequence with \( \lim D/L = 1/2 \). This seems not to be casual, \( D/L \) goes to \( 1/2 \) because the convex sets are “very thin” so it is not surprising that \( F/L \) goes to 0.

It seems natural to ask if a sequence of convex sets expanding over the whole plane with \( \lim D/L = 1/2 \) must have \( \lim F/L = 0 \). In the next sections we will look for bounds \( 0 \leq f(l) \leq g(l) \leq 1 \) such that if a sequence has \( \lim D/L = l \) then

\[
f(l) \leq \lim \frac{F_n}{L_n} \leq g(l).
\]

Taking into account the sequences of polygons used above , \( f(0) \) must be 0 and \( g(0) \) must be 1.

A first step, in order to find \( g \), is to find a bound for \( F \), the area of a convex set with fixed perimeter and diameter.

7. Extremal values of the area for a given perimeter and diameter

It is known that, given positive \( L_0 \), the compact domain with perimeter \( L_0 \) and maximum area is a circle. This is a consequence of the hyperbolic isoperimetric inequality (1). If we restrict to compact domains with diameter greater or equal than a given value \( D_0 \) and fixed perimeter \( L_0 < 2\pi \sinh D_0/2 \), circles are not allowed. Then we consider the problem of finding which of these domains has maximum area. As a first step we have

**Proposition 7.1.** If \( K \) is a compact domain with diameter greater or equal than \( D_0 \), fixed perimeter \( L_0 \) and maximum area, then \( K \) is convex.

**Proof.** Indeed, if \( K \) is not convex there must exist two boundary points \( x \) and \( y \) such that the geodesic segment \( s \) joining them is in \( K^c \). Let \( \gamma \) be the piece of \( \partial K \) between \( x \) and \( y \). Performing a reflection with respect to \( s \) of \( \gamma \) we can construct a new domain with perimeter \( L_0 \) and more area than \( K \). \( \square \)

Let \( \mathcal{C} \) be the set of all compact convex domains in the hyperbolic plane. If \( K \in \mathcal{C} \) we define its hyperbolic parallel convex sets at distance \( \epsilon \) as follows

\[
K^\epsilon = \{ p \in \mathbb{H}^2 \mid d(p, K) \leq \epsilon \}
\]

\[
K^{-\epsilon} = \{ p \in K \mid d(p, \partial K) \geq \epsilon \}.
\]

In \( \mathcal{C} \) we define the following distance:

\[
d(K_1, K_2) = \min \{ \epsilon > 0 \mid K_1^\epsilon \subset K_2 \text{ and } K_2^\epsilon \subset K_1 \}
\]

Now, \( \mathcal{C} \) is a metric space and we consider the induced metric topology. Distance \( d \) is the hyperbolic version of Hausdorff distance for convex sets in the euclidean plane (cf. for instance [4]). In fact, \( \mathcal{C} \) can be seen in the projective model as the
set of euclidean convex domains contained in the unit disk $\mathbb{D}$. If $d_e$ is the euclidean distance and $K_\varepsilon^e$, $K^{-\varepsilon}_e$ denote the euclidean parallel convex sets to $K$, we can define
\[ d_e(K_1, K_2) = \inf\{\varepsilon > 0 \mid (K_1)^e_\varepsilon \subset K_2 \text{ and } (K_2)^e_\varepsilon \subset K_1\} \]
where $K_1$ and $K_2$ are convex domains contained in $\mathbb{D}$. If $K$ is a convex subset of $\mathbb{D}$, the ball $B(K, d) = \{K' \in \mathbb{C} \mid d(K, K') \leq d\}$ contains the ball $B_e(K, d) = \{K' \in \mathbb{C} \mid d_e(K, K') \leq \varepsilon\}$ where
\[ \varepsilon = \inf\{d_e(\partial K, \partial K^d), d_e(\partial K, \partial K^{-d})\} \]
Indeed, if $K' \in B_e(K, d)$ then $K' \in B(K, d)$ since $K^{-d}_e \subset K^{-\varepsilon}_e \subset K' \subset K^e \subset K^d$.

Similarly, every euclidean ball contains a hyperbolic one. Therefore the topologies defined in $\mathbb{C}$ by $d$ and by $d_e$ are equivalent.

Let $B$ be the ball in $\mathbb{C}$ with radius $L_0$ and center the convex set containing only the origin. We are interested in convex domains belonging to $B$ because every convex domain with perimeter $L_0$ can be moved to be in $B$.

We have (cf. [4]) the following

**Theorem 2. [Blaschke Selection Theorem]** A bounded infinite family of euclidean convex sets has a sequence converging to some convex set.

**Corollary 7.1.** $B$ is compact.

**Proof.** In metric spaces a set $A$ is compact if and only if every infinite subset of $A$ contains an accumulation point. Since the euclidean and hyperbolic topologies are equivalent we can use theorem 2 to state that any infinite family in $B$ accumulates to a convex set. Since $B$ is closed we are done. □

**Proposition 7.2.** The diameter, $D$, perimeter, $L$, and area, $F$, functions are continuous over $\mathbb{C}$, the set of all compact convex domains in the hyperbolic plane with the Hausdorff topology.

**Proof.** Let $K \in \mathbb{C}$ and let $K_n$ be a sequence of convex domains such that $d(K_n, K)$ tends to $0$. By the definition of the distance between two convex sets
\[ K^{-d_n} \subset K_n \subset K^{d_n} \]
where $d_n = d(K_n, K)$. Therefore
\[ D(K) - 2d_n = D(K^{-d_n}) \leq D(K_n) \leq D(K_{d_n}) = D(K) + 2d_n \]
and $\lim D(K_n) = D(K)$. For the perimeter, using the hyperbolic Crofton formula we have
\[ \lim L(K_{\pm d_n}) = \lim \int_R \chi \{r \in R \mid r \cap K_{\pm d_n} \neq \emptyset\} dr = \int_R \lim \chi \{r \in R \mid r \cap K_{\pm d_n} \neq \emptyset\} dr = \int_R \chi \{r \in R \mid r \cap K \neq \emptyset\} dr = L(K) \]
where $R$ is the set of lines in $\mathbb{H}^2$ and $\chi$ is the characteristic function. Therefore $\lim L(K_n) = L(K)$. The continuity for the area follows analogously. □

Let $\mathcal{S} = \{K \in \mathcal{B} \mid D(K) \geq D \quad L(K) = L_0\} = D^{-1}(\{D_0, L_0/2\}) \cap L^{-1}(L_0) \cap \mathcal{B}$. $\mathcal{S}$ is a closed subset of $\mathcal{B}$ so it is compact. Then $F$ must have a maximum and a minimum over $\mathcal{S}$.

**Corollary 7.2.** In the set of hyperbolic convex domains with diameter bounded below by $D_0$ and fixed perimeter $L_0$ the area function attains its maximum value.
This proves the existence question. The uniqueness is discussed in the next theorem.

**Theorem 3.** Given $D_0$ and $L_0 < 2\pi \sinh D_0/2$, the compact convex domain with diameter greater or equal than $D_0$ and perimeter $L_0$ that maximizes the area is a \(\lambda\)-lens with diameter exactly $D_0$.

We need some previous results. Let $K$ be a convex set in \(\mathcal{B}\) maximizing the area. Let $c_1$ and $c_2$ be the endpoints of a diameter of $K$ and $\gamma(s)$ be the curve $\partial K$ parametrized by the arc.

First we see that $C^1$ points are locally equivalent.

**Lemma 7.1.** Let $p = \gamma(s)$ and $p' = \gamma(s')$ be two points different from $c_1$ and $c_2$. If $\gamma$ is $C^1$ in $p$ and $p'$ then there exists a rigid motion that moves a neighborhood of $p$ in $\gamma$ onto a neighborhood of $p'$.

**Proof.** Let $\gamma(s) = p$ and $\gamma(s') = p'$. Let $\epsilon$ be small enough to make $c_1, c_2 \notin \gamma([s-\epsilon, s + \epsilon])$ and $c_1, c_2 \notin \gamma([s', \epsilon, s' + \epsilon])$. For any $t \in (s, s + \epsilon)$ let $t' > s$ be the first one with $d(\gamma(s), \gamma(t)) = d(\gamma(s'), \gamma(t'))$. Swapping $\gamma([s, t])$ and $\gamma([s', t'])$ we obtain the border of a new domain $K'$ with the same area as $K$, the same perimeter and, perhaps, a greater diameter. The angle between $\gamma'(s)$ and $\gamma(s)\gamma(t)$ is equal to the angle between $\gamma'(s')$ and $\gamma(s')\gamma(t')$. Indeed, if one of these angles is greater than the other, in $\partial K'$ there would be an interior angle greater than $\pi$; contradicting the fact that $K'$ must be convex (cf. lemma 3.1)

So, in polar coordinates with center $p$ and direction $\gamma'(s)$, the curve $\gamma([s-\epsilon, s + \epsilon])$ has the same expression as $\gamma([s' - \epsilon, s' + \epsilon])$, in polar coordinates with center $p'$ and direction $\gamma'(s')$. If $g$ is the motion that moves $p$ on $p'$ and $\gamma'(s)$ on $\gamma'(s')$, then $g$ moves the neighborhood of $p$ onto the neighborhood of $p'$.

**Lemma 7.2.** $\gamma$ is of class $C^1$ except in $c_1$ and $c_2$.

**Proof.** Since $K$ is convex, $\partial K$ must be $C^1$ except from, at most, in a countable set of points. Let $\gamma(s) = x \neq c_1, c_2$ be one of these points. Let $(s_n)$ be a sequence such that $\lim s_n = s$ and $x_n = \gamma(s_n)$ are $C^1$ points. Let $\epsilon$ be such that $c_1, c_2 \notin U_n = \gamma([s_n - \epsilon, s_n + \epsilon])$ for any $n$. $U_n$ are not geodesic segments so $U_0$ must contain three non aligned points $p_1 = \gamma(s_0 + t_1)$, $p_2 = \gamma(s_0 + t_2)$ and $p_3 = \gamma(s_0 + t_3)$. Let $g_n$ be the motion that moves $U_0$ onto $U_n$. The group of rigid motions in \(\mathbb{H}^2\) can be identified with the set of triangles congruent to $p_1p_2p_3$. Since

$$\lim g_n(\gamma(s_0 + t_i)) = \lim \gamma(s_n + t_i) = \gamma(s + t_i), \quad i = 1, 2, 3$$

the sequence $(g_n)$ converges to a motion $g$ and for every $|t| < \epsilon$

$$g(\gamma(s_0 + t)) = (\lim g_n)(\gamma(s_0 + t)) = \lim \gamma(s_n + t) = \gamma(s + t).$$
So, \( g \) moves a neighborhood of \( x_0 \) (in \( \gamma \)) onto a neighborhood of \( x \). This contradicts the fact that, in \( x, \gamma \) is not \( C^1 \).

Now we can afford the

Proof of theorem 3. Let \( p = \gamma(s) \) and \( p' = \gamma(s') \) be two border points of \( K \) such that neither \( c_1 \) nor \( c_2 \) are in \( \gamma([s, s']) \). Let \( r \) and \( r' \) be the lines through \( p \) and \( p' \), respectively, orthogonal to \( \gamma \) in these points. Two cases are possible, \( r \) intersects \( r' \) or not.

If \( r \) and \( r' \) intersect in \( o \), let \( q = \gamma(t) \) be the point in \( \gamma([s, s']) \) such that \( d(p, q) = d(p', q) \) (see figure 7). The argument used in the proof of lemma 7.1 implies that the angles \( \langle pq, \gamma(s) \rangle, \langle \gamma(t), pq \rangle, \langle qp', \gamma'(t) \rangle \) and \( \langle \gamma'(s'), qp' \rangle \) must be equal. Then the triangles \( opq \) and \( op'q \) are isosceles with \( d(o, p) = d(o, q) = d(o, p') \). This argument could be repeated starting with \( p \) and \( q \) or with \( q \) and \( p' \). Repeating it indefinitely we get a dense subset \( \Omega \subset \gamma([s, s']) \) at a constant distance from \( o \). By continuity of the distance, \( \gamma([s, s']) \) must be an arc of circle with center \( o \).

If \( r \) and \( r' \) are nonsecant they have a common perpendicular line \( s \). In an analogous way we can see that \( \gamma([s, s']) \) must be a piece of an equidistant of \( s \) (see figure 7). Anyway, the curve between \( p \) and \( p' \) is \( C^2 \) and has constant curvature. Then, each piece of \( \partial K/\{c_1, c_2\} \) must be \( C^2 \) with constant curvature. Lemma 7.1 implies that the curvature must be the same in the two pieces.

For every pair \( (D_0, L_0) \) there exists a unique convex set with diameter \( D_0 \) and perimeter \( L_0 \) that maximizes the area and it is a \( \lambda \)-lens. It is important to remark that the value of this \( \lambda \) is a function on \( D_0 \) and \( L_0 \) and that

\[
\lambda < 1 \quad \text{if} \quad 2D_0 \leq L_0 < 4 \sinh \frac{D_0}{2}
\]

\[
\lambda \geq 1 \quad \text{if} \quad 4 \sinh \frac{D_0}{2} \leq L_0 \leq 2\pi \sinh \frac{D_0}{2}.
\]

Notice that in the first case the boundary lines are \( \lambda \)-geodesic segments and in the second case they are arcs of circumference.

Now we can treat the problem of finding the bounds in formula (8).

8. Upper and lower bounds for \( \lim F/L \) with respect \( \lim D/L \)

As usual, let \( (K_n) \) be a sequence of convex sets expanding over the whole hyperbolic plane. Let \( F_n, L_n \) and \( D_n \) be, respectively, the area, perimeter and diameter of \( K_n \). Let us suppose that \( \lim D_n/L_n = t \neq 0 \). The domain with diameter \( D_n \) and perimeter \( L_n \) of maximum area is a \( \lambda_n \)-lens with area \( F(D_n, L_n) \) and interior
angles $\beta_n$. In this case
\[
\lim \frac{F_n}{L_n} \leq \lim \frac{F(D_n, L_n)}{L_n} = \lim \frac{\lambda_n \cdot L_n - 2\beta_n}{L_n} = \lim \lambda_n.
\]
Notice that we can suppose, for $n$ big enough, $\lambda_n < 1$ and $\lim \lambda_n = 1$. Indeed, if $\liminf \lambda_n \geq 1$ then $L_n \geq 4\sinh D_n/2$ for $n$ arbitrarily big. Therefore $\lim D_n/L_n$ must be 0 and in this case all values of $\lim F_n/L_n$ between 0 and 1 can be attained (see section 6).

Using (2)
\[
\lim \frac{L_n}{D_n} = \lim \frac{4}{\sqrt{1 - \lambda_n^2}D_n} \cdot \text{arcsinh} \left( \sqrt{1 - \lambda_n^2} \sinh \frac{D_n}{2} \right) = \lim \frac{4}{\sqrt{1 - \lambda_n^2}D_n} \cdot \log(\sinh \frac{D_n}{2}) = \lim \frac{2}{\sqrt{1 - \lambda_n^2}}.
\]
So $\lim \lambda_n$ exists and its value is $\sqrt{1 - (2l)^2}$. We can state

**Theorem 4.** Let $(K_n)$ be a sequence of convex sets expanding over the whole hyperbolic plane. Let $F_n$, $L_n$ and $D_n$ denote their area, perimeter and diameter. If $\lim D_n/L_n = l$ then
\[
\lim \frac{F_n}{L_n} \leq \sqrt{1 - (2l)^2}.
\]

Sequences constructed in section 5 show that (9) could not be better. Taking $K_n = K_\lambda(e^{2n}, n)$ we know that
\[
\lim \left( \frac{F_n}{L_n} \right)^2 + \lim \left( \frac{2D_n}{L_n} \right)^2 = \lambda^2 + (1 - \lambda^2) = 1.
\]

The following proposition shows that the only lower bound for the $\lim F_n/L_n$ is $f(l) \equiv 0$.

**Proposition 8.1.** For every $0 \leq l \leq 1/2$ and every $0 \leq \lambda \leq \sqrt{1 - (2l)^2}$ there exists a sequence of convex sets expanding over the whole hyperbolic plane with $\lim D_n/L_n = l$ and $\lim F_n/L_n = \lambda$.

**Proof.** Let $K_n$ be the regular polygon with $3 \cdot 2^{n-1}$ sides inscribed in a circle of radius $n$. Let $K_n'$ be the polygon $K_n$ with two isosceles triangles of height $k \cdot 2^n$ ($k > 0$) attached in two opposite sides of $K_n$. Now, let $K_\lambda^n$ be the domain bounded by the exterior $\lambda$-geodesic segments corresponding to each side of $K_n'$ (see figure 8). Let $F_n$, $L_n$ and $D_n$ be the area, perimeter and diameter of $K_\lambda^n$ respectively. Let $l_n$ be the length of each side of $K_n$. Let $d_n$ be the length of the equal sides in
the attached triangles. We denote $l'_n$ and $d'_n$ the lengths of the $\lambda$-geodesic segments corresponding to $l_n$ and $d_n$ respectively. Let $\gamma_n$ and $\delta_n$ be the angles between $l_n$ and $l'_n$ and between $d_n$ and $d'_n$ respectively. Let $\beta_n$ be the half part of the interior angles in $\partial K_n$ and let $\tau_n$ be the value of the two equal angles in each attached triangle.

The domains $K_n^{\lambda}$ are convex for $n$ big enough. Indeed, $\theta_n$ and $\phi_n$ go to 0 and, taking (4) into account, $\theta_n$ and $\phi_n$ go to $\arccos \lambda < \pi/2$. Therefore, for $n$ big enough, the interior angles of $\partial K_n^{\lambda}$ are not greater than $\pi$ and the $K_n^{\lambda}$ are convex domains.

Using hyperbolic trigonometry,

$$l_n = 2 \text{arcsinh} \left( \sinh n \sin \left( \frac{\pi}{3 \cdot 2^{n-1}} \right) \right) \sim 2 n \log(e/2)$$

and

$$d_n = \text{arccosh} \left( \cosh l_n \cosh(k2^n n) \right) \sim k2^n n$$

when $n$ goes to infinity. Using (2) we have

$$l'_n \sim \frac{2n \log(e/2)}{\sqrt{1 - \lambda^2}} \quad d'_n \sim \frac{k2^n n}{\sqrt{1 - \lambda^2}}.$$  

Since $D_n - 2k2^n n < 2n$, $D_n \sim 2k2^n n$. Therefore

$$\lim_{n \to \infty} \frac{L_n}{D_n} = \lim_{n \to \infty} \frac{(3 \cdot 2^{n-1} - 2)l'_n + 4d'_n}{2k2^n n} = \frac{1}{\sqrt{1 - \lambda^2}} \left( \frac{3 \log(e/2)}{2k} + 2 \right)$$

which takes, depending on $k$ all the values between $2/\sqrt{1 - \lambda^2}$ and infinity.

Finally, using the Gauss-Bonnet formula

$$\lim_{n \to \infty} \frac{F_n}{L_n} = \lim_{n \to \infty} \frac{\lambda L_n + \sum \text{angles} - 2\pi}{L_n} = \lambda + \frac{(3 \cdot 2^{n-1} - n)(\pi - 2(\gamma_n + \beta_n))}{L_n} = \lambda.$$  

Notice that this computations give only the cases $0 < l < 1/2$ and $\lambda < \sqrt{1 - (2l)^2}$. We can use the examples given in section 5 for $l = 1/2$ and $\lambda = \sqrt{1 - (2l)^2}$ and those in section 6 for $l = 0$

So, we have seen that the upper bound $g(l)$ for $\lim F/L$ with respect $l = \lim D/L$ is the function $\sqrt{1 - (2l)^2}$. This value and all the lower ones are attained for every $l$ between 0 and 1/2 (see figure 9).
References


Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
E-mail address: egallego@mat.uab.es, solanes@mat.uab.es