Convexity in hyperbolic spaces

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joint work with

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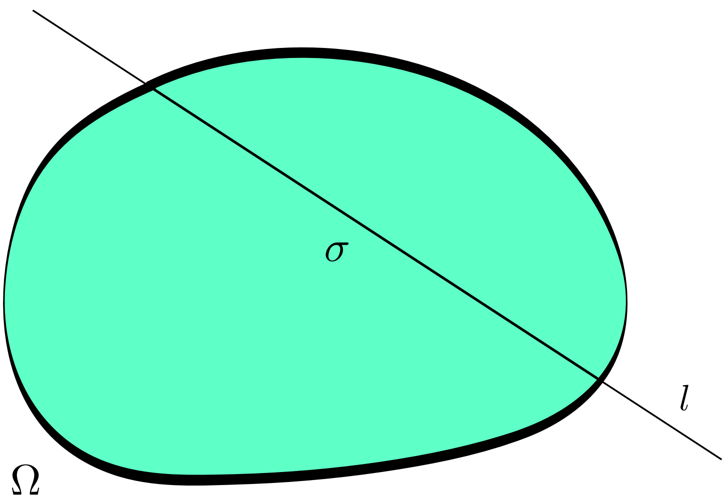
Introduction: euclidean plane

- Given a compact convex domain Ω in the euclidean plane and a random line l, the expected value of the length σ of the chord $l \cap \Omega$ is

$$E(\sigma) = \pi \frac{F}{L}$$

where F and L are the area and perimeter of Ω

- When Ω tends to cover the plane we have that $E(\sigma)$ tends to ∞



Introduction: hyperbolic plane

- Given a compact convex domain Ω in the hyperbolic plane and a random line l, the expected value of the length σ of the chord $l \cap \Omega$ is again

$$E(\sigma) = \pi \frac{F}{L}$$

  where F and L are the area and perimeter of Ω

- When Ω tends to cover the hyperbolic plane we don't have necessarily that $E(\sigma)$ tends to infinity

In each case we consider a rigid motion invariant density for geodesic lines.

Problem: *given a sequence Ω_n of compact convex domains expanding over the whole hyperbolic plane, find the possible values of*

$$\lim_n \frac{\mathit{area}(\Omega_n)}{\mathit{perimeter}(\Omega_n)}.$$

Introduction: hyperbolic plane

Consider the following curves in $\mathbb{H}^2$:

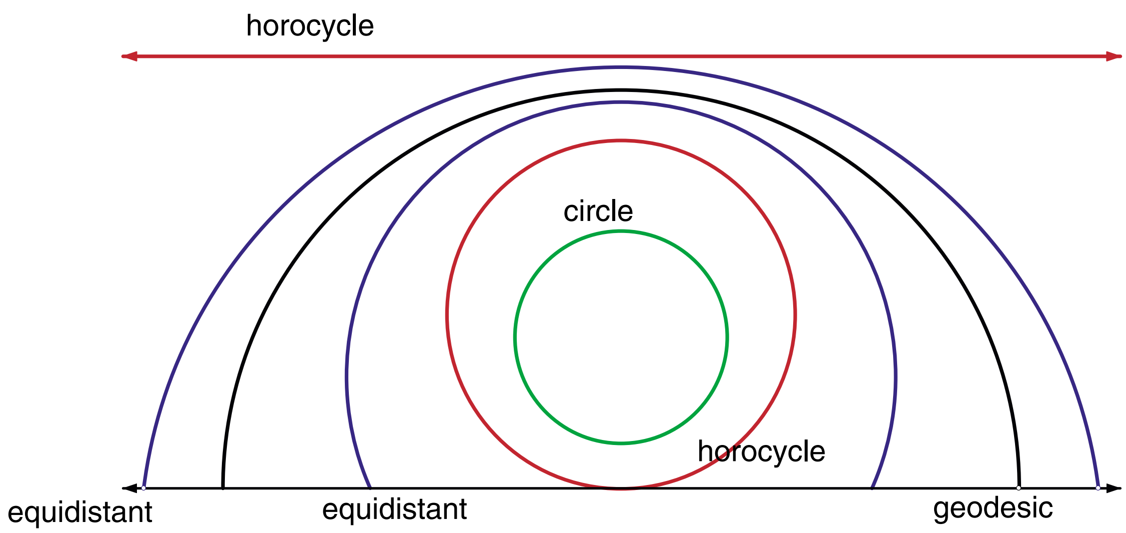
1. *Geodesics.* They have geodesic curvature equal to 0

2. *Horocycles.* Curves orthogonal to a pencil of parallel lines. They have geodesic curvature ± 1.

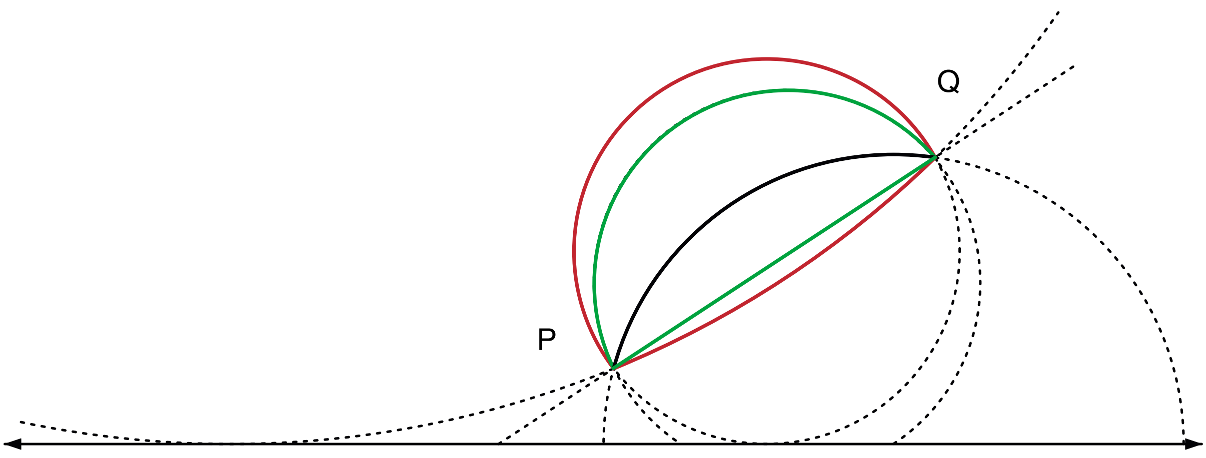
3. *Equidistants* or *λ-geodesics.* They are curves equidistant to geodesics. They have *absolute* geodesic curvature $\lambda \in (0, 1)$.

When $\lambda = 0$ we have geodesics, for $\lambda = 1$ horocycles.

Introduction: hyperbolic plane



Different curves in the hyperbolic plane



Special curves passing through two points P and Q

Introduction: hyperbolic plane

Definition. *Given* λ *in* $[0,1]$*, a set* Ω *in* $\mathbb{H}^2$ *is* λ-convex *when for every* $P, Q \in \Omega$ *the* λ*-geodesics joining them are contained in* Ω*.*

- 0-convex sets are ordinary convex sets

- 1-convex sets are also called h-convex sets or convex by horocycles

Using Gauss-Bonnet formula and isoperimetric formula $L^2 - 4\pi F - F^2 \geq 0$ it is true (Santaló-Yañez, 1972) that for every sequence Ω_n of *h-convex* sets expanding over the whole hyperbolic plane

$$\lim_n \frac{\text{area}(\Omega_n)}{\text{perimeter}(\Omega_n)} = 1$$

Introduction: hyperbolic plane

- For convex sets expanding over the whole hyperbolic plane it was proved (Gallego-Reventós, 85) that

$$
\begin{aligned}
0 &\leq \liminf_n \frac{\text{area}(\Omega_n)}{\text{perimeter}(\Omega_n)} \\
&\leq \limsup_n \frac{\text{area}(\Omega_n)}{\text{perimeter}(\Omega_n)} \leq 1
\end{aligned}
$$

  and *it is possible to find examples of sequences having as limit all the possible values between* 0 *and* 1*.*

  How the boundary bends has influence in the possible limit:

- For λ-convex sets expanding over the whole hyperbolic plane it is true (Gallego-Reventós, 99) that the above limit lies between λ and 1 and *it is possible to find examples of sequences having as limit all the possible values between* λ *and* 1*.*

Introduction: higher dimensions

- For $\mathbb{H}^{n+1}$ it was proved (Borisenko-Miquel, 99) for sequences of h-convex sets expanding over the whole hyperbolic space that

$$\lim_n \frac{\text{vol}(\Omega_n)}{\text{vol}(\partial\Omega_n)} = \frac{1}{n}$$

- For $\mathbb{H}^{n+1}$ it was proved (Borisenko-Vlasenko, 99) for sequences of λ-convex sets expanding over the whole hyperbolic space that

$$\begin{aligned} \frac{\lambda}{n} &\leq \liminf_n \frac{\text{vol}(\Omega_n)}{\text{vol}(\partial\Omega_n)} \\ &\leq \limsup_n \frac{\text{vol}(\Omega_n)}{\text{vol}(\partial\Omega_n)} \leq \frac{1}{n} \end{aligned}$$

Definitions

Definition. *A Hadamard manifold is a simply connected complete riemannian manifold with non-positive sectional curvature* K

We shall consider Hadamard manifolds such that K satisfies $-k_2^2 \leq K \leq -k_1^2 < 0$.

Definition. *A domain* Ω *with* C^2 *boundary is a* λ*-convex domain if normal curvatures with respect the interior normal are greater or equal than* λ *in every tangent direction of the boundary*

Definition. *An horosphere is the limit of a geodesic sphere when a point* P*is fixed and radius goes to* ∞ *following a geodesic through* P*.*

Definition. *A domain* Ω *is* h*-convex if every point in the boundary has a locally supporting horosphere.*

Definitions: remarks

- In the non-regular case we say that a domain Ω is λ-convex if for every point in the boundary there is a regular λ-convex hypersurface locally supporting Ω.

- Every λ-convex domain is ordinary convex.

- For geodesic spheres of radius r:

$$k_1 \coth(k_1 r) \leq k_n \leq k_2 \coth(k_2 r).$$

  Then for every $\lambda \leq k_1$ spheres are λ-convex.

- If Ω is λ-convex with $\lambda > k_2$ then the inner radius r must be less than

$$\frac{1}{k_2}\text{arctanh}\left(\frac{k_2}{\lambda}\right).$$

  There are no λ-convex domains with $\lambda > k_2$ and arbitrary inner radius. λ-convex domains with $\lambda > k_2$ are h-convex.

- Horospheres have normal curvature between k_1 and k_2.

Statement of the problem

Let Ω be a convex compact domain in M a Hadamard manifold with $-k_2^2 \leq K \leq -k_1^2 < 0$.

$$\begin{aligned} \text{vol}(\Omega) &= \int_\Omega \eta \\ \text{vol}(\partial\Omega) &= \int_{\partial\Omega} i_{\mathbf{n}}\eta \end{aligned}$$

where $\mathbf{n}$ is the outward normal and η the volume element of M.

This can be written as

$$\begin{aligned} \text{vol}(\Omega) &= \int_{S^n}\int_0^{l(u)} J_u(t)t^n dt dS \\ \text{vol}(\partial\Omega) &= \int_{S^n} \frac{J_u(l(u))l(u)^n}{< \partial_t, \mathbf{n} >} dS \end{aligned}$$

Where

- $J_u(t)$ is the jacobian of $\exp_O$ in the point tu for $u \in S^n \simeq (T_O M)^1$

- ∂_t is the radial direction in $\exp_O(l(u)u)$

Statement of the problem

Consider

$$g(u) = \int_0^{l(u)} \frac{J_u(t)t^n}{J_u(l(u))l(u)^n} \mathrm{d}t.$$

As geodesics have no conjugate points, comparing the jacobian of M with jacobians of spaces of constant curvature we have

Lemma. *If r and R are the inradius and the circumradius,*

$$f(r) \leq g(u) \leq h(R)$$

with $f(r) \to 1/nk_2$ *and* $h(R) \to 1/nk_1$.

Then

$$f(r)\alpha \leq \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(\partial\Omega)} \leq h(R)\alpha$$

where

$$\alpha = \frac{\int_{S^n} J_u(l(u))l(u)^n dS}{\int_{S^n} \frac{J_u(l(u))l(u)^n}{\cos\varphi} dS}$$

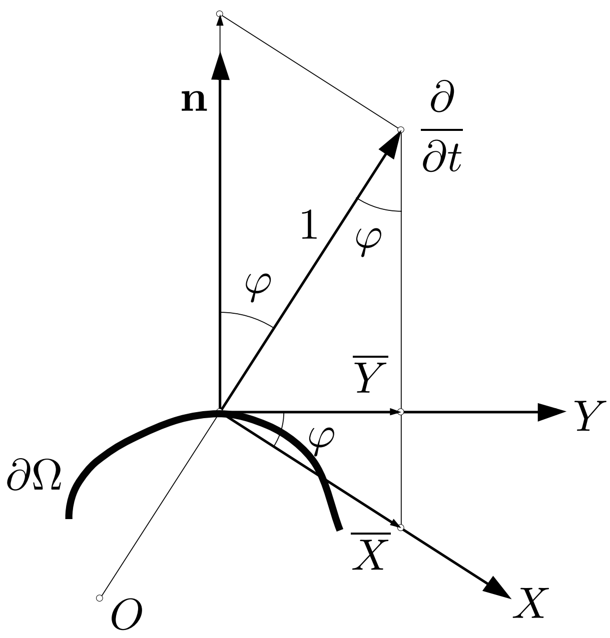
Problem. *We need a bound* $\cos\varphi \geq C(r)$.

Fundamental lemma

Lemma. *Let N be the boundary of a λ-*convex *domain Ω defined by $t = \rho(\theta)$ with ρ the distance to the interior point O. If k_n is the normal curvature in the direction of the gradient of ρ and μ_n the normal curvature of the geodesic sphere of radius $\rho(\theta)$ in the direction X (see figure) we have*

$$k_n = \mu_n \cos\varphi + \frac{d\varphi}{ds}$$

with s the arc parameter of the integral curves of $Y = \mathrm{grad}(\rho)$.



Remark. *This is some kind of Liouville formula.*

Bound for $\cos\varphi$

Now, as a consequence of the previous lemma and comparing with the constant curvature case we can prove a bound for $\cos\varphi$.

Proposition. *Let* Ω *be a* λ*-convex as above and* $\lambda < k_2$*.*

- *When* $d(O,N) \leq \frac{1}{k_2}\text{arctanh}(\frac{\lambda}{k_2})$ *we have*

$$\cos\varphi \geq \frac{1}{k_2}\sqrt{\lambda^2 \cosh^2 k_2 s - k_2^2 \sinh^2 k_2 s}.$$

where $s = \frac{1}{k_2}\text{arctanh}(\frac{\lambda}{k_2}) - d(O,N)$.

- *When* $d(O,N) \geq \frac{1}{k_2}\text{arctanh}(\frac{\lambda}{k_2})$ *we have*

$$\cos\varphi \geq \frac{\lambda}{k_2}.$$

The quotient of volumes in the general case

Finally, using the bound for $\cos\varphi$ we obtain

Theorem. *Let M be a $(n+1)$-dimensional Hadamard manifold with sectional curvature K such that*

$$-k_2^2 \leq K \leq -k_1^2 \qquad k_1, k_2 > 0.$$

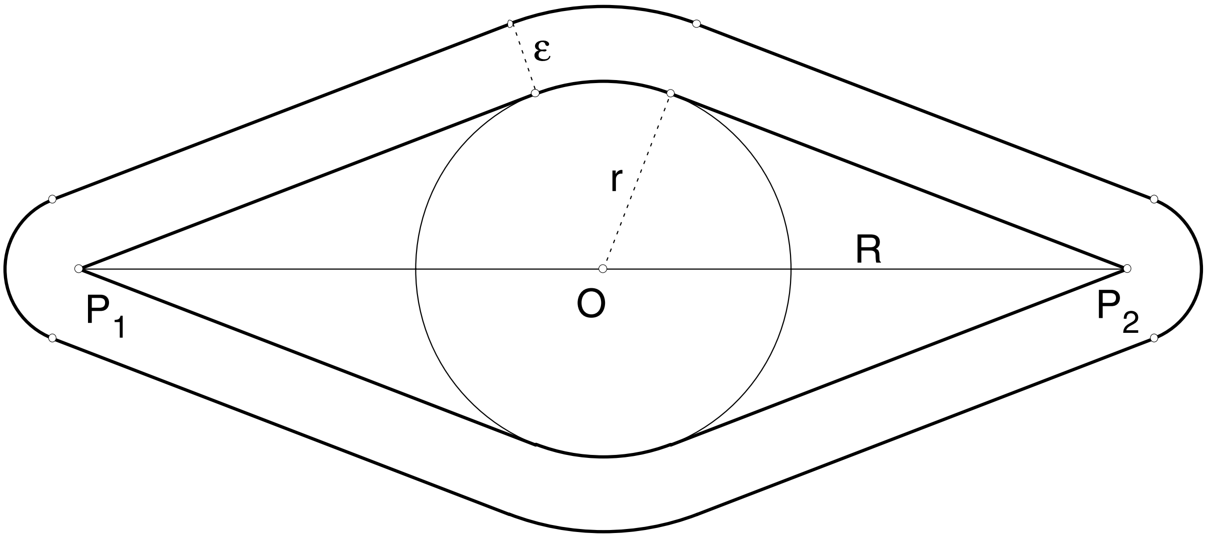
Let $\{\Omega(t)\}_{t\in\mathbb{R}^+}$ be a family of λ-convex compact domains expanding over the whole space. Then, if $\lambda \leq k_2$

$$\begin{aligned}\frac{\lambda}{nk_2^2} &\leq \liminf_t \frac{vol(\Omega_n)}{vol(\partial\Omega_n)} \\ &\leq \limsup_t \frac{vol(\Omega_n)}{vol(\partial\Omega_n)} \leq \frac{1}{nk_1}\end{aligned}$$

When $\lambda \geq k_2$ the limits take values between $1/nk_2$ and $1/nk_1$.

Some examples in $\mathbb{H}^{n+1}$

Consider a geodesic ball with radius $r > 0$ and center in a fixed point $O \in \mathbb{H}^{n+1}$. Let P_1 and P_2 be two points defining a geodesic segment of length $2R > r$ such that O is the midpoint. The convex hull of the ball $B_O(r)$ and the points P_1, P_2 will be denoted $K(R, r)$. Let $K_\epsilon(R, r)$ be the set of the points at a distance from $K(R, r)$ smaller than ϵ. It is a λ-convex set for $\lambda = \tanh \epsilon$.



Putting $R = \exp(2r)$ it can be shown (Gallego-Reventós-Solanes, 2000) that

$$\lim_{r\to\infty} \frac{\mathrm{vol}(K_r)}{\mathrm{vol}\partial(K_r)} = \frac{\tanh \epsilon}{n} = \frac{\lambda}{n}.$$

Note that the value $1/n$ can be obtained considering a sequence of balls.

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