A KINEMATIC FORMULA FOR THE TOTAL ABSOLUTE CURVATURE OF INTERSECTIONS

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Abstract. Given two surfaces in three dimensional euclidean space, one fixed and the other moved by rigid motions, we consider the total absolute curvature of the intersection curves. In this paper we investigate the integral of these total absolute curvatures over all motions. Under some geometric conditions we obtain kinematic formulas, and with weaker conditions we get upper and lower bounds. Finally, as applications, we obtain upper bounds for the average number of connected components of the intersections, and we give Hadwiger conditions for a convex domain to be able to contain another one.

1. Introduction.

Kinematic formulas play a central role in Integral Geometry. For instance let S and S' be two surfaces in $\mathbb{R}^3$. Let S' be moved by all rigid motions g and consider a geometric quantity $I(S \cap gS')$ of the intersection. Then kinematic formulas express the integral of $I(S \cap gS')$, with respect to the invariant measure dg of the group G of motions, in terms of the geometry of S and S'. As a concrete example take the length $\ell(S \cap gS')$ of the intersection curve as $I(S \cap gS')$. Then the well-known Poincaré formula (cf. [San76]) states

$$\int_G \ell(S \cap gS') dg = 2\pi^3 A A', \tag{1}$$

where A and A' are the areas of S and S'.

C.-S. Chen considered in [Che73] the integral of the total square curvature of the intersection curve as $I(S \cap gS')$, and he obtained

$$\int_G \int_{S \cap gS'} \kappa^2(s) ds\ dg = 2\pi^3 \left(A' \int_S (3H^2 - K) dS + A \int_{S'} (3H'^2 - K') dS' \right), \tag{2}$$

where H and K are the mean curvature and the Gauss curvature of S, and similarly for S'.

Formulas (1) and (2) can be seen as particular instances of a very general kinematic formula obtained by R. Howard in [How93]. There he solves the case where I is the integral of any invariant polynomial of the second fundamental form of the intersection. The square curvature κ^2 is such a polynomial but the curvature κ is not. Therefore a kinematic formula for the total absolute curvature is not covered by R. Howard's result. In fact, C.-S. Chen [Che73] already mentioned that this problem is more involved.

In this paper we study precisely this case where $I(S \cap gS')$ is the total absolute curvature of the intersection curve. Under some geometric hypothesis (such as S, S' are strictly convex

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and S can roll freely inside S') we have the following kinematic formula (see Proposition 1)

$$I_{S,S'} := \int_G \int_{\gamma_g} \kappa(s)\ dsdg = \frac{4\pi^2}{3} \int_S (3H^2 - K) dS \int_{S'} \frac{dS'}{\sqrt{K'}} + 8\pi^2 A \int_{S'} H' dS'.$$

Under weaker assumptions we can state inequalities, namely upper and lower bounds for $I_{S,S'}$, see (4), (10) and (14).

As an application, we get upper bounds (15) for the average number of connected components of $S \cap gS'$. Also, in the style of Hadwiger, we give sufficient conditions for a convex domain to contain another one in three dimensional euclidean space.

We remark that kinematic formulas for powers of κ higher than 2 do not exist, because the integrals diverge, as can be seen from (7).

In higher dimensions one can consider the total absolute Lipschitz-Killing curvature of intersections of say two hypersurfaces, but corresponding kinematic formulas seem to be more complicated than ours in this paper.

2. A kinematic formula and upper bounds.

Let S and S' be compact smooth oriented regular surfaces, maybe with boundaries, in $\mathbb{R}^3$. We move S' by the elements g of the group G of orientation preserving rigid motions in $\mathbb{R}^3$ while keeping S fixed. If the intersection of S and gS' is non-empty, then $S \cap gS'$ is generically a regular curve γ_g (maybe non-connected) in $\mathbb{R}^3$. The total absolute curvature of γ_g is

$$\int_{\gamma_g} \kappa(s)\ ds$$

where $\kappa(s)$ is the curvature of γ_g in $\mathbb{R}^3$ and s is the arc-length parameter. Note that from the definition of the curvature of space curves we always have $\kappa(s) \geq 0$.

We are interested in kinematic formulas involving these total absolute curvatures, namely we look for the value of

$$I_{S,S'} = \int_G \int_{\gamma_g} \kappa(s)\ dsdg \tag{3}$$

where dg is the kinematic density (i.e. the Haar measure) of G (see for instance [San76]). If $S \cap gS'$ is empty, then we set the inner integral to zero.

Remark 1. The integral $I_{S,S'}$ is convergent as can be seen from the following argument. As $\kappa \leq \kappa^2 + 1$, we have

$$I_{S,S'} \leq \int_G \int_{\gamma_g} \kappa^2(s) dsdg + \int_G \int_{\gamma_g} dsdg.$$

The convergence of the first integral was established by Chen (see (2)) and the existence of the second one follows from Poincaré's formula (see (1)).

Proposition 1. *Let S and S' be compact smooth oriented regular surfaces in $\mathbb{R}^3$. Let $k'_{\min}$ denote the minimum of the normal curvatures k'_n of S'.*

(i) If $k'_{\min} > 0$ and $k_n \geq -k'_{\min}$ for all normal curvatures k_n of S, then we have the following inequality

$$I_{S,S'} \leq \frac{4\pi^2}{3} \int_S (3H^2 - K) dS \int_{S'} \frac{dS'}{\sqrt{K'}} + 8\pi^2 A \int_{S'} H' dS' \tag{4}$$

where H, H' are the mean curvatures, K, K' are the Gauss curvatures, dS and dS' are the area elements of the surfaces, and A is the area of S.

(ii) If $k'_{\min} > 0$ and $-k'_{\min} \leq k_n \leq k'_{\min}$ for all normal curvatures k_n of S, then we have

$$I_{S,S'} = \frac{4\pi^2}{3} \int_S (3H^2 - K) dS \int_{S'} \frac{dS'}{\sqrt{K'}} + 8\pi^2 A \int_{S'} H' dS'. \tag{5}$$

Remark 2. The assumption $k'_{\min} > 0$ means that S' is strictly convex. In case of S and S' strictly convex the conditions in (i) are fulfilled.

If S, S' are strictly convex, the condition in (ii) says $k_{\max} \leq k'_{\min}$. If moreover they are closed, then by the Blaschke rolling theorem S' can roll freely inside S (see [Bla49]).

Proof. Given $g \in G$ and a point $p \in S \cap gS'$, let $q = g^{-1}(p) \in S'$. Let $u \in T^1_p S$ be the unit tangent vector of γ_g at p, and let $v \in T^1_q S'$ be the associated unit vector with $g_*(v) = u$. Let φ denote the angle between the inner normals of S and gS' at p. Then the kinematic density fulfills (see formula (15.35) in [San76]):

$$ds\ dg = \sin^2(\varphi) d\varphi\ dS_p du\ dS'_q dv.$$

From this we have

$$I_{S,S'} = \int_{T^1 S'} \int_{T^1 S} \int_0^\pi \kappa(s) \sin^2(\varphi) d\varphi\ dS_p du\ dS'_q dv. \tag{6}$$

Now we compute κ which is the norm of $d^2\gamma_g/ds^2$. We try

$$\frac{d^2\gamma_g}{ds^2} = \alpha N + \beta N'$$

where N and N' are the inner normals of S and gS' at p and α, β unknown coefficients. In order to compute α and β we note that

$$\left\langle \frac{d^2\gamma_g}{ds^2}, N \right\rangle = k_n \quad \text{and} \quad \left\langle \frac{d^2\gamma_g}{ds^2}, N' \right\rangle = k'_n$$

where k_n and k'_n are the normal curvatures of S and gS' at p in direction u. This yields the expression

$$\frac{d^2\gamma_g}{ds^2} = \frac{(k_n - k'_n \cos\varphi) N + (k'_n - k_n \cos\varphi) N'}{\sin^2\varphi}. \tag{7}$$

From (6) and (7) we get

$$I_{S,S'} = \int_{T^1 S} \int_{T^1 S'} \left(\int_0^\pi \sin\varphi \sqrt{k_n^2 + k_n'^2 - 2k_n k'_n \cos\varphi}\ d\varphi \right) dS_p du\ dS'_q dv. \tag{8}$$

The inner integral of (8) gives

$$J := \int_0^\pi \sin\varphi \sqrt{k_n^2 + k_n'^2 - 2k_n k'_n \cos\varphi}\ d\varphi = \frac{|k_n + k'_n|^3 - |k_n - k'_n|^3}{3k_n k'_n}.$$

In particular, by the assumption we get

$$J \leq \frac{(k_n + k'_n)^3 - (k'_n - k_n)^3}{3k_n k'_n} = 2k'_n + \frac{2k_n^2}{3k'_n}. \tag{9}$$

In order to integrate (9) we first take

$$\int_{T^1 S} \int_{T^1 S'} k'_n dS'_q dv\ dS_p du = 2\pi A \int_{S'} \int_0^{2\pi} k'_n(v) dv dS'_q = 4\pi^2 A \int_{S'} H' dS'.$$

On the other hand we compute

$$\int_{T^1S'}\int_{T^1S}\frac{k_n^2}{k_n'}dS_p du\ dS_q' dv = \left(\int_{T^1S} k_n^2 dS_p du\right)\cdot\left(\int_{T^1S'}\frac{1}{k_n'}dS_q' dv\right).$$

For the first factor we use the Euler formula, and we get

$$\int_{T^1S} k_n^2 dS_p du = \int_S\int_0^{2\pi}(\kappa_1\cos^2\theta+\kappa_2\sin^2\theta)^2 d\theta dS_p = \int_S(3\pi H^2-\pi K)dS.$$

For the second factor, taking into account the strong convexity of S', the integral converges and gives

$$\int_{T^1S'}\frac{1}{k_n'}dS_q' dv = 2\pi\int_{S'}\frac{dS'}{\sqrt{K}}.$$

Putting all these integrals into (9) and (8) gives the inequality (4).

With the additional assumption of (ii) we have $k_n \le k_n'$ everywhere and therefore equality in (9). This proves formula (5). □

Corollary 1. *If S, S' are strictly convex, then we have the following inequality*

$$I_{S,S'} \le \frac{4\pi^2}{3}\int_S(3H^2-K)dS\int_{S'}\frac{dS'}{\sqrt{K'}}+8\pi^2 A\int_{S'}H'dS' + \frac{4\pi^2}{3}\int_{S'}(3H'^2-K')dS'\int_S\frac{dS}{\sqrt{K}}+8\pi^2A'\int_S HdS. \tag{10}$$

The equality holds if and only if S and S' are parts of spheres of the same radius.

Proof. From Proposition 1, changing the roles of S and S', we get two inequalities. By taking the arithmetic mean of both we get (10). Finally, the equality case occurs exactly when the normal curvatures of S and S' are constant and coincide. Hence S and S' are umbilical and therefore parts of spheres of the same radius. □

Remark 3. One can improve inequality (10) by using the geometric mean instead of the arithmetic mean.

Remark 4. Let S be strictly convex and let S' be congruent to λS with $\lambda \in \mathbb{R}_+$. In particular, when $\lambda = 1$, S' is congruent to S. Then the inequality (4) becomes

$$I_{S,S'} = I_{S,\lambda S} \le \frac{4\pi^2\lambda^3}{3}\int_S(3H^2-K)dS\int_S\frac{dS}{\sqrt{K}}+8\pi^2\lambda A\int_S HdS. \tag{11}$$

Specially, if the maximum of the normal curvatures $k_{\max}$ and the minimum of the normal curvatures $k_{\min}$ of S fulfill $\lambda \le k_{\min}/k_{\max}$ then formula (5) gives equality in (11).

Remark 5. Let S be a sphere of radius R. Then the right-hand side of (4) is a polynomial of degree 2 in R.

Specially, for $R \ge 1/k_{\min}$ the equality (5) becomes

$$I_{S,S'} = 32\pi^3\left(\frac{1}{3}\int_{S'}\frac{dS'}{\sqrt{K'}}+R^2\int_{S'}H'dS'\right).$$

If R tends to infinity, normalizing the kinematic measure, we obtain the well known fact that the measure of planes intersecting S' is proportional to the integral of its mean curvature.

If $R \le 1/k_{\max}$, then formula (5) applies in the same line by switching the roles of S and S'.

Remark 6. In the style of Howard's transfer principle (see [How93]), Proposition 1 holds with the same proof in 3-dimensional spherical and hyperbolic spaces.

Remark 7. Let S and S' be closed surfaces ($\partial S = \partial S' = \emptyset$) in 3-dimensional space of constant curvature $\epsilon = -1, 0, 1$. Using $K = \kappa_1\kappa_2 = K_{\text{int}} - \epsilon$, with K_{int} the intrinsic Gauss curvature and the Gauss-Bonnet formula

$$\int_S K_{\text{int}} dS = 2\pi\chi(S)$$

one can rewrite formulas (4), (5) and (10).

Remark 8. Since $H \geq \sqrt{K}$, Schwartz inequality gives

$$A^2 \leq \left(\int_S \sqrt{\frac{H}{\sqrt{K}}}\, dS\right)^2 \leq \int_S H dS \cdot \int \frac{dS}{\sqrt{K}}$$

Hence we have the following lower bound, sharp for spheres, for the integral of the inverse of the square root of the curvature

$$\int_S \frac{dS}{\sqrt{K}} \geq \frac{A^2}{M},$$

where M is the total mean curvature of S.

3. Lower bounds.

Let now S, S' be smooth regular surfaces bounding compact convex domains Ω, Ω' in $\mathbb{R}^3$. We give here lower bounds for the integral $I_{S,S'}$.

By the inequality of Fenchel (see [Fen29]), the total absolute curvature of a closed regular curve in $\mathbb{R}^3$ is always greater than or equal to 2π. The equality holds if and only if the curve is plane and convex. In our situation, for non-empty intersection curves $\gamma_g = S \cap gS'$, we have

$$\int_{\gamma_g} \kappa(s) ds \geq 2\pi. \tag{12}$$

On the other hand the kinematic formula of Blaschke (see [San76]) states that

$$\int_G \chi(\Omega \cap g\Omega') dg = 8\pi^2(\chi(\Omega')V + \chi(\Omega)V') + 2\pi(AM' + A'M), \tag{13}$$

where V is the volume of Ω, while A, M are the area and the total mean curvature of S and V', A', M' are the analogous quantities for S'. Here χ is the Euler characteristic, that for non-empty convex domains is equal to one.

Proposition 2. *Let $U = \{g \in G : g\Omega' \subset \Omega \quad \text{or} \quad g\Omega' \supset \Omega\}$. We suppose that its measure $m(U)$ is zero. Then we have the following inequality*

$$I_{S,S'} \geq 16\pi^3(V + V') + 4\pi^2(AM' + A'M). \tag{14}$$

The equality case holds if and only if S and S' are congruent spheres.

Proof. Because of $m(U) = 0$ we have $S \cap gS' \neq \emptyset$ if and only if $\chi(\Omega \cap g\Omega') = 1$ for almost every $g \in G$. Therefore

$$\int_{\gamma_g} \kappa(s) ds \geq 2\pi\chi(\Omega \cap g\Omega')$$

for almost every $g \in G$. Now we get from formula (3)

$$I_{S,S'} \geq 2\pi \int_G \chi(\Omega \cap g\Omega')dg.$$

And by Blaschke's formula (13) we get the inequality (14).

The equality case is a consequence of the following lemma. □

Lemma 1. *Let S, S' be two strictly convex surfaces such that $S \cap gS'$ is a plane curve for almost all $g \in G$. Then S, S' are parts of spheres.*

Proof. Take $p \in S$ and $p' \in S'$ such that at least one of them is not umbilical. Let v_1, v_2 (resp. v_1', v_2') be principal directions of S at p (resp. S' at p'), and $0 < k_1 \leq k_2$ (resp. $0 < k_1' \leq k_2'$) the principal curvatures with respect to the normal vector n (resp. n'). Take a motion g such that $g(p') = p$, $g_*(n') = n$, and $g_*(v_2') = v_1$. We choose cartesian coordinates in $\mathbb{R}^3$ so that $p = 0$, and $n = (0, 0, 1)$. Then S, gS' are locally the graphs of two functions $f(x, y), f'(x, y)$.

$$f(x, y) = \frac{k_1}{2}x^2 + \frac{k_2}{2}y^2 + O((x^2 + y^2)^{3/2})$$

$$f'(x, y) = \frac{k_2'}{2}x^2 + \frac{k_1'}{2}y^2 + O((x^2 + y^2)^{3/2})$$

Case 1: $(k_1, k_2) \cap (k_1', k_2') \neq \emptyset$. We have $(k_2' - k_1)(k_1' - k_2) < 0$. Consider the function $h(x, y, z) = z - f'(x, y)$ defined on $\mathbb{R}^3$. The level sets of h are given by $z = a + f'(x, y), a \in \mathbb{R}$. Their intersections with the surface S are given by $a = f(x, y) - f'(x, y)$, i.e. they are the intersections of S and $gS' - an$. Now, p is a non-degenerate critical point of index 1 of $h|_S$, i.e. locally $h(x, y) = f(x, y) - f'(x, y)$. Thus, by Morse lemma, we can choose local coordinates (u, v) on S such that $h(u, v) = u^2 - v^2$. Hence there are two different tangent vectors in T_pS of $gS' \cap S$. Because S is strictly convex, this intersection $gS' \cap S$ is not a plane curve. By continuity, for a small enough the intersection curve $\{a = h(u, v)\} \subset S$ is still not planar. Moreover, for small $a \neq 0$ the surfaces S and $gS' - an$ intersect transversely (at least near p). By stability, there is an open neighborhood of g in G such that the associated neighboring positions of S' will intersect S in a non-planar set; a contradiction.

Case 2: $(k_1, k_2) \cap (k_1', k_2') = \emptyset$. We can assume, $k_1 < k_2 \leq k_1' < k_2'$. We have

$$\frac{k_2'}{k_1} > \frac{k_1'}{k_2}$$

Analyzing the sections of S, gS' with the planes $x = 0$, $y = 0$ one can find four points in $S \cap gS'$of the form

$$\left(0, \sqrt{\frac{2h}{k_1' - k_2}} + O(h), \frac{k_1'}{k_1' - k_2}h + O(h^2)\right), \left(0, -\sqrt{\frac{2h}{k_1' - k_2}} + O(h), \frac{k_1'}{k_1' - k_2}h + O(h^2)\right)$$

$$\left(\frac{\sqrt{2h}}{k_2' - k_1} + O(h), 0, \frac{k_2'}{k_2' - k_1}h + O(h^2)\right), \left(\frac{\sqrt{2h}}{k_2' - k_1} + O(h), 0, \frac{k_2'}{k_2' - k_1}h + O(h^2)\right).$$

For small values of h, the first pair of points have a bigger third coordinate than the second pair. Then, it is geometrically clear that these four points are not coplanar; a contradiction.

We are only left with the case $k_1 = k_2$ or $k_1' = k_2'$ which can be treated similarly.

Therefore all points of S and S' must be umbilical, hence S and S' are part of spheres. □

Remark 9. If Ω is congruent to Ω', then $m(U) = 0$. In this case

$$I_{S,S'} \geq 8\pi^2(4\pi V + AM).$$

Remark 10. Using the monotonicity of the Quermaßintegrale (see [San76]), we get sufficient conditions for Proposition 2:

$$\begin{aligned} V = V' &\implies m(U) = 0, \\ A = A' &\implies m(U) = 0, \\ M = M' &\implies m(U) = 0. \end{aligned}$$

Remark 11. There is an analog of Fenchel inequality (12) in hyperbolic space (see [Sze68], [BH74] and [Tsu74]). Therefore Proposition 2 holds there.

4. Applications.

Using the previous results we find upper bounds for the average number of connected curves of the intersection $S \cap gS'$ of closed, smooth, oriented, regular surfaces S, S'. We also give bounds for the average number of components of the intersection $\Omega \cap gS'$ of a domain Ω and a moving surface S'. Finally we give, in the style of Hadwiger (cf. [Had41]), sufficient conditions for a convex domain to contain another one in three dimensional euclidean space.

Proposition 3. *Let $c(g)$ be the number of connected components of $\gamma_g = S \cap gS'$. If S is closed and S' bounds a strictly convex domain, then*

$$\int_G c(g)dg \leq \frac{2\pi}{3}\int_S (3H^2 - K)dS \int_{S'} \frac{dS'}{\sqrt{K'}} + 4\pi A \int_{S'} H' dS'. \tag{15}$$

The equality holds if and only if S and S' are spheres.

Proof. By the Fenchel inequality (12) applied to each connected component of the intersection γ_g we have

$$\int_G c(g)dg \leq \frac{1}{2\pi}\int_G \int_{\gamma_g} k(s)\ dsdg.$$

Now formula (4) yields (15). The equality case comes from the equality case in the Fenchel inequality and Lemma 1. □

Remark 12. In case both S, S' are strictly convex one can also find an upper bound where S and S' play symmetric roles (as in Corollary 1).

Proposition 4. *Let $n(g)$ be the number of connected components of $\Omega \cap gS'$. If the boundary of Ω is a smooth and regular surface S, and if S' bounds a strictly convex domain, then*

$$\int_G n(g)dg \leq 8\pi^2 V + \pi A'M + 2\pi AM' + \frac{\pi}{3}\int_S (3H^2 - K)dS \int_{S'} \frac{dS'}{\sqrt{K'}}. \tag{16}$$

The equality holds if and only if S and S' are spheres.

Proof. For almost every g, we have that $\Omega \cap gS'$ is a domain of the topological sphere gS'. For such domains one has the following relation between the number of components of $\Omega \cap gS'$ and its boundary:

$$\chi(\Omega \cap gS') = 2n(g) - c(g). \tag{17}$$

The proof of (17) for $n(g) = 1$ is an elementary exercise using the properties of the Euler characteristic. The case $n(g) > 1$ follows then immediately. Applying Blaschke's formula (13) to Ω and an ϵ-tube around S', and making ϵ tend to 0, one gets in the limit

$$\int_G \chi(\Omega \cap gS')dg = 8\pi^2\chi(S')V + 2\pi A'M.$$

Combining this with the relation (17), and Proposition 3 we get the result. □

Remark 13. One has a similar inequality in the case where Ω is strictly convex and S' is a topological sphere.

Remark 14. For closed convex surfaces S, S' formulas (15) and (16) lead to inequalities for the expectation values of c and n. Here the measure dg must be normalized by Blaschke's formula in order to get the probability measure on the space of intersection positions.

Proposition 5. *Let S, S' bound strictly convex domains Ω, Ω'. Then the set $U = \{g \in G : g\Omega' \subset \Omega \quad \text{or} \quad g\Omega' \supset \Omega\}$ has measure $m(U)$ fulfilling the following inequalities*

$$m(U) \geq 8\pi^2(V + V') + 2\pi(A'M - AM') - \frac{2\pi}{3}\int_S (3H^2 - K)dS \int_{S'} \frac{dS'}{\sqrt{K'}}, \qquad (18)$$

$$m(U) \geq 8\pi^2 V' + \pi A'M - \frac{\pi}{3}\int_S (3H^2 - K)dS \int_{S'} \frac{dS'}{\sqrt{K'}}. \qquad (19)$$

In particular if the right hand side of (18) or (19) is positive, then there are positions where one of the domains contains the other.

Proof. Beginning with

$$\int_G \chi(\Omega \cap g\Omega')dg \leq m(U) + \int_G c(g)dg,$$

and using (15) we get (18).

Starting with

$$\int_G \chi(\Omega \cap g\Omega')dg \leq m(U) + \int_G n(g)dg,$$

and using (16) we get (19). □

Remark 15. An alternative Hadwiger condition in dimension 3 was given by J. Zhou in [Zho98]. Our inequality (18) is sharp for spheres, while inequality in [Zho98] is not.

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