LIE FLOWS OF CODIMENSION 3

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Abstract. We study the following realization problem: given a Lie algebra of dimension 3 and an integer $q$, $0 \leq q \leq 3$, is there a compact manifold endowed with a Lie flow transversely modeled on $G$ and with structural Lie algebra of dimension $q$? We give here a quite complete answer to this problem but some questions remain still open (cf. §2).

0 Introduction

Among the class of foliations with a transverse structure Lie foliations stand out. These are foliations transversely modeled on Lie groups. They have been studied by several authors, mainly by Fedida (cf. [3]). Apart from its intrinsic interest the importance of this study is increased by the fact that they arise naturally in Molino’s classification of Riemannian foliations ([6]).

To each Lie foliation are associated two Lie algebras, the Lie algebra $G$ of the Lie group on which it is modeled and the structural Lie algebra $H$. The latter algebra is the Lie algebra of the Lie foliation $F$ restricted to the closure of any one of its leaves. In particular it is a subalgebra of $G$. We remark that although $H$ is canonically associated to $F$, $G$ is not.

Thus, one natural and interesting question is to know which pairs of Lie algebras $(G, H)$, with $H$ a subalgebra of $G$, can arise as transverse algebra and structural Lie algebra respectively of a Lie foliation $F$ on a compact manifold $M$.

We shall study here a particular but interesting case, namely given a Lie algebra of dimension 3 and an integer $q$, $0 \leq q \leq 3$, is there a compact manifold endowed with a Lie flow transversely modeled on $G$ and with structural Lie algebra of dimension $q$? For simplicity’s sake we shall say that the pair $(G, q)$ is (or is not) realizable.

By using the classification of the 3-dimensional Lie algebras and the fact that the structural Lie algebra of a Lie flow is abelian (cf. [1]) it becomes apparent that certain pairs $(G, q)$ are not realizable (for instance

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(sl(2), 2) and (so(3), 2) are not realizable because sl(2) and so(3) have no abelian subalgebras of dimension two).

Nevertheless in some cases the obstruction for certain pairs to be realizable is rooted in the compactness of \( M \) and not based on purely algebraic reasons (for instance the pair \((affine, 0)\) is not realizable (cf. Theorem 1)).

We classify the 3-dimensional Lie algebras in 6 algebras \( G_1, \ldots, G_6 \) and two families \( G_7 \) (parametrized by \( k \in \mathbb{R}, k \neq 0 \)) and \( G_8 \) (parametrized by \( h \in \mathbb{R}, h^2 < 4 \)) (cf. §1). We obtain

**Theorem 1.** If the structural Lie algebra is zero, i.e. \( \mathcal{F} \) is a compact foliation, then \( G_1, G_2, G_3 \) and \( G_4 \) are realizable. \( G_5 \) and \( G_6 \) are not realizable. \( G_7 \) is realizable if and only if \( k = -1 \) and \( G_8 \) is realizable if and only if \( h = 0 \).

**Theorem 2.** If the structural Lie algebra has dimension 1, then \( G_1, G_2, G_3, G_4 \) and \( G_5 \) are realizable. \( G_6 \) and \( G_7 \) are not realizable and \( G_8 \) with \( h = 0 \) is realizable.

We do not know any realization of \( G_8 \) with \( h \neq 0 \) and 1-dimensional structural Lie algebra of dimension 1.

Finally it is remarkable that the realization of the pair \((G_7, 2)\) depends on \( k \). In fact we have

**Theorem 3.** If the structural Lie algebra has dimension 2 then \( G_1, G_5 \) and \( G_8 \) with \( h = 0 \) are realizable. \( G_2, G_3, G_4, G_6 \) and \( G_7 \) with \( k \in \mathbb{Q} \) are not realizable.

We give a realization of \( G_7 \) with \( k \notin \mathbb{Q} \). A characterization of those \( k \) for which \( G_7 \) is realizable and the \( G_8 \) case, are still open.

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### 1 Preliminary definitions and results

Let \( \mathcal{F} \) be a smooth foliation of codimension \( n \) on a smooth manifold \( M \) given by an integrable subbundle \( L \subset TM \). We denote by \( \mathcal{L}(M, \mathcal{F}) \) the Lie algebra of foliated vector fields, i.e. \( X \in \mathcal{L}(M, \mathcal{F}) \) if and only if \([X, Y] \in \Gamma L\) for all \( Y \in \Gamma L \). Thus, the set of sections of \( L, \Gamma L \), is an ideal of \( \mathcal{L}(M, \mathcal{F}) \). The elements of \( \mathcal{X}(M, \mathcal{F}) \) are called basic vector fields.

If there is a family \( \{X_1, \ldots, X_n\} \) of foliated vector fields of \( M \) such that the corresponding family \( \{\overline{X}_1, \ldots, \overline{X}_n\} \) of basic vector fields has rank \( n \) everywhere, the foliation is called transversely parallelizable and \( \{\overline{X}_1, \ldots, \overline{X}_n\} \) a transversely parallelism. If the vector subspace \( \mathcal{G} \) of
\( \mathcal{X}(M, \mathcal{F}) \) generated by \( \{ \overline{X}_1, \ldots, \overline{X}_n \} \) is a Lie subalgebra, the foliation is called a Lie foliation.

We shall use the following structure theorems (cf. [3] and [6]):

**Theorem A.** Let \( \mathcal{F} \) be a transversally parallelizable foliation on a compact manifold \( M \) of codimension \( n \). Then:

a) There is a Lie algebra \( \mathcal{H} \) of dimension \( q \leq n \).

b) There is a locally trivial fibration \( \pi : M \to W \) with compact fibre \( F \) and \( \dim W = n - q = m \).

c) There is a dense Lie \( \mathcal{H} \)-foliation on \( F \) such that:

(i) The fibres of \( \pi \) are the closures of the leaves of \( \mathcal{F} \).

(ii) The foliation induced by \( \mathcal{F} \) on each fibre of \( \pi \) is isomorphic to the \( \mathcal{H} \)-foliation on \( F \).

\( \mathcal{H} \) is called the structural Lie algebra of \( (M, \mathcal{F}) \), \( \pi \) the basic fibration and \( W \) the basic manifold. The foliation given by the fibres of \( \pi \) is denoted by \( \overline{\mathcal{F}} \). Note that \( \text{codim} \overline{\mathcal{F}} + q = \text{codim} \mathcal{F} \).

**Theorem B.** Let \( \mathcal{F} \) be a \( G \)-foliation on a compact manifold \( M \) and let \( G \) be the connected simply connected Lie group with Lie algebra \( \mathcal{G} \). Let \( p : \tilde{M} \to M \) be the universal covering of \( M \). Then there is a locally trivial fibration \( D : \tilde{M} \to G \) equivariant by \( \text{Aut}(p) \) (i.e. if \( D(x) = D(y) \) then \( D(gx) = D(gy) \) for all \( x, y \in \tilde{M} \) and \( g \in \text{Aut}(p) \)) such that the foliation \( \tilde{\mathcal{F}} = p^* \mathcal{F} \) is given by the fibres of \( D \).

The natural morphism \( h : \pi_1(M) \to \text{Diff}(G) \) is such that \( \Gamma = \text{im}(h) \subset G \), where the inclusion \( G \subset \text{Diff}(G) \) is by left translations.

We shall also use some cohomological properties of the foliation. Recall that the basic forms complex is given by the forms \( \alpha \in \Omega^*(M) \) such that \( \mathcal{L}_X \alpha = 0 \) and \( i_X \alpha = 0 \) for all \( X \in \Gamma \mathcal{L} \). The cohomology of this complex, \( H^*(M, \mathcal{F}) \), is the basic cohomology of the foliated manifold \( (M, \mathcal{F}) \). If \( H^n(M, \mathcal{F}) \neq 0 \) we say that \( \mathcal{F} \) is homologically orientable or unimodular. We have (cf. [5]):

**Theorem C.** Let \( \mathcal{F} \) be an unimodular Lie \( G \)-foliation on a compact manifold \( M \). Then the Lie algebra \( \mathcal{G} \) is unimodular.

Finally we recall that the 3-dimensional Lie algebras can be classified in eight families

- \( \mathcal{G}_1 \) (Abelian):
  \[ [e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0 \]

- \( \mathcal{G}_2 \) (Heisenberg):
  \[ [e_1, e_2] = [e_1, e_3] = 0, \quad [e_2, e_3] = e_1 \]
- $G_3$ (so(3)):
  \[ [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2 \]

- $G_4$ (sl(2)):
  \[ [e_1, e_2] = e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2 \]

- $G_5$ (Affine):
  \[ [e_1, e_2] = e_1, \quad [e_1, e_3] = [e_2, e_3] = 0 \]

- $G_6$:
  \[ [e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2 \]

- $G_7$:
  \[ [e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = ke_2 \quad k \neq 0 \]

- $G_8$:
  \[ [e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1 + he_2 \quad h^2 < 4 \]

Notice that $G_1$, $G_2$, $G_3$ and $G_4$ are unimodular, $G_5$ and $G_6$ are not unimodular, $G_7$ is unimodular only if $k = -1$ and $G_8$ only if $h = 0$.

**Remark:** We can think that $G_7$ is parametrized by $k \in [-1, 0) \cup (0, 1]$. In fact two of these algebras are isomorphic if and only if $k \cdot k' = 1$.

### 2 Lie flows of codimension 3

Let $\mathcal{F}$ be a Lie flow of codimension 3 on a compact manifold $M$. Since the closures of the leaves of $\mathcal{F}$ are the fibres of a bundle (cf. Theorem A), there are four possible cases.

1 **Case** codim $\mathcal{F} = 3$.

   In this case $\mathcal{F}$ is compact and the basic bundle is $M \to M/\mathcal{F}$. Thus the basic cohomology coincides with the de Rham cohomology of the compact manifold $M/\mathcal{F}$ and hence $H^3(M/\mathcal{F}) \neq 0$. By Theorem C, if such a flow exists it is transversely modeled on a unimodular Lie algebra. So $G_5$ and $G_6$ are not realizables, $G_7$ is realizable (a priori) only if $k = -1$ and $G_8$ only if $h = 0$.

   We give now examples for each one of the remainder algebras.

   - $G_1$: Just consider the trivial bundle $T^1 \times T^3 \to T^3$. 


- \( \mathcal{G}_2 \): Consider the trivial bundle \( T^1 \times M \rightarrow M \) where \( M \) is the homogeneous space \( N/\Gamma \) of the Heisenberg group

\[
N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbb{R} \right\}
\]

by the discret uniform subgroup \( \Gamma \) of \( N \) given by the matrices of \( N \) with integer coefficients.

- \( \mathcal{G}_3 \): Just consider the trivial bundle \( T^1 \times S^3 \rightarrow S^3 \).

- \( \mathcal{G}_4 \): Consider the trivial bundle \( T^1 \times T_1W \rightarrow T_1W \) where \( T_1W \) is the homogeneous space \( PSL(2, \mathbb{R})/\pi_1(W) \) and therefore we have the desired example.

- \( \mathcal{G}_7 \) (with \( k = -1 \)): Let \( A \in SL(2, \mathbb{Z}) \) be a matrix with eigenvalues \( \lambda, 1/\lambda \) (being \( \lambda > 0 \) and \( \lambda \neq 1 \)). We can give a solvable Lie group structure on \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2 \) by

\[
(t, u)(s, v) = (t + s, A^t \cdot v + u)
\]

The Lie algebra of this group is \( \mathcal{G}_7 \) with \( k = -1 \) (cf. [4]). Moreover the points of \( \mathbb{R}^3 \) with integer coordinates constitute a uniform discret subgroup \( \Gamma \) of \( \mathbb{R}^3 \). The quotient is usually denoted by \( T_3^3 \). Then, one example of a Lie flow transversely modeled on \( \mathcal{G}_7 \), with \( k = -1 \), is given by the trivial bundle \( T^1 \times T_3^3 \rightarrow T_3^3 \).

- \( \mathcal{G}_8 \) (with \( h = 0 \)) (P. Molino): Let us consider the flow given by the fibres of the trivial bundle \( T^1 \times T^3 \rightarrow T^3 \). Let \( \theta^0, \theta^1, \theta^2, \theta^3 \) denote the canonical coordinates in \( T^1 \times T^3 \). The parallelism given by \( \partial/\partial \theta^1, \partial/\partial \theta^2, \partial/\partial \theta^3 \) makes the fibres of the bundle an abelian Lie foliation. But we have basic functions enough to modify this parallelism. In fact, we can take

\[
\begin{align*}
e_1 &= \cos \theta^1 \cdot \partial/\partial \theta^2 + \sin \theta^1 \cdot \partial/\partial \theta^3 \\
e_2 &= -\sin \theta^1 \cdot \partial/\partial \theta^2 + \cos \theta^1 \cdot \partial/\partial \theta^3 \\
e_3 &= -\partial/\partial \theta^1
\end{align*}
\]

to obtain a new parallelism with \( [e_1, e_2] = 0, [e_1, e_3] = e_2, [e_2, e_3] = -e_1 \) i.e. the flow is also transversely modeled on \( \mathcal{G}_8 \) (with \( h = 0 \)).

2 Case \( \text{codim} \mathcal{F} = 2 \).
In this case we give examples for $G_1$, $G_2$, $G_3$, $G_4$, $G_5$ and $G_8$ (with $h = 0$). We also prove that $G_6$ and $G_7$ are not realizable.

- $G_1$: One example is given by the flow $(X, 0)$ on $T^2 \times T^2$ where $X$ is a dense linear flow on $T^2$.

- $G_2$: Let $M$ be the homogeneous space of the Heisenberg group considered before. The flow on $M \times T^1$ whose integral curves are given by

$$\varphi_t(p) = \begin{pmatrix} 1 & a & b + t \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, t + d$$

where

$$p = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, d \text{ and } d \in \mathbb{R} \setminus \mathbb{Q}$$

is transverse to $M$ and the closure of each leaf is $T^2$. Hence it is one example of a $G_2$–Lie flow with codim $\mathcal{F} = 2$.

- $G_3$: As $S^3 = SU(3)$ an example can be constructed by suspending the representation

$$h : \pi_1(S^1) \longrightarrow \text{Diff}(S^3)$$

given by

$$h(1) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

- $G_4$ (A. ElKacimi): Let $\mathcal{F}_0$ be the transverse affine Lie flow on $T^3_A$ (cf.[1]). Using the fact that the affine group $GA$ can be considered, lifting the map

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

of $GA$ in $SL(2, \mathbb{R})$, as a Lie subgroup of $SL(2, \mathbb{R})$ and using also that the unfolding diagram of $\mathcal{F}_0$ (cf. Theorem B), $D_0 : \tilde{T}^3_A \longrightarrow GA$, $\rho_0 : \pi_1(T^3_A) \longrightarrow GA$ the desired foliation can be constructed as follows:

Let $\tilde{M} = \tilde{T}^3_A \times \mathbb{R}$ be the universal covering of $M = T^3_A \times S^1$ and define $D : \tilde{M} \longrightarrow \tilde{SL}(2, \mathbb{R})$, $\rho : \pi_1(M) \longrightarrow \tilde{SL}(2, \mathbb{R})$...
by \( D(x, t) = D_0 x \cdot \bar{\varphi}(t) \) and \( \rho(\gamma, n) = \rho_0(\gamma) \cdot \varphi(n) \) where \( \bar{\varphi} : \mathbb{R} \to SL(2, \mathbb{R}) \) is a lift of the uniparametric subgroup \( \varphi : \mathbb{R} \to SL(2, \mathbb{R}) \) given by

\[
\varphi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}
\]

It turns out, using that \( \bar{\varphi}(n) \) is in the center of \( SL(2, \mathbb{R}) \), that \( \varphi \) is an homomorphism and \( D \) is equivariant (i.e. \( D((\gamma, n), (x, t)) = \rho(\gamma, n).D(x, t) \)). Thus the fibres of \( D \) induce the desired Lie foliation on \( M \) (cf. [2] for details).

\(- \mathcal{G}_5 : \) Let \( X \) be the generator of the transversely affine Lie flow on \( T^3_A \). As we have that \( \mathcal{G}_5 = \mathcal{A} + \mathbb{R} \), where \( \mathcal{A} \) is the affine Lie algebra of dimension 2, the vector field \( (X, 0) \) on \( T^3_A \times S^1 \) is transversely modeled on \( \mathcal{G}_5 \) and \( \text{codim} \mathcal{F} = 2 \).

\(- \mathcal{G}_6 \) and \( \mathcal{G}_7 \) are not realizables : Let \( \mathcal{F} \) be a \( \mathcal{G}_6 \) or a \( \mathcal{G}_7 \) Lie flow on a compact manifold \( M \). Fix a generator \( X \) of \( \mathcal{F} \) and a transverse parallelism \( Y_1, Y_2, Y_3 \) such that \( [Y_1, Y_2] = 0, [Y_1, Y_3] = Y_2, [Y_2, Y_3] = Y_1 + Y_2 \) for \( \mathcal{G}_6 \) and \( [Y_1, Y_2] = 0, [Y_1, Y_3] = Y_1, [Y_2, Y_3] = k Y_2 \) for \( \mathcal{G}_7 \). Let \( g \) be a Riemannian metric on \( M \). Then we have the orthogonal decomposition \( TM = T\mathcal{F} + T\mathcal{F}^\perp \) and we shall denote by \( Z^t \) and \( Z^n \) the tangent and the orthogonal parts of a vector field \( Z \) on \( M \).

The set \( T = \{ p \in M; Y_1^n(p) = 0 \} \) is open. In fact, if \( p \in T \), \( Y_1 \) is tangent to \( \mathcal{F} \) in \( p \) therefore \( Y_2^n, Y_3^n \) are independent in \( p \). Hence they are independent in an open neighborhood \( U \) of \( p \) and we can write \( Y_1^n = \lambda Y_2^n + \mu Y_3^n \) where \( \lambda, \mu \) are basic functions on \( U \). Computing now \([Y_1^n, Y_2^n]\) and \([Y_1^n, Y_3^n]\) we deduce the following system of differential equations:

\[
\begin{align*}
Y_2^n(\lambda) + \mu \lambda + \mu &= 0 \\
Y_2^n(\mu) + \mu^2 &= 0 \\
Y_3^n(\lambda) - \lambda^2 &= 0 \\
Y_3^n(\mu) - \mu \lambda + \mu &= 0
\end{align*}
\]

for \( \mathcal{G}_6 \) and

\[
\begin{align*}
Y_2^n(\lambda) + k \mu &= 0 \\
Y_2^n(\mu) &= 0 \\
Y_3^n(\lambda) + (1 - k)\lambda &= 0 \\
Y_3^n(\mu) + \mu &= 0
\end{align*}
\]
for $\mathcal{G}_7$, with the initial conditions $\lambda(p) = \mu(p) = 0$.

This implies that $\mu = 0$ on the integral curves of $Y_3$ and $Y_2$. Due to transverse transitivity $\mu = 0$ on $U$. It follows in a similar way that $\lambda = 0$ on $U$. Thus $Y = 0$ on $U$ and $T$ is open.

As it is also closed and $M$ is supposed to be connected, $T = \emptyset$ or $T = M$.

But if $T = M$ we arrive in both cases ($\mathcal{G}_6$ and $\mathcal{G}_7$) to a contradiction. In fact, if we denote by $\theta^0, \theta^1, \theta^2, \theta^3$ the dual basis of $X, Y_1, Y_2, Y_3$ we have $d\theta^2 = -\theta^2 \wedge \theta^3$ in $\mathcal{G}_6$ and $d\theta^2 = k\theta^2 \wedge \theta^3 (k \neq 0)$ in $\mathcal{G}_7$. As $\theta^2(Z) = \theta^3(Z) = d\theta^2(Z, \cdot) = d\theta^3(Z, \cdot) = 0$ for each vector field $Z$ tangent to $\mathcal{F}$, the 1-forms $\theta^2$ and $\theta^3$ are projectable on the basic manifold $W = M/\mathcal{F}$.

So we would have an exact volume element on the compact manifold $W$, which is a contradiction.

Therefore $T = \emptyset$.

Next we consider the set $Q = \bigcup_{a \in \mathbb{R}} Q_a$ where $Q_a = \{ p \in M; Y_2^n(p) = aY_1^n(p) \}$.

$Q$ is open: If $p \in Q$, there is $a \in \mathbb{R}$ such that $Y_2^n(p) = aY_3^n(p)$ and hence $Y_3^n$ and $Y_2^n$ are independent in $p$. So $Y_2^n = \lambda Y_1^n + \mu Y_2^n$ in an open neighborhood $U$ of $p$ with $\lambda(p) = a$ and $\mu(p) = 0$. Computing now $[Y_1, Y_2^n], [Y_3, Y_2^n]$ and considering their tangent and normal parts one obtains the equations:

$$\begin{align*}
Y_1(\lambda) + \mu &= 0 \\
Y_1(\mu) &= 0 \\
Y_3(\lambda) + 1 &= 0 \\
Y_3(\mu) - \mu &= 0
\end{align*}$$

As before, this yields $\mu = 0$ i.e. $Y_2^n = \lambda Y_1^n$ on $U$. Thus every point $x \in U$ is in $Q_{\lambda(x)} \subset Q$ and $Q$ is open.

$Q$ is closed: If $p \notin Q$, for each $a \in \mathbb{R}$, $Y_2^n(p) \neq aY_1^n(p)$. In particular $Y_2^n(p) = 0$. As we have proved that $Y_2^n \neq 0$, the vector fields $Y_1, Y_2$ are linearly independent on $p$. Hence they are independent in an open neighborhood $U$ of $p$, i.e. $U \subset M \setminus Q$ and $Q$ is closed.

As $M$ is connected $Q = \emptyset$ or $Q = M$.

If $Q = \emptyset$, $Y_1^n$ and $Y_2^n$ are linearly independent in each point. So there are differentiable functions $\lambda$ and $\mu$ globally defined on $M$, such that $Y_3^n = \lambda Y_1^n + \mu Y_2^n$. Computing now $[Y_1, Y_3^n]$ we obtain $Y_1(\lambda) = 1$, but as $M$ is compact this is impossible.

If $Q = M$, for each $p \in M$ there is $a(p) \in \mathbb{R}$ such that $Y_2^n(p) = a(p)Y_1^n(p)$. This gives rise to a differentiable basic function $a$ on
M with $Y_2^n = a \cdot Y_1^n$. Equivalently $Y_2 - a \cdot Y_1$ is everywhere tangent to $\mathcal{F}$. Since $[Y_3, Y_2 - a \cdot Y_1]$ must be in $\mathcal{F}$ we obtain $Y_3(a) = -1$ for $\mathcal{G}_6$, which is again a contradiction, and $Y_3(a) = (1 - k)a$ for $\mathcal{G}_7$. If $k \neq 1$, the only possibility is $a = 0$ and so $Y_2$ is everywhere tangent to $\mathcal{F}$. As before, this yields a contradiction because $d\theta^1 = -\theta^1 \wedge \theta^3$, with $\theta^1$ and $\theta^3$ projectables on $W = M/\mathcal{F}$. If $k = 1$ it follows that $a$ is constant over the integral curves of $Y_1, Y_2, Y_3$, i.e. $a$ is constant. Being $\omega^0, \omega^1, \omega^2, \omega^3$ the dual basis of $X, Y_2 - aY_1, Y_1, Y_3$ we obtain $d\omega^2 = -\omega^2 \wedge \omega^3$ with $\omega^2, \omega^3$ projectables on $W$, again a contradiction. This proves that $\mathcal{G}_6$ and $\mathcal{G}_7$ are not realizables.

$\mathcal{G}_8$ (with $h = 0$) : The same construction as before. If $\theta^0, \theta^1, \theta^2, \theta^3$ are the canonical coordinates on $T^2 \times T^2$, the vector field $X = \partial/\partial \theta^0 + \alpha \partial/\partial \theta^1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is transversely abelian for the parallelism $\partial/\partial \theta^1, \partial/\partial \theta^2, \partial/\partial \theta^3$ and has codim $\mathcal{F} = 2$.

We modify this parallelism by taking

$$
eq 1 = \cos \theta^2 \cdot \partial/\partial \theta^1 + \sin \theta^2 \cdot \partial/\partial \theta^3 \quad \neq 2 = -\sin \theta^2 \cdot \partial/\partial \theta^1 + \cos \theta^2 \cdot \partial/\partial \theta^3 \quad \neq 3 = -\partial/\partial \theta^2 \quad \neq 3$$

Thus $X$ is also transversely modeled on $\mathcal{G}_8$ (with $h = 0$).

3 Case codim $\mathcal{F} = 1$.

In this case the structural Lie algebra has dimension 2. As this algebra is abelian (cf. [2]), $\mathcal{G}_3$ and $\mathcal{G}_4$ are not realizables because they do not have abelian subalgebras of dimension 2. Examples for the algebras $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_5$ and $\mathcal{G}_8$ ($h = 0$) are given. For the algebra $\mathcal{G}_7$ we prove that the only realizables cases are when $k \notin \mathbb{Q}$, an example will be given. We also prove that $\mathcal{G}_6$ is not realizable.

$\mathcal{G}_1$ : Consider the flow $(X, 0)$ on $T^3 \times T^1$ where $X$ is a dense linear flow on $T^3$.

$\mathcal{G}_2$ is not realizable : As $\mathcal{G}_2$ is unimodular and codim $\mathcal{F} = 1$, $\mathcal{F}$ is unimodular (cf. [5]) and it follows, from the results by Molino (cf. [6]), that the central transfers sheaf $\mathcal{C}$ admits a global trivialization, i.e. there are independent foliated vector fields $v, w$ tangents to the $\mathcal{F}$ closure which commute, as transfers fields, with every global foliated vector field. In particular $[v, e_i] = [w, e_i] = 0$. Writing

$$v = \lambda e_1 + \mu e_2 + \nu e_3$$
$$w = \alpha e_1 + \beta e_2 + \gamma e_3$$
we obtain \( v = \lambda e_1 \) and \( w = \alpha e_1 \) which is a contradiction.

- \( G_5 \) : Let \( X \) be the generator of the transversely affine Lie flow on \( T^3_x \). The vector field \( (X, \alpha \partial/\partial \theta) \) on \( T^3_x \times S^1 \), with \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and \( \theta \) the coordinate function on \( S^1 \), is transversely modeled on \( G_5 = A + \mathbb{R} \) and \( \text{codim} \mathcal{F} = 1 \).

- \( G_8 (h = 0) \) : The same construction as before. If \( \theta^0, \theta^1, \theta^2, \theta^3 \) are the canonical coordinates on \( T^3 \times T^1 \), the vector field \( X = \partial/\partial \theta^0 + \alpha \partial/\partial \theta^1 + \beta \partial/\partial \theta^2 \) with \( \alpha, \beta \) rationally independent, admits

\[
\begin{align*}
e_1 &= \cos \theta^3 \cdot \partial/\partial \theta^0 + \sin \theta^3 \cdot \partial/\partial \theta^1 \\
e_2 &= -\sin \theta^3 \cdot \partial/\partial \theta^0 + \cos \theta^3 \cdot \partial/\partial \theta^1 \\
e_3 &= -\partial/\partial \theta^3
\end{align*}
\]

as a transverse parallelism. But \( e_1, e_2, e_3 \) is a basis of \( G_8 \) with \( h = 0 \).

- Next we study the remainder algebras \( G_6, G_7 \) and \( G_8 (h \neq 0) \). As the center of these algebras are trivial, the corresponding connected simply connected groups \( G_6, G_7, G_8 \) can be obtained as \( e^{t \cdot \text{ad} \alpha} \), \( \alpha \in G_i \) with \( i = 1, 2, 3 \). We find that these groups can be though as \( \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \) with the product \( (p, t) \cdot (p', t') = (p + e^{ht} \cdot p', t + t') \) and \( \Lambda \) depending on the algebra.

For \( G_6 \)

\[
\Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^{-\Lambda t} = \begin{pmatrix} e^{-t} & -t e^{-t} \\ 0 & e^{-t} \end{pmatrix}
\]

For \( G_7 \)

\[
\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \quad e^{-\Lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-kt} \end{pmatrix}
\]

For \( G_8 \)

\[
\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix}, \quad e^{-\Lambda t} = C(t) \cdot \begin{pmatrix} \cos(\varphi + t) & -\sin t \\ \sin t & \cos(\varphi - t) \end{pmatrix}
\]

where \( C(t) = \frac{2}{\alpha} e^{\beta t} \) and \( \alpha = \sqrt{4 - h^2}, \beta = \tan \varphi = h/\alpha, (\sin \varphi = h/2, \cos \varphi = \alpha/2) \).

The basis to define the algebras are given by the following left invariant fields.
For $\mathcal{G}_6$

\[
\begin{align*}
  e_1 &= e^{-t} \frac{\partial}{\partial x} \\
  e_2 &= -te^{-t} \frac{\partial}{\partial x} + e^{-t} \frac{\partial}{\partial y} \\
  e_3 &= \frac{\partial}{\partial t}
\end{align*}
\]

For $\mathcal{G}_7$

\[
\begin{align*}
  e_1 &= e^{-t} \frac{\partial}{\partial x} \\
  e_2 &= e^{-kt} \frac{\partial}{\partial y} \\
  e_3 &= \frac{\partial}{\partial t}
\end{align*}
\]

For $\mathcal{G}_8$

\[
\begin{align*}
  e_1 &= \frac{2}{\alpha} e^{-t} \left( \cos(t + \varphi) \frac{\partial}{\partial x} + \sin(t) \frac{\partial}{\partial y} \right) \\
  e_2 &= \frac{2}{\alpha} e^{-t} \left( -\sin(t) \frac{\partial}{\partial x} + \cos(t - \varphi) \frac{\partial}{\partial y} \right) \\
  e_3 &= -\frac{\alpha}{2} \frac{\partial}{\partial t}
\end{align*}
\]

Suppose now that we have a codim $\mathcal{F} = 1$ realization on a compact manifold $M$ of one of these algebras. We shall denote the algebra by $\mathcal{G}$ and the corresponding group by $G$. The basic fibration is:

\[T^3 \longrightarrow M \longrightarrow T^1\]

and, as $\pi_1(T^3) = \mathbb{Z}^3$, $\pi_1(T^1) = \mathbb{Z}$ and $\pi_2(T^1) = 0$ the corresponding homotopy exact sequence is

\[0 \longrightarrow \mathbb{Z}^3 \longrightarrow \pi_1(M) \longrightarrow \mathbb{Z} \longrightarrow 0\]

Since this exact sequence has a section, $\pi_1(M)$ is the semidirect product of $\mathbb{Z}^3$ with $\mathbb{Z}$, i.e. $\pi_1(M)$ is the product $\mathbb{Z}^3 \ltimes \mathbb{Z}$ with the operation $(x, t) \cdot (y, s) = (x + t \cdot y, t + s)$ where $t \cdot y$ represents the natural action of $\mathbb{Z}$ on $\mathbb{Z}^3$. To be precise, if $\varphi : T^3 \longrightarrow T^3$ is the diffeomorphism which gives the bundle, then the action is $t \cdot y = \varphi^t \cdot y$ where $\varphi^t : \pi_1(T^3) \longrightarrow \pi_1(T^3)$ is the morphism induced by $\varphi$. We shall denote this group by $\mathbb{Z}^3 \ltimes \varphi \mathbb{Z}$. 
Since $\mathcal{F}$ is a Lie foliation we have the unfolding diagram:

$$
\begin{array}{c}
\tilde{M} \\
\downarrow \\
M
\end{array}
\xrightarrow{D} \begin{array}{c}
G
\end{array}
$$

and the holonomy representation $h : \pi_1(M) \to h(\pi_1(M)) = \Gamma \subset G$ with $D(\gamma \cdot \tilde{x}) = h(\gamma) \cdot D\tilde{x}$, $\tilde{x} \in \tilde{M}$, $\gamma \in \pi_1(M)$.

As $M/\mathcal{F}$ is diffeomorphic to $G/\Gamma$ (cf., for instance, [5]) we have that $\Gamma$ is a two dimensional closed subgroup of $G$.

The Lie algebra $\mathcal{H}$ of $\Gamma_e$ (the identity component of $\Gamma$) is named the structural Lie algebra of $\mathcal{F}$ and, in the case of flows it is abelian (cf. [1]).

But it is easy to see that the only two dimensional abelian subalgebra of $G$ is $\langle e_1, e_2 \rangle$, thus $\mathcal{H} = \langle e_1, e_2 \rangle$. Looking at the expressions for $e_1$ and $e_2$ in $\mathcal{G}_6$, $\mathcal{G}_7$ and $\mathcal{G}_8$ we see that $\mathcal{H} = \langle \partial/\partial x, \partial/\partial y \rangle$ and hence $\Gamma \simeq \mathbb{R}^2 \times \mathbb{Z} \varepsilon$, $\varepsilon > 0$.

Notice that $\Gamma_e = \mathbb{R}^2 \times \{0\}$ is abelian.

**Lemma.** Let $A$ be an abelian subgroup of $\Gamma$. Then $A$ is contained in $\mathbb{R}^2 \times \{0\}$ or there is an element $a = (a_1, a_2, a_3)$ with $a_3 \neq 0$ such that $A = \{a^n, n \in \mathbb{Z}\}$

**Proof.** If $A$ is not in $\mathbb{R}^2 \times \{0\}$, then $A \cap (\mathbb{R}^2 \times \{0\}) = 0 \in \mathbb{R}^3$.

Otherwise there is $(p, 0) \in A$, $p \neq 0$, and $(q, t) \in A$, $t \neq 0$. As $A$ is abelian we have that $(p, 0)(q, t) = (q, t)(p, 0)$. Then,

$$q + e^{-\lambda t} \cdot p = q + p$$

and this implies that $t = 0$, except for $\mathcal{G}_8$ with $h = 0$, but this case is not considered here. Therefore $A \cap (\mathbb{R}^2 \times \{0\}) = 0 \in \mathbb{R}^3$.

In particular $A$ has at most one element in each level $\mathbb{R}^2 \times \{m\varepsilon\}$, $m \in \mathbb{Z}$. In fact, $a_1 \cdot a_2^{-1} \in A \cap (\mathbb{R}^2 \times \{0\}) = 0$ and $a_1 = a_2$.

Let $a = (a_1, a_2, n\varepsilon)$ be the element of $A$ in the lower level. For each $b = (b_1, b_2, m\varepsilon) \in A$, we put $m = nd + r$, then $ba^{-d}$ is an element of $A$ in the $r\varepsilon$ level and hence $r = 0$, i.e. $b = a^d$ and this proves the lemma.

**Proposition 1.** Let the notation be as above. Then

$$(\mathbb{R}^2 \times \{0\}) \cap \Gamma = h(\mathbb{Z}^3)$$

**Proof.** Applying the lemma we have four possibilities:

(i) $h(\mathbb{Z}^3)$ and $h(\mathbb{Z})$ are both contained in $\mathbb{R}^2 \times \{0\}$. Then $\Gamma$, generated by $h(\mathbb{Z}^3)$ and $h(\mathbb{Z})$, is contained in $\mathbb{R}^2 \times \{0\}$ which contradicts $\mathbb{R}^3/\Gamma = S^1$. 


(ii) \( h(\mathbb{Z}^3) \) is contained in \( \mathbb{R}^2 \times \{0\} \) and \( h(\mathbb{Z}) = \{a^n, n \in \mathbb{Z}\} \) with \( a \notin \mathbb{R}^2 \times \{0\} \). As \( h(\mathbb{Z}^3) \) is a normal subgroup of \( \Gamma \), for each \( b \in h(\mathbb{Z}^3) \) we have \( aba^{-1} = b' \) which is in \( h(\mathbb{Z}^3) \). Hence, the elements of \( \mathbb{R}^2 \times \{0\} \cap \Gamma \) can be written as

\[
\sigma = b_1 a^{r_1} b_2 a^{r_2} b_3 a^{r_3} \cdots b_k a^{r_k}
\]

with \( \Sigma r_i = 0 \) and \( b_i \in h(\mathbb{Z}^3) \). That is \( \sigma = \tilde{b} \cdot a^{\Sigma r_i} = \tilde{b} \in h(\mathbb{Z}^3) \), i.e. \( (\mathbb{R}^2 \times \{0\}) \cap \Gamma = h(\mathbb{Z}^3) \)

(iii) \( h(\mathbb{Z}) \) is contained in \( \mathbb{R}^2 \times \{0\} \) and \( h(\mathbb{Z}^3) = \{a^n, n \in \mathbb{Z}\} \) with \( a \notin \mathbb{R}^2 \times \{0\} \). In this case \( \Gamma \) is abelian because if we let \( h(1) = b \) we have \( bab^{-1} = a^k \). This implies \( k = 1 \) and \( ab = ba \). As in (ii) this implies that \( \Gamma \cap \mathbb{R}^2 \times \{0\} = h(\mathbb{Z}) \) which is not dense in \( \mathbb{R}^2 \times \{0\} \).

(iv) \( h(\mathbb{Z}) = \{a^n, n \in \mathbb{Z}\} \) and \( h(\mathbb{Z}^3) = \{b^n, n \in \mathbb{Z}\} \) with \( a, b \notin \mathbb{R}^2 \times \{0\} \).

As before \( aba^{-1} = b^k \) and therefore \( \Gamma \) is abelian. So the elements of \( \mathbb{R}^2 \times \{0\} \cap \Gamma \) can be written as \( a^n \cdot b^{-n} = (a \cdot b^{-1})^n \) and this is not dense in \( \mathbb{R}^2 \times \{0\} \).

**Remark:** Three elements \( u, v, w \in \mathbb{R}^3 \) can generate a dense subgroup of \( \mathbb{R}^2 \). In fact it suffices to take \( u = \lambda v + \mu w \) with \( \lambda, \mu \) and \( \lambda/\mu \in \mathbb{R} \setminus \mathbb{Q} \). So, a priori, it is possible to have \( h(\mathbb{Z}) = \mathbb{R}^2 \times \{0\} \).

**Proposition 2.** \( \mathcal{G}_0 \) is not realizable.

**Proof.** If such a realization exists the subgroup \( h(\mathbb{Z}^3) \) is normal in \( \Gamma \). Let \( h(\mathbb{Z}^3) = \langle (p_1, 0), (p_2, 0), (p_3, 0) \rangle \) and \( h(\mathbb{Z}) = \langle (p, t) \rangle \) with \( t > 0 \) then the normality condition can be written as:

\[
e^{-\Lambda t} \cdot p_i = \sum_{j=1}^{3} \lambda^j \cdot p_j
\]

where \( \lambda^j \in \mathbb{Z} \).

The matrix \( A = (\lambda^j) \) corresponds in fact to \( \varphi_* : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3 \) then it is invertible, and, as we are assuming orientability, we have \( \det A = 1 \). Let \( v_1 = (a_1, b_1, c_1) \) and \( v_2 = (a_2, b_2, c_2) \) where \( p_1 = (a_1, a_2), p_2 = (b_1, b_2) \) and \( p_3 = (c_1, c_2) \). From the above equations we have that:

\[
\begin{align*}
Av_1 &= a \cdot v_1 + a \log a \cdot v_2 \\
Av_2 &= a \cdot v_2
\end{align*}
\]

where \( a = e^{-t} \).
Completing \( v_1, v_2 \) to a basis \( \{v_1, v_2, v_3\} \) the matrix \( A \) can be written:

\[
\begin{pmatrix}
a & 0 & \alpha \\
 \log a & a & \beta \\
 0 & 0 & a^{-2}
\end{pmatrix}
\]

and satisfies that:

\[
\begin{align*}
2a + \frac{1}{a^2} &= p \\
a^2 + \frac{2}{a} &= q
\end{align*}
\]

with \( p, q \in \mathbb{Z}, 0 < a < 1, \) and \( a \in \mathbb{R} \setminus \mathbb{Q} \). But this is impossible, because this equations imply that \( pa^2 - 2qa + 3 = 0 \) and hence \( a = (q \pm \sqrt{q^2 - 3p})/p \). In particular \( \sqrt{q^2 - 3p} \in \mathbb{R} \setminus \mathbb{Q} \). Substituting now \( a \) in the first of the above equations we conclude, after a short computation, that \( p = q = 3 \) which is in contradiction with \( a \in \mathbb{R} \setminus \mathbb{Q} \). So \( G_6 \) is not realizable.

**Proposition 3.** The Lie algebras of the \( G_7 \) family with \( k \in \mathbb{Q} \) are not realizable.

**Proof.** Proceeding as in Proposition 2 we obtain that \( e^{-t} \) and \( e^{-kt} \) are eigenvalues of \( A \).

The characteristic polynomial of \( A, x^3 - px^2 + qx - 1 \), has three roots, \( \xi, \xi^k \) and \( \xi^{-(k+1)} \) with \( \xi = e^{-t} \). As \( t > 0 \) we have \( 0 < \xi < 1 \). This implies, from standard arguments in Galois theory (see lemma below), that \( k \notin \mathbb{Q} \); i.e. the Lie algebras of \( G_7 \) with \( k \notin \mathbb{Q} \) are not realizable.

The authors are grateful to P. Ara for his remarks about the following lemma.

**Lemma.** Let \( f(x) = x^3 - px^2 + qx - 1 \) be a polynomial with \( p, q \in \mathbb{Z} \). If there are \( k \in \mathbb{R} \) and \( \xi \in (0, 1) \) such that the roots of \( f(x) \) can be written as \( \xi, \xi^k, \xi^{-(k+1)} \), then \( k \in \mathbb{R} \setminus \mathbb{Q} \).

**Proof.** First we observe that any rational root of this polynomial must be 1 or \(-1\), and so it is irreducible over \( \mathbb{Q} \). Hence the Galois group of \( f(x) \) over \( \mathbb{Q} \) is \( \mathbb{Z}_3 \) or the symmetric group \( S_3 \). In both cases there is an automorphism \( \sigma \) of the splitting field \( K \) of order 3 which is the identity over \( \mathbb{Q} \). This automorphism permutes the roots, i.e.

\[
\begin{align*}
\sigma(\xi) &= \xi^k, & \sigma(\xi^k) &= \xi^{-(k+1)}, & \sigma(\xi^{-(k+1)}) &= \xi \\
\text{or} & \\
\sigma(\xi) &= \xi^{-(k+1)}, & \sigma(\xi^k) &= \xi, & \sigma(\xi^{-(k+1)}) &= \xi^k
\end{align*}
\]
If $k = p/q$, using that $\sigma(x^{1/q}) = \pm \sigma(x)^{1/q}$, we obtain $\xi^{-k-1} = \sigma(\xi^k) = \sigma(\xi^{p/q}) = \sigma((\xi^p)^{1/q}) = (\sigma(\xi^p))^{1/q} = \xi^{k^2}$ in the first case and $\xi = \sigma(\xi^p) = \xi^{(-k-1)k}$ in the second. This implies that $k^2 + k + 1 = 0$ which is impossible. Thus $\xi \notin \mathbb{Q}$ and the lemma is proved.

**Example:** Now we give an example of a Lie flow on a compact manifold $M$ transversely modeled over a Lie algebra $G$ of the family $G_7$ with structural Lie algebra of dimension 2.

Let

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$

be an element of $SL(3; \mathbb{R})$.

The eigenvalues are $\lambda_j = 2 + 2 \cos(\frac{6\pi j - 4\pi}{9})$ where $j = 1, 2, 3$. We have $2 + 2 \cos(\frac{8\pi}{9}) < 1 < 2 + 2 \cos(\frac{14\pi}{9}) < 2 + 2 \cos(\frac{2\pi}{9})$. If we let $\xi = \lambda_2$, there is a $k < 0$ such that $\xi^k = \lambda_3$. In this case $\lambda_1 = \xi^{-k-1}$. Here $k$ is the quotient of logarithms of algebraic numbers. Notice that this is a necessary condition for the corresponding algebra to be realizable.

Thus we have the eigenvectors

$u_j = (\lambda_j - 3, \lambda_j(\lambda_j - 3) - 1, 1)$

A computation shows that the components of this vectors are irrational numbers with irrational quotient, i.e. they induce dense linear flows in $T^3$.

Now we consider the compact manifold $T^4_A = T^3 \times \mathbb{R}/ \sim$ where $(x, t) \sim (A \cdot x, t + 1)$. As the direction given by $u_1$ is invariant by $A$ it induces a global flow in $T^4_A$. This flow is transversely modeled over the Lie algebra of $G_7$ with $k = \log \lambda_3/\log \lambda_2 < 0$. To verify this we observe that an invariant tranverse parallelism in $T^3 \times \mathbb{R}$ is given by

\[
\begin{align*}
e_1 &= \xi^t u_2 \\
e_2 &= \xi^{kt} u_3 \\
e_3 &= -\frac{1}{\log \xi} \frac{\partial}{\partial t}
\end{align*}
\]

and it satisfies that $[e_1, e_2] = 0$, $[e_2, e_3] = e_1$, $[e_1, e_3] = ke_2$.

**Remark:** We do not know any realization of $G_8$ with $h \neq 0$ and codim $\mathcal{F} = 1$. 

4 Case $\text{codim} \mathcal{F} = 0$.

This is a trivial case because the transverse algebra coincides with the structural algebra and so it is abelian. Only $\mathcal{G}_1$ is realizable (a linear dense flow on $T^4$).

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