INTEGRAL GEOMETRY AND GEOMETRIC INEQUALITIES IN HYPERBOLIC SPACE.

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Abstract. Using results from integral geometry, we find inequalities involving mean curvature integrals of convex hypersurfaces in hyperbolic space. Such inequalities generalize the Minkowski formulas for euclidean convex sets.

1. Introduction and results

In hyperbolic space the following isoperimetric-like inequality is well-known (cf.[5])

$$\mathrm{vol}(\partial Q) > (n-1)\mathrm{vol}(Q) \tag{1}$$

for any convex domain $Q \subset \mathbb{H}^n$. This shows a strong contrast with euclidean geometry where these two volumes can not be *lineraly* compared, since for instance they are affected differently by homothetical transformations of Q. Indeed, the isoperimertic inequality in euclidean space is

$$(\mathrm{vol}(\partial Q))^n \geq c(\mathrm{vol}(Q))^{n-1} \tag{2}$$

for a constant c. More generally, the Minkowski inequalities for euclidean convex domains Q take the form

$$(W_r(Q))^s > c(W_s(Q))^r \qquad r > s \tag{3}$$

where W_i are the so-called *Quermassintegrale* whose definition is recalled a few lines below. Again, the exponents correct the different dimensions of the magnitudes, and again this will not be necessary in hyperbolic space. Indeed, the aim of this paper is to generalize (1) by finding *linear* geometric inequalities for convex domains in hyperbolic space analogue to (3). Before, let us recall how the Quermassintegrale of an euclidean convex domain are defined in the frame of integral geometry

$$W_r(Q) = \frac{(n-r)\cdot O_{n-1}}{n \cdot O_{n-r-1} \cdot \mathrm{vol}(G(n-r,n))} \int_{G(n-r,n)} \mathrm{vol}_{n-r}(\pi_V(Q))\mathrm{d}V$$

where π_V is the orthogonal projection onto the $(n-r)$-dimensional linear subspace V, and $\mathrm{d}V$ is the natural (invariant) measure in the Grassmannian of such subspaces. Here and in the following $O_i = \mathrm{vol}(\mathbb{S}^i)$. Alternatively,

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the Quermassintegrale are, up to constants, the measure of the set of affine subspaces intersecting the convex body (cf. [8]). Namely,

$$W_r(Q) = \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}}\int_{\mathcal{L}_r}\chi(L\cap Q)\mathrm{d}L \tag{4}$$

where $\mathcal{L}_r$ is the space of r-dimensional affine subspaces L, endowed with its natural (invariant) measure $\mathrm{d}L$. Here and in the following the function χ is just given by $\chi(Q)=1$ whenever $Q\neq\emptyset$, and $\chi(\emptyset)=0$. In case that ∂Q is C^2-differentiable, the Quermassintegrale coincide with the *total mean curvatures* of the boundary

$$W_r(Q) = nM_r(\partial Q) := n\int_{\partial Q}\sigma_r(x)\mathrm{d}x$$

where σ_r and $\mathrm{d}x$ are respectively the r-th mean curvature and the volume element of ∂Q.

Therefore, in order to generalize (3), the first point is to clarify the notion of Quermassintegrale for hyperbolic convex domains. It is easy to see that the average of the projections onto geodesic subspaces by some origin, depends on the choice of this origin. However, one can take (4) as a definition. For a (geodesically) convex domain $Q\subset\mathbb{H}^n$ we define

$$W_r(Q) := \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}}\int_{\mathcal{L}_r}\chi(L\cap Q)\mathrm{d}L$$

where $\mathcal{L}_r$ is the space of r-dimensional totally geodesic subspaces $L\subset\mathbb{H}^n$, and $\mathrm{d}L$ is the natural (invariant) measure on it (cf. [8]). As in the euclidean case we take $W_0(Q)=\mathrm{vol}(Q)$, and $W_n(Q)=O_{n-1}/n$. With these definitions, the Quermassintegrale do not coincide with the total mean curvatures, but they are closely related (cf. [11])

$$M_r(\partial Q) = n\left(W_{r+1}(Q)+\frac{r}{n-r+1}W_{r-1}(Q)\right). \tag{5}$$

Therefore we are concerned with inequalities between Quermassintegrale, and also between total mean curvatures. The main results are the following.

Theorem 1.1. *For any convex domain $Q\subset\mathbb{H}^n$*

$$W_r(Q) > \frac{n-r}{n-s}\,W_s(Q) \qquad r>s. \tag{6}$$

When ∂Q is C^2-differentiable

$$M_r(\partial Q) > c\ \mathrm{vol}(\partial Q) \tag{7}$$

with $c=1$ if $r>1$, and $c=(n-2)/(n-1)$ for $r=1$.

The inequalities (6) will be obtained in a very geometric way, and they will be used to get (7) by means of the relation (5). Note that the mean curvature integrals are differential geometric invariants with interest outside the field of integral geometry. For instance they appear in Weyl's tube formula as well as in Steiner's formula. However, in order to get inequalities between them, we will make strong use of their relation to the Quermassintegrale, which come from the field of integral geometry. Inequalities of the form (7) have also been studied in [2, 3], but only for stronger notions of convexity (such as convexity with respect to horospheres).

As an application of (6), we prove that the expected volume of a random r-dimensional totally geodesic slice $L \cap Q$ of any given domain Q in hyperbolic space is bounded above

$$E[\mathrm{vol}(L \cap Q)] \leq \frac{O_{n-1}}{O_{n-r-1}}.$$

This surprising fact illustrates the importance of the *linearity* of (6).

2. Inequalities between Quermassintegrale

In this section we prove inequalities of the form (6). The first step is to get similar inequalities for convex domains in $\mathbb{S}^n$. The totally geodesic r-dimensional spheres in $\mathbb{S}^n$, which are obtained by intersecting with $(r+1)$-dimensional linear subspaces of $\mathbb{R}^{n+1}$, will be denoted S_r. The space of such S_r is the Grassmann manifold $G(r+1, n+1)$, and has a unique (up to constants) measure $\mathrm{d}S_r$ invariant under rotations (cf.[8]).

Proposition 2.1. *Let Q be a convex set in $\mathbb{S}^n$. Then, for $r \leq n-1$ and $s \leq n-r-1$*

$$\int_{G(r+1,n+1)} \chi(S_r \cap Q)\, \mathrm{d}S_r \leq \frac{O_{r+s} \ldots O_{r+1}}{O_{n-r-1} \cdots O_{n-r-s}} \int_{G(r+s+1,n+1)} \chi(S_{r+s} \cap Q)\, \mathrm{d}S_{r+s}$$

and equality holds only when Q is a hemisphere of $\mathbb{S}^n$.

Proof. Denote $G(r+1, r+s+1, n+1)$ the flag space consisting of pairs $S_r \subset S_{r+s}$ of geodesic spheres of $\mathbb{S}^n$. It is known (cf.[8]) that

$$\mathrm{d}S_{(r+s)[r]} \mathrm{d}S_r = \mathrm{d}S_{[r+s]r} \mathrm{d}S_{r+s} \tag{8}$$

where $\mathrm{d}S_{[r+s]r}$ is the measure on the grassmannian of great r-spheres contained in S_{r+s} and $\mathrm{d}S_{(r+s)[r]}$ is the measure of $(r+s)$-spheres containing S_r.

Since $S_r \cap Q \subset S_{r+s} \cap Q$, we have

$$\begin{aligned}
\mathrm{vol}(G(r+1, r+s+1)) &\int_{G(r+s+1,n+1)} \chi(S_{r+s} \cap Q)\, \mathrm{d}S_{r+s} = \\
&= \int_{G(r+1,r+s+1,n+1)} \chi(S_{r+s} \cap Q)\, \mathrm{d}S_{[r+s]r} \mathrm{d}S_{r+s} \geq \\
&\geq \int_{G(r+1,r+s+1,n+1)} \chi(S_r \cap Q)\, \mathrm{d}S_{(r+s)[r]} \mathrm{d}S_r = \\
&= \mathrm{vol}(G(s, n+1-r)) \int_{G(r+1,n+1)} \chi(S_r \cap Q)\, \mathrm{d}S_r.
\end{aligned}$$

□

To prove the analogues in hyperbolic space we need a proper expression of the measure of geodesic planes $\mathrm{d}L_r$. Fix an origin $o \in \mathbb{H}^n$. Now every r-plane in $\mathbb{H}^n$ is determined by the $(n-r)$-plane through o orthogonal to L_r, and by the intersection point $x = L_r \cap L_{n-r}$. This way, $\mathcal{L}_r$ is identified

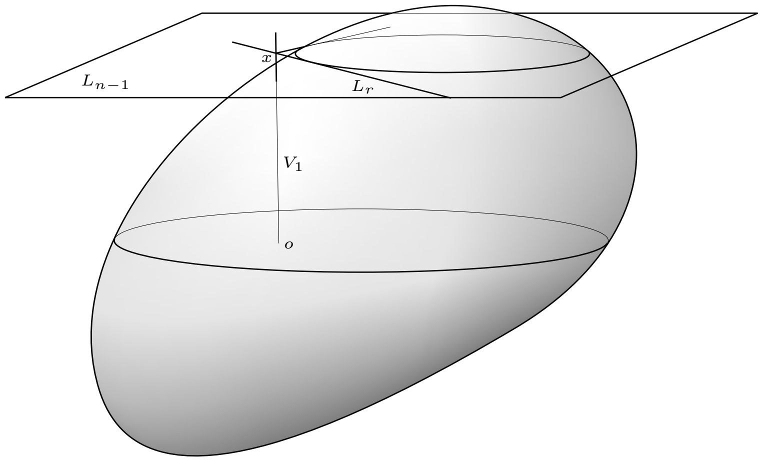

FIGURE 1. r-planes meeting Q

to the tautological bundle over $G(n-r,n)$, and the invariant measure is written (cf.[8])

$$\mathrm{d}L_r = \cosh^r \rho \, \mathrm{d}x \, \mathrm{d}V_{n-r} \tag{9}$$

where ρ is the distance from x to o, $\mathrm{d}x$ is the volume element on L_{n-r}, and $\mathrm{d}V_r$ is the volume element on $G(n-r,n)$ corresponding to $V_{n-r} = T_o L_{n-r}$.

Proposition 2.2. *Let Q be a convex domain in $\mathbb{H}^n$ contained in a ball of radius R. Then, for $r \leq n-1$ and $1 \leq s \leq n-r-1$*

$$\int_{\mathcal{L}_r} \chi(L_r \cap Q) \, \mathrm{d}L_r < \tanh^s(R) \frac{O_{r+s-1} \ldots O_r}{O_{n-r-2} \cdots O_{n-r-s-1}} \int_{\mathcal{L}_{r+s}} \chi(L_{r+s} \cap Q) \, \mathrm{d}L_{r+s}.$$

Proof. We can assume the center o of the sphere to be in the interior of Q. Using the expression (9) for the measure of r-planes,

$$\int_{\mathcal{L}_r} \chi(L_r \cap Q) \, \mathrm{d}L_r = \int_{G(n-r,n)} \int_{V_{n-r}} \chi(L_r \cap Q) \cosh^r \rho \, \mathrm{d}x \mathrm{d}V_{n-r}.$$

Let us write $\mathrm{d}x$, the volume element of L_{n-r}, in polar coordinates. Since L_{n-r} is isometric to $\mathbb{H}^{n-r}$,

$$\int_{\mathcal{L}_r} \chi(L_r \cap Q) \, \mathrm{d}L_r = \int_{G(n-r,n)} \int_{\mathbb{RP}^{n-r-1}} \int_{\mathbb{R}} \chi(L_r \cap Q) \cosh^r \rho |\sinh^{n-r-1} \rho| \, \mathrm{d}\rho \mathrm{d}u \mathrm{d}V_{n-r}$$

where $\mathrm{d}u$ is the volume element of $\mathbb{RP}^{n-r-1}$ corresponding to the initial vector of the ray going from o to x. The formula (8) gives in this setting that $\mathrm{d}u \mathrm{d}V_{n-r} = \mathrm{d}V_{(n-r)[1]} \mathrm{d}V_1$ where $\mathrm{d}V_{(n-r)[1]}$ is the measure of the V_{n-r} containing V_1. Then,

$$\int_{\mathcal{L}_r} \chi(L_r \cap Q) \, \mathrm{d}L_r = \int_{\mathbb{RP}^{n-1}} \int_{G(n-r-1,(V_1)^\perp)} \int_{\mathbb{R}} \chi(L_r \cap Q) \cosh^r \rho |\sinh^{n-r-1} \rho| \, \mathrm{d}\rho \mathrm{d}V_{(n-r)[1]} \mathrm{d}V_1$$

$$= \int_{\mathbb{RP}^{n-1}} \int_{\mathbb{R}} \left(\int_{G(r,(V_1)^\perp)} \chi(L_r \cap Q) \, \mathrm{d}V_r \right) \cosh^r \rho |\sinh^{n-r-1} \rho| \, \mathrm{d}\rho \mathrm{d}V_1.$$

Now, given V_1 and ρ (i.e. given x), we projectivize (from x) the hyperplane L_{n-1} orthogonal to V_1 in x. The integral between parenthesis is the measure of the set of geodesic $(r-1)$-planes meeting a convex set in $\mathbb{S}^{n-2}$. Applying proposition 2.1 this measure is bounded in terms of the measure of $(r+s-1)$-planes meeting this convex set in $\mathbb{S}^{n-2}$. We get

$$\int_{G(r,(V_1)^\perp)} \chi(L_r \cap Q)\,\mathrm{d}V_r \leq \frac{O_{r+s-1}\ldots O_r}{O_{n-r-2}\cdots O_{n-r-s-1}} \int_{G(r+s,(V_1)^\perp)} \chi(L_{r+s}\cap Q)\,\mathrm{d}V_{r+s}.$$

And the proof is finished since $-R \leq \rho \leq R$, and thus

$$\cosh^r \rho |\sinh^{n-r-1}\rho| \leq \tanh^s R \cosh^{r+s}\rho |\sinh^{n-r-s-1}\rho|.$$

□

In terms of Quermassintegrale, the previous inequality becomes

$$W_r(Q) < \tanh^s R \frac{n-r}{n-r-s} W_{r+s}(Q).$$

In particular we have the inequality (6).

Corollary 2.3. *If $Q \subset \mathbb{H}^n$ is convex, then*

$$W_r(Q) < \frac{n-r}{n-r-s} W_{r+s}(Q). \tag{10}$$

In the case $r = 0$, the inequalities can be improved.

Proposition 2.4. *Let $Q \subset \mathbb{H}^n$ be a convex set contained in a ball $B(R)$ with radius R. Then*

$$\frac{\mathrm{vol}(Q)}{W_r(Q)} \leq \frac{\mathrm{vol}(B(R))}{W_r(B(R))} \tag{11}$$

with equality only for $Q = B(R)$.

Proof. Let us compute the volume of Q in polar coordinates from the center of $B(R)$

$$\mathrm{vol}(Q) = \int_{\mathbb{S}^{n-1}} \int_0^{l(u)} \sinh^{n-1}\rho \mathrm{d}\rho \mathrm{d}u$$

where $l(u)$ is the length of the geodesic segment $\gamma(u)$ starting at the origin with tangent vector $u \in \mathbb{S}^{n-1}$ and ending at ∂Q. Since all the hyperplanes orthogonal to $\gamma(u) \cap Q$ meet Q, we have

$$W_r(Q) \geq \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n \cdot O_{n-2}\cdots O_{n-r-1}} \int_{\mathbb{S}^{n-1}} \int_0^{l(u)} \cosh^r \rho \sinh^{n-r-1}\rho \mathrm{d}\rho \mathrm{d}u$$

On the other hand it is easy to see that the function

$$f(R) = \frac{W_r(B(R))}{\mathrm{vol}(B(R))} = \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}} \frac{\int_0^R \cosh^r\rho \sinh^{n-r-1}\rho \mathrm{d}\rho}{\int_0^R \sinh^{n-1}\rho \mathrm{d}\rho}$$

is increasing. Thus, since $l(u) \leq R$, we have $f(l(u)) \leq f(R)$ and then

$$\begin{aligned} W_r(Q) \geq \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}} \int_{\mathbb{S}^{n-1}} \int_0^{l(u)} \cosh^r \rho \sinh^{n-r-1} \rho \mathrm{d}\rho = \\ = \int_{\mathbb{S}^{n-1}} f(l(u)) \int_0^{l(u)} \sinh^{n-1} \rho \mathrm{d}\rho \mathrm{d}u \geq \\ \geq \int_{\mathbb{S}^{n-1}} f(R) \int_0^{l(u)} \sinh^{n-1} \rho \mathrm{d}\rho \mathrm{d}u = \frac{W_r(B(R))}{\mathrm{vol}(B(R))} \mathrm{vol}(Q). \end{aligned}$$

□

Remark. The previous inequalities run in the only possible direction. Indeed, an inequality of the form $W_{r+s}(Q) \leq cW_r(Q)$ can not be true. To see this, take a convex domain Q contained in a geodesic $(n-r-s)$-plane. Since Q is an $(n-r-s)$-dimensional submanifold, by the Cauchy-Crofton formula (cf.[8]), $W_{r+s}(Q)$ is a multiple of its $(n-r-s)$-dimensional volume, while the set of r-planes meeting Q has null measure. Thus $W_r(Q) = 0$, and $W_{r+s}(Q) > 0$.

3. Inequalities for the mean curvature integrals

Now we are ready to find inequalities involving the mean curvature integrals of convex hypersurfaces in $\mathbb{H}^n$. The most interesting case is that of (6) which we prove next.

Proposition 3.1. *If $Q \subset \mathbb{H}^n$ is convex, and ∂Q is C^2-differentiable then, for $r > 1$*

$$\frac{M_r(\partial Q)}{\mathrm{vol}(\partial Q)} > 1,$$

and the bound is sharp. For $r = 1$,

$$\frac{M_1(\partial Q)}{\mathrm{vol}(\partial Q)} > \frac{n-2}{n-1}.$$

In other words, the mean value of the higher order mean curvatures of a convex hyperbolic hypersurface is greater than 1, and the mean value of the first mean curvature is also bounded below.

Proof. Thanks to equation (5), which relates the mean curvature integrals to the Quermassintegrale, and thanks to inequality (10), we have

$$\frac{M_r(\partial Q)}{M_0(\partial Q)} = \frac{W_{r+1}(Q) + \frac{r}{n-r+1} W_{r-1}(Q)}{W_1(Q)} > \frac{n-r-1}{n-1} + \frac{r}{n-r+1}\frac{n-r+1}{n-1} = 1.$$

To prove the sharpness, consider a ball $B(R)$ of radius R. Then (cf. [8])

$$M_r(\partial B(R)) = O_{n-1} \cosh^r R \sinh^{n-1} R,$$

and

$$\frac{M_r(\partial B(R))}{\mathrm{vol}\partial B(R)} = \coth^r R$$

which is arbitrarily close to 1 for R big enough.

For $r = 1$, we use (5) and (10)

$$\frac{M_1(\partial Q)}{M_0(\partial Q)} = \frac{W_2(Q) + \frac{1}{n} W_0(Q)}{W_1(Q)} > \frac{W_2(Q)}{W_1(Q)} > \frac{n-2}{n-1}.$$

□

Remark. Although it is not clear if the bound for $M_1(\partial Q)/\mathrm{vol}(\partial Q)$ is sharp, it must be noticed that, at least in $\mathbb{H}^3$, the greatest lower bound is smaller than 1. Indeed, take a plane disk $Q \subset L_2 \subset \mathbb{H}^3$ of radius R,

$$M_1(\partial Q) = 3 \left(W_2(Q) + \frac{1}{3} W_0(Q) \right) = \pi^2 \sinh R$$

$$\mathrm{vol}(\partial Q) = 4\pi(\cosh R - 1)$$

then $M_1(\partial Q)/\mathrm{vol}(\partial Q)$ goes to $\pi/4$ (which is smaller than 1) when R grows.

We can also compare the mean curvature integrals with the volume of the interior.

Corollary 3.2. *The following inequality holds for convex sets in* $\mathbb{H}^n$

$$\frac{M_r(\partial Q)}{\mathrm{vol}(Q)} > n - 1$$

and the bound is sharp.

Proof. The case $r = 0$ coincides with (1). Note also that it follows immediately from (10), and we have thus given a new proof of this know inequality. For $r > 1$, by applying proposition 3.1 and (1), we get

$$\frac{M_r(\partial Q)}{\mathrm{vol}(Q)} = \frac{M_r(\partial Q)}{\mathrm{vol}(\partial Q)} \cdot \frac{\mathrm{vol}(\partial Q)}{\mathrm{vol}(Q)} > 1 \cdot (n-1).$$

For $r = 1$

$$\frac{M_1(\partial Q)}{\mathrm{vol}(Q)} = \frac{n(W_2(Q) + \frac{1}{n}\mathrm{vol}(Q))}{\mathrm{vol}(Q)} > n\frac{n-2}{n} + 1 = n - 1.$$

The bound can be approximated by taking balls of big radius. □

In a similar way, we can find estimations for any quotient of mean curvature integrals.

Proposition 3.3. *If* $Q \subset \mathbb{H}^n$ *is convex then, for* $r \geq 0$ *and* $1 < s \leq n-r-1$

$$\frac{M_{r+s}(\partial Q)}{M_r(\partial Q)} > 1$$

and the bound is sharp. For $s = 1$*,*

$$\frac{M_{r+1}(\partial Q)}{M_r(\partial Q)} > \frac{n-r-2}{n-r-1}.$$

Proof. Use again the equation (5) and the inequality (10)

$$\frac{M_{r+s}(\partial Q)}{M_r(\partial Q)} = \frac{W_{r+s+1}(Q) + \frac{r+s}{n-r-s+1}W_{r+s-1}(Q)}{W_{r+1}(Q) + \frac{r}{n-r+1}W_{r-1}(Q)} > \tag{12}$$

$$= \frac{\frac{n-r-s-1}{n-r-s+1}W_{r+s-1}(Q) + \frac{r+s}{n-r-s+1}W_{r+s-1}(Q)}{W_{r+1}(Q) + \frac{r}{n-r+1}\frac{n-r+1}{n-r-1}W_{r+1}(Q)} =$$

$$= \frac{n-r-1}{n-r-s+1}\frac{W_{r+s-1}(Q)}{W_{r+1}(Q)} > \frac{n-r-1}{n-r-s+1}\frac{n-r-s+1}{n-r-1} = 1.$$

This is sharp since for a sequence of balls the quotient M_{r+s}/M_r approaches 1 as the radius grows to ∞.

For $s = 1$,

$$\frac{M_{r+1}(\partial Q)}{M_r(\partial Q)} = \frac{W_{r+2}(Q) + \frac{r+1}{n-r}W_r(Q)}{W_{r+1}(Q) + \frac{r}{n-r+1}W_{r-1}(Q)}.$$

And we finish since

$$\frac{W_{r+2}(Q)}{W_{r+1}(Q)} > \frac{n-r-2}{n-r-1} \qquad \frac{\frac{r+1}{n-r}W_r(Q)}{\frac{r}{n-r+1}W_{r-1}(Q)} > \frac{r+1}{r} > 1 > \frac{n-r-2}{n-r-1}.$$

□

Remark. Note that the bound for the quotient M_{n-1}/M_{n-2} is 0. Below we construct convex domains for which this quotient actually takes arbitrarily small values.

In short, we have found lower bounds for all the quotients M_{r+s}/M_r. Except from the case M_{n-1}/M_{n-2}, these bounds are strictly positive, and they are sharp when $s > 1$.

It is natural to look for upper bounds of such quotients (or equivalently lower bounds of M_r/M_{r+s}). However, it is immediate to see that these quotients are not bounded from above. For instance, for a radius ball R one has $M_{r+s}/M_r = \coth^r R$, which is arbitrarily big if R is small enough. One could also argue noting that in euclidean space there are examples of arbitrarily small convex sets with arbitrarily big M_{r+s}/M_r. Since in *small* neighborhoods of a point the metrics of $\mathbb{H}^n$ and $\mathbb{R}^n$ are very *similar*, there must be convex bodies in hyperbolic space with big M_{r+s}/M_r.

However, if we restrict ourselves to convex bodies that are *big* in some sense, it is possible to find some upper bounds for M_{r+s}/M_r.

Proposition 3.4. *Let* (Q_t) *be a sequence of convex sets such that* $\mathrm{vol}(\partial Q_t)$ *goes to infinity. Then*

$$i) \quad \lim_{n\to\infty} \frac{M_{n-1}(\partial Q_t)}{M_{n-2}(\partial Q_t)} \leq n-1$$

$$ii) \quad \lim_{n\to\infty} \frac{M_{n-1}(\partial Q_t)}{M_{n-3}(\partial Q_t)} \leq \frac{n-1}{2}$$

Remark. For stronger notions of convexity (such as convexity with respect to horospheres), the limit values of the quotients $M_i(\partial Q)/\mathrm{vol}(\partial Q)$ and $\mathrm{vol}(\partial Q)/\mathrm{vol}(Q)$ have been studied (also for general negatively curved manifolds) in [10, 6, 7, 4, 1, 3].

Proof. We have that

$$\frac{M_{n-1}(\partial Q_t)}{M_{n-2}(\partial Q_t)} = \frac{W_n(Q_t) + \frac{n-1}{2}W_{n-2}(Q_t)}{W_{n-1}(Q_t) + \frac{n-2}{3}W_{n-3}(Q_t)}$$

But $W_n(Q_t)$ is constant O_{n-1}/n. On the other hand $W_{n-1}(Q_t)$, $W_{n-2}(Q_t)$ and $W_{n-3}(Q_t)$ go to infinity if $\mathrm{vol}(\partial Q_t)$ expands over $\mathbb{H}^n$. Therefore

$$\lim_{n\to\infty} \frac{M_{n-1}(\partial Q_t)}{M_{n-2}(\partial Q_t)} = \frac{n-1}{2} \lim_{n\to\infty} \frac{W_{n-2}(Q_t)}{W_{n-1}(Q_t) + \frac{n-2}{3}W_{n-3}(Q_t)} \leq$$

$$\leq \frac{n-1}{2} \lim \frac{W_{n-2}(Q_t)}{W_{n-1}(Q_t)}$$

Bearing in mind that $W_{n-2}/W_{n-1} < 2$, we have proved *i*). Analogously one proves *ii*).

$$\frac{M_{n-1}(\partial Q_t)}{M_{n-3}(\partial Q_t)} \sim \frac{n-1}{2} \frac{W_{n-2}(Q_t)}{W_{n-2}(Q_t) + \frac{n-3}{4}W_{n-4}(Q_t)} \leq \frac{n-1}{2}.$$

□

The second inequality is sharp. Moreover, these are the only cases where upper bounds are possible. To see this, consider the following sequence of convex sets. Given $t > 0$ consider a geodesic segment of length t and the set Q_t^ϵ consisting of points at a distance smaller than ϵ from the segment. Except from the two spherical caps centered at the endpoints, the boundary of Q_t^ϵ has one normal curvature equal to $\coth\epsilon$, and the rest are equal to $\tanh\epsilon$. The volume of this part of the boundary is $O_{n-2}t\sinh\epsilon\cosh^{n-2}\epsilon$. Hence, for big t

$$M_r(\partial Q_t^\epsilon) \sim O_{n-2}t\frac{\binom{n-2}{r-1}\coth^{r-2}\epsilon + \binom{n-2}{r}\coth^r\epsilon}{\binom{n-1}{r}}\sinh^{n-1}\epsilon\cosh\epsilon =$$

$$= \frac{tO_{n-2}}{n-1}((r\coth^{r-2}\epsilon + (n-r-1)\coth^r\epsilon)\sinh^{n-2}\epsilon\cosh\epsilon.$$

so that

$$\frac{M_r(\partial Q_t^\epsilon)}{M_s(\partial Q_t^\epsilon)} \sim \frac{r\coth^{r-2}\epsilon + (n-r-1)\coth^r\epsilon}{s\coth^{s-2}\epsilon + (n-s-1)\coth^s\epsilon}.$$

Now, for $r > s$ and small ϵ, this quotient takes arbitrarily big values, except for the cases $r = n-1, s = n-2$ and $r = n-1, s = n-3$. In these cases the limit can take any value in $[0, 1]$ and $[1, (n-1)/2]$ respectively.

4. Slice expectation for random geodesic planes

We end by giving an application of the inequalities for Quermassintegrale to the following problem of geometric probability. Throw randomly (according to the invariant measure $\mathrm{d}L_r$) a geodesic r-plane L_r of $\mathbb{H}^n$ to intersect a given (not necessarily convex) domain $Q \subset \mathbb{H}^n$. Consider the random variable given by the r-dimensional volume of the intersection of L_r with Q. We are concerned with the expectation of this random variable

$$E[\mathrm{vol}(L_r \cap Q)] = \frac{\int_{\mathcal{L}_r} \mathrm{vol}(L_r \cap Q)\,\mathrm{d}L_r}{\int_{\{L_r\cap Q\neq\emptyset\}} \mathrm{d}L_r}.$$

A surprising fact is that this expectation is bounded.

Proposition 4.1. *The expectation for the volume of the intersection of a random r-plane with a domain Q in $\mathbb{H}^n$ is bounded by*

$$E[\mathrm{vol}(L_r \cap Q)] < E[\mathrm{vol}(L_r \cap B)] < \frac{O_{n-1}}{O_{n-r-1}}.$$

where B is a ball containing Q.

Proof. Take $\overline{Q}$ the convex hull of Q. Then

$$E[\mathrm{vol}(L_r \cap Q)] < E[\mathrm{vol}(L_r \cap \overline{Q})] = \frac{(n-r)O_{n-1}O_0}{nO_{n-r-1}} \frac{\mathrm{vol}(\overline{Q})}{W_r(\overline{Q})}.$$

And we finish using the previous inequality. □

Remark. If $Q \subset Q'$ are convex domains, it is not clear whether $E[\mathrm{vol}(L_r \cap Q)] \leq E[\mathrm{vol}(L_r \cap Q')]$.

In hyperbolic plane it was known that the expectation of a random chord is below π (cf. [9]). In higher dimensions the previous estimations seem to be new. As an example, let us mention that the expectation of a random chord in $\mathbb{H}^3$ is below π or that the expected area of a random plane slice is below 2π.

References

[1] A. A. Borisenko, E. Gallego, and A. Reventós. Relation between area and volume for λ-convex sets in Hadamard manifolds. *Differential Geom. Appl.*, 14(3):267–280, 2001.

[2] A. A. Borisenko and V. Miquel. Total curvatures of convex hypersurfaces in hyperbolic space. *Illinois J. Math.*, 43(1):61–78, 1999.

[3] A. A. Borisenko and V. Miquel. Comparison theorems on convex hypersurfaces in Hadamard manifolds. *Ann. Global Anal. Geom.*, 21(2):191–202, 2002.

[4] A. A. Borisenko and D. I. Vlasenko. Convex surfaces in Lobachevskiĭ space. *Mat. Fiz. Anal. Geom.*, 4(3):278–285, 1997.

[5] Y. D. Burago and V. A. Zalgaller. *Geometric inequalities*, volume 285 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1988.

[6] E. Gallego and A. Reventós. Asymptotic behavior of convex sets in the hyperbolic plane. *J. Differential Geom.*, 21(1):63–72, 1985.

[7] E. Gallego and A. Reventós. Asymptotic behaviour of λ-convex sets in the hyperbolic plane. *Geom. Dedicata*, 76(3):275–289, 1999.

[8] L. A. Santaló. *Integral geometry and geometric probability.* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 1.

[9] L. A. Santaló. Notes on the integral geometry in the hyperbolic plane. *Portugal. Math.*, 39(1-4):239–249, 1980.

[10] L. A. Santaló and I. Yañez. Averages for polygons formed by random lines in Euclidean and hyperbolic planes. *J. Appl. Probability*, 9:140–157, 1972.

[11] G. Solanes. Integrals de curvatura i geometria integral a l'espai hiperblic. *Univ.Aut.Barclona, PhD. Thesis*, 2003.

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