Continuous Fields of C*-algebras, their Cuntz Semigroup and the Geometry of Dimension Functions

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*Figure: Sea Star*
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2. Let \( A \) be a finite dimensional C*-algebra. Then there exist \( n_1, \ldots, n_r \in \mathbb{N} \) such that

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A \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C}).
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Theorem

1. Any unital commutative $C^*$-algebra is isomorphic to $C(X)$ for some compact space $X$.

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Property: The category of C*-algebras has inductive limits.
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Example

$AF$-algebras is the class of C*-algebras built as inductive limits of finite-dimensional C*-algebras. An important subclass of them are UHF-algebras.
Dimension theory for C*-algebras

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For C*-algebras there are two well-known notions of dimension called 
stable rank (Rieffel, 83) and 
real rank (Brown, Pedersen, 91).

Definition

We say that a unital C*-algebra $A$ has 
stable rank one, $sr(A) = 1$, if the set of 
invertibles in $A$ is dense in $A$. And $A$ has 
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Examples

If $A = C(X)$ for a compact zero-dimensional space $X$, it follows that 
$RR(A) = 0$, and $sr(A) = 1$.

If $A = M_n(C)$, then $RR(A) = 0$. And, furthermore, 
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The range of the above invariant consists of the class of dimension groups (Effros-Handelman-Shen)
Classification of C*-algebras

Conjecture (Elliott, circa 1989-Elliott Program)

There is a complete functor $F$ from the category of separable, simple and nuclear C*-algebras which is constructed from $K$-theory and the simplex of traces $\mathcal{T}(A)$. 

Usual form of the invariant:

$\text{Ell}(A) = (\langle K_0(A), K_0(A) + \mathbb{Z}, [1_A] \rangle, K_1(A), \mathcal{T}(A), r_A)$

where $r_A : \mathcal{T}(A) \times K_0(A) \rightarrow \mathbb{R}$ is the pairing between $K_0(A)$ and $\mathcal{T}(A)$ given by evaluation of a trace on a $K_0$-class.
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Kirchberg Algebras: unital, purely infinite, simple, separable and nuclear C*-algebras.

unital, purely infinite, simple: $\forall a \neq 0 \in A \exists x, y \in A$ such that $xay = 1$.

Non-stably finite and simple $\Rightarrow \mathcal{T}(A) = \emptyset$, but is non-stably finite=purely infinite? (simple case)
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Counterexamples

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*A simple, nuclear C*-algebra which is neither stably finite nor purely infinite. (Contains a finite and an infinite projection.)*
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**Example (Rørdam, 2003)**

A simple, nuclear $C^*$-algebra which is *neither stably finite nor purely infinite*. (Contains a finite and an infinite projection.)

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Two unital simple $C^*$-algebras that agree on: *Elliott invariant*, real rank, stable rank and other continuous isomorphism invariants.

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These algebras were distinguished by their Cuntz semigroup $W(\_)$.
Introduction

The Cuntz Semigroup

It was introduced by Cuntz in 1978 modelling the construction of the Murray-von Neumann semigroup $V(A)$.

**Definition (V(A)-The Murray-von Neumann semigroup)**

Let $A$ be a C*-algebra and denote by $P_n(A) = \{p \text{ a projection in } M_n(A)\}$. $p$ is M.-v.N. equivalent to $q$ in $P_n(A)$ ($p \sim_0 q$) if there exists $v \in M_n(A)$ such that $p = vv^*$ and $q = v^*v$.

Extending this relation to $P_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A)$, one defines the Murray-von Neumann semigroup of $A$ as $V(A) = P_\infty(A)/\sim_0$.

Denote the equivalence classes by $[p]$. The operation and order are given by $[p] + [q] = [p \circ 0 q]$, $[p] \leq [q]$ if $p \sim_0 p' \leq q$ (i.e. $p'q = p'$).

The order in $V(A)$ is algebraic. (i.e. if $[p] \leq [q] = \Rightarrow \exists [r]$ s.t. $[p] + [r] = [q]$.)
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**Definition (W(A)-The Cuntz semigroup)**

Let $A$ be a C*-algebra and $a, b \in A_+$. 

| $a$ is *Cuntz subequivalent* to $b$ $(a \lesssim b)$ | $\exists$ a sequence $(x_n)$ in $A$ such that $\|x_n b x_n^* - a\| \to 0$ |

---

$a \sim b$ if $a \lesssim b$ and $b \lesssim a$.
Definition (W(A)-The Cuntz semigroup)

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$a$ and $b$ are Cuntz equivalent if $a \preceq b$ and $b \preceq a$ (denoted $a \sim b$).
**Introduction**

**The Cuntz Semigroup**

**Definition (W(A)-The Cuntz semigroup)**

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**Definition (W(\(A\))-The Cuntz semigroup)**

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\[ a \text{ is Cuntz subequivalent to } b \quad (a \preccurlyeq b) \iff \exists \text{ a sequence } (x_n) \text{ in } A \text{ such that } \|x_nbx_n^* - a\| \to 0 \]

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The order in $W(A)$ is usually not the algebraic order.
Relation between $V(A)$ and $W(A)$
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Remark

- There is a natural map $V(A) \to W(A)$ defined by $[p] \mapsto \langle p \rangle$, which is injective if $A$ is stably finite.
- When $A$ is finite dimensional, it follows that $W(A) = V(A)$. 
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Definition

If we consider the Grothendieck group construction, we have the following:

$$G(V(A)) = K_0(A) \text{ (unital case)} \quad G(W(A)) = K_0^*(A).$$
Ell(A) and W(A), are they related?
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Theorem (Brown-Perera-Toms, ’08)

The Cuntz semigroup can be recovered from the Elliott invariant for a large class of C*-algebras.
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In fact, for simple, unital and finite C*-algebras A that are exact and \( \mathcal{Z} \)-stable, where \( \mathcal{Z} \) is the Jiang-Su algebra, it was proved that...
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**Theorem (Brown-Perera-Toms, '08)**

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In fact, for simple, unital and finite C*-algebras $A$ that are exact and $\mathcal{Z}$-stable, where $\mathcal{Z}$ is the Jiang-Su algebra, it was proved that

$$W(A) \cong V(A) \sqcup \text{LAff}(T(A))^{++}.$$
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Theorem (Antoine-Dadarlat-Perera-Santiago, '13, Tikuisis, '12)

The Elliott invariant can be recovered from the Cuntz semigroup after tensoring with the circle for the same class of C*-algebras as above.
Introduction

The Cuntz Semigroup

Continuity of $W(A)$
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- If $A$ is a C*-algebra of the form $A = \lim(A_i)$, then in general $W(A) \neq \lim W(A_i)$. 
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The assignment $A \mapsto W(A)$ does not preserve inductive limits.

Coward-Elliott-Ivanescu in 2008 defined $Cu(A)$ for any C*-algebra, which is a modified version of the Cuntz semigroup. In fact, $Cu(A)$ can be identified with $W(A \otimes K)$. 
Continuous Fields of C*-algebras, their Cuntz Semigroup and the Geometry of Dimension Functions

Joan Bosa Puigredon

Introduction

The Cuntz Semigroup

Continuity of $W(A)$

- If $A$ is a C*-algebra of the form $A = \lim_{i \to \infty} (A_i)$, then in general $W(A) \neq \lim_{i \to \infty} W(A_i)$.

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The assignment $A \mapsto W(A)$ does not preserve inductive limits.

Coward-Elliott-Ivanescu in 2008 defined $Cu(A)$ for any C*-algebra, which is a modified version of the Cuntz semigroup. In fact, $Cu(A)$ can be identified with $W(A \otimes K)$.

Properties

- $Cu(A)$ belongs to a category of semigroups called Cu that admits inductive limits that are not algebraic.
- The assignment $A \mapsto Cu(A)$ is sequentially continuous.
The category $\mathcal{Cu}$
The category $\mathbf{Cu}$

**Definition**

Let $a, b$ be elements in a partially ordered set $S$. Then, we will say that $a \ll b$ (way-below) if for any increasing sequence $\{y_n\}$ with supremum in $S$ such that $b \leq \sup(y_n)$, there exists $m$ such that $a \leq y_m$. 
The category $\text{Cu}$

**Definition**

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**Definition ($\text{Cu}$)**

An object of $\text{Cu}$ is a partially ordered semigroup with zero element $S$ such that:

- The order, in $S$, is compatible with the addition, i.e., if $x_i \leq y_i$, $i \in \{1, 2\}$ then $x_1 + x_2 \leq y_1 + y_2$,
- every increasing sequence in $S$ has a supremum,
- for all $x \in S$ there exists a sequence $\{x_n\}$ such that $x = \sup(x_n)$ where $x_n \ll x_{n+1}$,
- the relation $\ll$ and suprema are compatible with addition.

The maps of $\text{Cu}$ are those morphisms which preserve the order, the zero, the suprema of increasing sequences and the relation $\ll$. 
Remark

In fact, $\langle (a - \varepsilon)_+ \rangle \ll \langle a \rangle$ in Cu$(A)$ for all $\varepsilon > 0$ and for all $a \in A_+$. 
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In fact, $\langle (a - \varepsilon)_+ \rangle \ll \langle a \rangle$ in $\text{Cu}(A)$ for all $\varepsilon > 0$ and for all $a \in A_+$. 

Example

Let $X$ be a compact metric space. Then, if $\mathcal{O}(X)$ is the set of open sets in $X$ ordered by inclusion, it follows that $\mathcal{O}(X) \in \text{Cu}$. In this, we have that $U \ll V$ for $U, V \in \mathcal{O}(X)$, if there exists a compact subset $K \subseteq X$ such that $U \subseteq K \subseteq V$. 
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In fact, $\langle (a - \varepsilon)_+ \rangle \ll \langle a \rangle$ in $\text{Cu}(A)$ for all $\varepsilon > 0$ and for all $a \in A_+$.

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- Let $X$ be a finite-dimensional compact metric space, then $\text{Lsc}(X, \overline{\mathbb{N}}) \in \text{Cu}$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. 
Remark

In fact, $\langle (a - \varepsilon)_+ \rangle \ll \langle a \rangle$ in $\text{Cu}(A)$ for all $\varepsilon > 0$ and for all $a \in A_+$.

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Remark

The main difference between the classical and the stabilized Cuntz semigroup is that $W(A)$ is not necessarily closed with respect to suprema of increasing sequences.
1 Introduction

2 The Cuntz Semigroup of Continuous Fields of C*-algebras
   - Continuous Fields
   - Sheaves of semigroups
   - The sheaves on Cu
   - The sheaf $Cu_A(\_)$

3 The geometry of Dimension Functions

4 Local triviality for Continuous Fields of C*-algebras
Continuous Fields of C*-algebras, their Cuntz Semigroup and the Geometry of Dimension Functions

The Cuntz Semigroup of Continuous Fields of C*-algebras

C(X)-algebras

Definition

Let $X$ be a compact metric space. A unital $C(X)$-algebra is a C*-algebra $A$ together with a unital $*$-homomorphism $\theta: C(X) \to \mathbb{Z}(A)$, where $\mathbb{Z}(A)$ is the center of $A$.

Remark

A $C(X)$-algebra has the structure of $C(X)$-module. In particular, we write $f_\theta a$ instead of $\theta(f)a$ where $f \in C(X)$ and $a \in A$.

Notation

If $U \subset X$ is an open set, we denote $A(U) = C_0(U)A$, which is a closed ideal of $A$. (Cohen)

If $Y \subseteq X$ is a closed set, $A(Y)$, is the quotient of $A$ by the ideal $A(X \setminus Y)$, which becomes a $C(Y)$-algebra. The quotient map is denoted by $\pi_Y$.

If $Y$ reduces to a point $x$, we write $A_x$, denote by $\pi_x$ the quotient map. The C*-algebra $A_x$ is called the fiber of $A$ at $x$. 


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Lemma (Blanchard)

Let $A$ be a $C(X)$-algebra and $a \in A$. Then the following conditions are satisfied:

(i) $\|a\| = \sup_{x \in X} \|a_x\|.$

(ii) The map $x \mapsto \|a_x\|$ is upper semicontinuous.
Continuous Fields

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A $C(X)$-algebra such that the map $x \mapsto \|a_x\|$ is continuous for all $a \in A$ is called a continuous field of $C^*$-algebras.
Continuous Fields

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A $C(X)$-algebra such that the map $x \mapsto \|a_x\|$ is continuous for all $a \in A$ is called a **continuous field of C*-algebras**.

A continuous field is called **trivial** if there exists a C*-algebra $D$ such that $A \cong C(X, D)$. 
Sheaves of semigroups

Definition (Presheaves)

A **presheaf** over \( X \) is a contravariant functor \( S: \mathcal{V}_X \to C \)

where \( \mathcal{V}_X \) is the **category of closed sets of** \( X \) **with non-empty interior** and \( C \) is a subcategory of the category of sets which is **closed under sequential inductive limits**.
Sheaves of semigroups

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**Definition (Sheaves)**

A presheaf is a **sheaf** if for all $V, V' \in \mathcal{V}_X$ such that $V \cap V' \in \mathcal{V}_X$, the map

$$\pi^V_{V \cup V'} \times \pi^{V'}_{V \cup V'} : S(V \cup V') \to \{(f, g) \in S(V) \times S(V') \mid \pi^V_{V \cap V'}(f) = \pi^{V'}_{V \cap V'}(g)\}$$

is bijective.
A presheaf (respectively a sheaf) is **continuous** if for any decreasing sequence \((V_i)_{i=1}^{\infty}\) in \(\mathcal{V}_X\) whose intersection \(\cap_{i=1}^{\infty} V_i = V\) belongs to \(\mathcal{V}_X\), the limit \(\lim_{\longrightarrow} S(V_i)\) is isomorphic to \(S(V)\).
A presheaf (respectively a sheaf) is **continuous** if for any decreasing sequence $(V_i)_{i=1}^\infty$ in $\mathcal{V}_X$ whose intersection $\bigcap_{i=1}^\infty V_i = V$ belongs to $\mathcal{V}_X$, the limit $\lim_{\to S} (V_i)$ is isomorphic to $S(V)$.

**Definition**

Let $S$ be a continuous presheaf over $X$. For any $x \in X$, we define the fiber of $S$ at $x$ as

$$S_x := \lim_{x \in V_n} S(V_n),$$

with respect to the restriction maps, where $\{V_n\}_{n}$ is any decreasing sequence in $\mathcal{V}_X$ such that $\bigcap_{n=1}^\infty V_n = \{x\}$. 
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**Examples**

Let $X$ be a compact metric space, and let $A$ be a $C(X)$-algebra. Then:
A presheaf (respectively a sheaf) is **continuous** if for any decreasing sequence $(V_i)_{i=1}^\infty$ in $\mathcal{V}_X$ whose intersection $\bigcap_{i=1}^\infty V_i = V$ belongs to $\mathcal{V}_X$, the limit $\lim_{\to} S(V_i)$ is isomorphic to $S(V)$.

**Definition**

Let $S$ be a continuous presheaf over $X$. For any $x \in X$, we define the fiber of $S$ at $x$ as

$$S_x := \lim_{x \in V_n} S(V_n),$$

with respect to the restriction maps, where $\{V_n\}_n$ is any decreasing sequence in $\mathcal{V}_X$ such that $\bigcap_{n=1}^\infty V_n = \{x\}$.

**Examples**

Let $X$ be a compact metric space, and let $A$ be a $C(X)$-algebra. Then:

$$\text{Cu}_A : \mathcal{V}_X \to \text{Cu}$$

$$U \mapsto \text{Cu}_A(U) = \text{Cu}(A(U))$$

$$\mathcal{V}_A : \mathcal{V}_X \to \mathcal{V}_A(U) = \mathcal{V}(A(U))$$

$$\text{Sg}_A : \mathcal{V}_X \to \mathcal{Sg}$$

are continuous presheaves.
Sheaf of sections
Sheaf of sections

What is a sheaf of sections?
Sheaf of sections

What is a sheaf of sections?

Let $S$ be a sheaf over a space $X$. 

Sheaf of sections

What is a sheaf of sections?

Let $S$ be a sheaf over a space $X$ and define $F_S = \bigsqcup_{x \in X} S_x$ and $r : F_S \to X$ be the natural projection taking elements in $S_x$ to $x$. 
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We shall call section any map $f : V \subseteq X \rightarrow F_{S(V)}$ such that $r \circ f = 1_V$. 
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Taking $\{\hat{s}(U)\}$, where $U$ is open in $V$ and $s \in S(V)$, as a basis for the topology of $F_{S(V)}$, all the functions $\hat{s}$ are continuous.
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Taking $\{\hat{s}(U)\}$, where $U$ is open in $V$ and $s \in S(V)$, as a basis for the topology of $F_{S(V)}$, all the functions $\hat{s}$ are continuous.

One can define $\Gamma(V, F_{S(V)}) = \{ f : V \to F_{S(V)} \mid f \text{ is a continuous section} \}$.
Relation between a sheaf and the sheaf of sections
Relation between a sheaf and the sheaf of sections

**Theorem (Classical Result)**

Let $S$ be an algebraic sheaf over $X$. Then,

$$S \text{ and } \Gamma(-, F_{S(-)}) \text{ are isomorphic sheaves.}$$
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**algebraic sheaf** = Inductive limits in the target category are algebraic limits.
The Cuntz Semigroup of Continuous Fields of C*-algebras

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**Problem**

The inductive limits in the category $\text{Cu}$ are not algebraic.
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**algebraic sheaf** = Inductive limits in the target category are algebraic limits.

**Problem**

The inductive limits in the category $Cu$ are not algebraic.

**Example**

Let $A = C([0, 1], \mathbb{C})$ and $\{U_m = [\frac{1}{2} - \frac{1}{m}, \frac{1}{2} + \frac{1}{m}]\}_{m \geq 2}$, which is a sequence of decreasing closed subsets of $[0, 1]$ whose intersection is $\{1/2\}$.

It follows $Cu(A) \cong Lsc([0, 1], \overline{N})$, where $\overline{N} = \mathbb{N} \cup \{\infty\}$. So one has

$$\lim Lsc(U_n, \overline{N}) = \lim Cu(A(U_n)) = Cu(\lim A(U_n)) = Cu(A(1/2)) = \overline{N}.$$

However, the computation of the above direct limit in $Sg$ is not $\overline{N}$. 
The sheaves of sections on $\text{Cu}$
The sheaves of sections on $\mathcal{Cu}$

**Question**

*How do we recover $S$ on $\mathcal{Cu}$ from the sheaf of sections $F_S \to X$?*
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*How do we recover $S$ on $\mathcal{C}u$ from the sheaf of sections $F_S \to X$?*

Let $S : \mathcal{V}_X \to \mathcal{C}u$ be a sheaf on $\mathcal{C}u$ and $X$ be a compact metric space.
The sheaves of sections on $\mathcal{Cu}$

**Question**

*How do we recover $S$ on $\mathcal{Cu}$ from the sheaf of sections $F_S \to X$?*

Let $S : \mathcal{V}_X \to \mathcal{Cu}$ be a sheaf on $\mathcal{Cu}$ and $X$ be a compact metric space.

- We define a topology on $F_S$ generated by

  $$U_s^{\ll} = \{y \in F_S \mid \hat{s}(x) \gg y \text{ for some } x \in U\}. $$

  The induced sections are continuous with this topology.
The sheaves of sections on $\mathcal{C}u$

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- We equip the set of sections with pointwise addition and order. Moreover, the set of sections is closed under pointwise suprema of increasing sequences (by properties of $\mathcal{C}u$).
The sheaves of sections on \( \text{Cu} \)

**Question**

*How do we recover \( S \) on \( \text{Cu} \) from the sheaf of sections \( F_S \rightarrow X \)?*

Let \( S : \mathcal{V}_X \rightarrow \text{Cu} \) be a sheaf on \( \text{Cu} \) and \( X \) be a compact metric space.

- We define a topology on \( F_S \) generated by
  \[
  U_s^{\ll} = \{ y \in F_S \mid \hat{s}(x) \gg y \text{ for some } x \in U \}.
  \]
  The induced sections are continuous with this topology.

- We equip the set of sections with pointwise addition and order. Moreover, the set of sections is closed under pointwise suprema of increasing sequences (by properties of \( \text{Cu} \)).

**Theorem**

*Let \( X \) be a one-dimensional compact metric space, and let \( S : \mathcal{V}_X \rightarrow \text{Cu} \) be a surjective sheaf. Then \( \Gamma(X, F_S) \) is a semigroup in \( \text{Cu} \).*
When do we have a sheaf on $\text{Cu}$?
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**Theorem**

For a continuous field $A$ over a one-dimensional compact metric space $X$ whose fibers have no $K_1$ obstructions, the presheaves

$$
\text{Cu}_A(\_): \mathcal{V}_X \to \text{Cu} \\
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**Definition**

A C*-algebra $A$ is said to have no $K_1$ obstructions, if $sr(A) = 1$ and $K_1(I) = \{0\}$ for any ideal $I$ of $A$. 
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A C*-algebra $A$ is said to have no $K_1$ obstructions, if $sr(A) = 1$ and $K_1(I) = \{0\}$ for any ideal $I$ of $A$.

**Examples**

- If $sr(A) = 1$, $A$ is simple and $K_1(A) = \{0\}$, then $A$ has no $K_1$ obstructions.
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$$
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Cu_A(\_): \mathcal{V}_X & \rightarrow Cu \\
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A C*-algebra $A$ is said to have no $K_1$ obstructions, if $sr(A) = 1$ and $K_1(I) = \{0\}$ for any ideal $I$ of $A$.

**Examples**

- If $sr(A) = 1$, $A$ is simple and $K_1(A) = \{0\}$, then $A$ has no $K_1$ obstructions.
- (Lin) If $sr(A) = 1$, $RR(A) = 0$ and $K_1(A) = \{0\}$, then $A$ has no $K_1$ obstructions.
The sheaf $\mathcal{C}u_A(\_)$
The sheaf $\mathcal{Cu}_A(\_)$

**Theorem**

Let $X$ be a one-dimensional compact metric space, and let $A$ be a continuous field over $X$ whose fibers have no $K_1$ obstructions. Consider the functors

$$
\mathcal{Cu}_A(\_): \mathcal{V}_X \rightarrow \mathcal{Cu} \quad \text{and} \quad \Gamma(\_, F_{\mathcal{Cu}_A(\_)}) : \mathcal{V}_X \rightarrow \mathcal{Cu}
$$

$V \mapsto \mathcal{Cu}(A(V)) \quad \text{and} \quad V \mapsto \Gamma(V, F_{\mathcal{Cu}_A(V)})$.

Then, $\mathcal{Cu}_A(\_)$ and $\Gamma(\_, F_{\mathcal{Cu}_A(\_)})$ are isomorphic sheaves.
Relation between $\text{Cu}(A)$ and the sheaves $\text{Cu}_A(-)$, $\mathcal{V}_A(-)$

Considering an induced action of $\text{Cu}(C(X))$ on $\text{Cu}(A)$, we obtained that:
Relation between \( \text{Cu}(A) \) and the sheaves \( \text{Cu}_A(-) \), \( \mathbb{V}_A(-) \)

Considering an induced action of \( \text{Cu}(C(X)) \) on \( \text{Cu}(A) \), we obtained that:

**Theorem**

Let \( X \) be a compact metric space, and let \( A \) and \( B \) be \( C(X) \)-algebras such that all fibers have stable rank one. Consider the following conditions:

1. \( \text{Cu}(A) \cong \text{Cu}(B) \) preserving the action of \( \text{Cu}(C(X)) \),
2. \( \text{Cu}_A(-) \cong \text{Cu}_B(-) \),
3. \( \mathbb{V}_A(-) \cong \mathbb{V}_B(-) \).

Then (i) \( \implies \) (ii) \( \implies \) (iii). If \( X \) is one-dimensional, then also (ii) \( \implies \) (i). If, furthermore, \( A \) and \( B \) are continuous fields such that for all \( x \in X \) the fibers \( A_x \), \( B_x \) have real rank zero and \( K_1(A_x) = K_1(B_x) = \{0\} \), then (iii) \( \implies \) (ii) and so all three conditions are equivalent.
Classification result (Dadarlat-Elliott-Niu)

**Theorem**

Let $A, B$ be separable unital continuous fields of AF-algebras over $[0, 1]$. Any isomorphism $\tilde{\phi} : Cu(A) \to Cu(B)$ that preserves the action by $Cu(C(X))$ and such that $\tilde{\phi}(\langle 1_A \rangle) = \langle 1_B \rangle$ lifts to an isomorphism $\phi : A \to B$ of continuous fields of $C^*$-algebras.
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**Question**

*Can the above result be extended when the fibers are simple $\text{AI}$-algebras?*
1 Introduction

2 The Cuntz Semigroup of Continuous Fields of C*-algebras

3 The geometry of Dimension Functions
   - Stable rank of Continuous Fields of C*-algebras
   - The Blackadar-Handelman conjectures

4 Local triviality for Continuous Fields of C*-algebras
In the case of trivial fields:
Stable rank of Continuous Fields

In the case of trivial fields:

Theorem (Nagisa, Osaka, Phillips, 2001)

Let $A$ be a $C^*$-algebra.

1. If $K_1(A) = \{0\}$, $sr(A) = 1$, $RR(A) = 0$, then $sr(C([0,1], A)) = 1$.

2. If $sr(C([0,1], A)) = 1$, then $K_1(A) = \{0\}$ and $sr(A) = 1$. 
Stable rank of Continuous Fields

In the case of trivial fields:

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$(\text{Lin}) \implies A$ has no $K_1$ obstructions.
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Is no $K_1$ obstructions the optimal hypothesis to obtain $\iff$ ?
Trivial fields

**Theorem**

Let $A$ be any C*-algebra and $X$ be a compact metric space. Then

$$\text{sr}(C(X, A)) = 1 \iff A \text{ has no } K_1 \text{ obstructions and } \dim(X) \leq 1.$$
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**Corollary**

Let $A$ be a simple C*-algebra with $\text{sr}(A) = 1$ and $K_1(A) = \{0\}$. Then

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**Corollary**

Let $A$ be a C*-algebra with no $K_1$ obstructions. Then the stable rank of $A \otimes \mathbb{Z}$ is one.
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Is $\text{sr}(A \otimes \mathcal{Z}) = 1$ when $\text{sr}(A) = 1$?

M. Rørdam : $A$ is simple.  
L. Santiago : $A$ is commutative.
Non-trivial continuous fields
Non-trivial continuous fields

**Theorem**

Let $X$ be a one-dimensional, compact metric space, and let $A$ be a continuous field over $X$ such that each fiber $A_x$ has no $K_1$ obstructions. Then $sr(A) = 1$. 

In this case, we provide an example which shows that the converse is not true.

Example

There is a continuous field $A$ over $[0,1]$ such that $sr(A) = 1$ and $K_1(A_x) \neq \{0\}$ for $x$ in a dense subset of $[0,1]$. 


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The Blackadar-Handelman conjectures
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What are the dimension functions?
The Blackadar-Handelman conjectures

What are the dimension functions?

**Definition**

The set of **dimension functions** is $\text{St}(W(A), \langle 1_A \rangle)$ (normalized positive linear functionals), denoted by $\text{DF}(A)$.

We denote by $\text{LDF}(A)$ the subset of $\text{DF}(A)$ such that the dimension functions are **lower semicontinuous**.

(If $a_n \to a$ in $M_\infty(A)_+$, then $d(\langle a \rangle) \leq \lim \inf d(\langle a_n \rangle)$ for $d \in \text{LDF}(A)$)
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Remark

*It follows by the construction of the Grothendieck group that $St(W(A), \langle 1_A \rangle) = St(K_0^*(A), [1_A])$.*
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**Remark**

*It follows by the construction of the Grothendieck group that\* $\text{St}(\mathcal{W}(A), \langle 1_A \rangle) = \text{St}(K_0^*(A), [1_A])$.

**Theorem (Blackadar, Handelman, 1982)**

*There is an affine bijection between the set of traces of $A$ and $\text{LDF}(A)$, when $A$ is exact.*
Blackadar-Handelman conjectures (1982)

1. The set $\text{DF}(A)$ of dimension functions is a simplex.
The Blackadar-Handelman conjectures (1982)

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History

- (1997): Perera proved that 1rst conjecture holds for unital C*-algebras with stable rank one and real rank zero.
- (2008): Brown-Perera-Toms proved both conjectures hold for all unital simple exact and $\mathcal{Z}$-stable C*-algebras.
Proof 1st conjecture (Strategy)

- We study when $\left( K_0^*(A), [1_A] \right)$ is an interpolation group.
Proof 1st conjecture (Strategy)

- We study when \((K^*_0(A), [1_A])\) is an interpolation group.

\[
\begin{array}{cccc}
  x_1 & y_1 \\
  \leq & \Rightarrow & \exists \ z & | \ x_i \leq z \leq y_j \ \text{for} \ i, j = 1, 2 \\
  x_2 & y_2
\end{array}
\]
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- (Goodearl-Handelman-Lawrence) If \((G, u)\) is an interpolation group with an order-unit \(u\), then \(St(G, u)\) is a Choquet simplex.
Proof 1st conjecture (Strategy)

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   \[
   x_1 \leq y_1 \quad \Rightarrow \quad \exists \ z \mid x_i \leq z \leq y_i \text{ for } i, j = 1, 2
   \]

2. (Goodearl-Handelman-Lawrence) If \((G, u)\) is an interpolation group with an order-unit \(u\), then \(St(G, u)\) is a Choquet simplex.

Question

When \((K_0^*(A), [1_A])\) is an interpolation group?
Theorem

Let $X$ be a compact metric space, and let $A$ be a unital continuous field over $X$. Then, $(K_0^*(A), [1_A])$ is an interpolation group in the following cases:

(i) If $X$ is a one-dimensional and $A$ is a continuous field over $X$ such that, for all $x \in X$, $A_x$ has stable rank one, trivial $K_1$, and is either of real rank zero or simple and $\mathcal{Z}$-stable.
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Moreover, in the above cases, the set of dimension functions is a Choquet Simplex.
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- If $Cu(A)$ has interpolation and $W(A) \subseteq Cu(A)$ is hereditary, then $W(A)$ has interpolation.
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Theorem

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$sr(A) = 1 \quad Cu(A) = \Gamma(X, \bigsqcup_{x \in X} Cu(A_x))$
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(i) If $X$ is a one-dimensional and $A$ is a continuous field over $X$ such that, for all $x \in X$, $A_x$ has \textit{stable rank one}, \textit{trivial $K_1$}, and is either of \textit{real rank zero} or \textit{simple and $\mathbb{Z}$-stable}.

(ii) If $X$ is finite dimensional and $A = C(X, B)$, where $B$ is a unital, \textit{simple}, non-type $I$, ASH algebra with slow dimension growth. ($\implies \mathbb{Z}$-stable)

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- If $W(A)$ has interpolation, then $K_0^*(A)$ does.
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$sr(A) = 1$ \hspace{1cm} $Cu(A) = \Gamma(X, \bigsqcup_{x \in X} Cu(A_x))$

either $sr(A_x) = 1$ and $RR(A_x) = 0$ or $A_x$ is simple and $\mathbb{Z}$-stable.
Blackadar-Handelman conjectures

1. The set $\text{DF}(A)$ of dimension functions is a simplex.
2. The set $\text{LDF}(A)$ of lower semicontinuous dimension functions is dense in $\text{DF}(A)$. 
Theorem

Let $X$ be a finite dimensional, compact metric space, and let $A$ be a unital, separable infinite dimensional and exact $C^*$-algebra of stable rank one such that $T(A)$ is a Bauer simplex. Then $\text{LDF}(\text{C}(X, A))$ is dense in $\text{DF}(\text{C}(X, A))$ in the following cases:

1. $\dim X \leq 1$, $A$ is simple with $K_1(A) = 0$ and $W(A)$ is almost unperforated.
2. $A$ is a non-type I, simple, unital ASH algebra with slow dimension growth.
1 Introduction

2 The Cuntz Semigroup of Continuous Fields of $C^*$-algebras

3 The geometry of Dimension Functions

4 Local triviality for Continuous Fields of $C^*$-algebras
   - Nowhere locally trivial continuous fields
   - Local triviality
Nowhere locally trivial continuous fields

Definition

A point $x \in X$ is called singular for $A$ if $A(U)$ is nontrivial for any open set $U$ that contains $x$ (i.e. $A(U)$ is not isomorphic to $C_0(U, D)$ for some C*-algebra $D$).
Nowhere locally trivial continuous fields

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If all points of $X$ are singular for $A$ we say that $A$ is **nowhere locally trivial**.
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Cuntz Algebras \( \mathcal{O}_n \)

If \( n \geq 2 \). The **Cuntz Algebras** are defined as the universal C*-algebras generated by isometries \( s_1, \ldots, s_n \) with orthogonal ranges such that \( \sum_{i=1}^{n} s_is_i^* = 1 \).
Local triviality

Example (Dadarlat, Elliott-’08)

A nowhere locally trivial continuous field over $[0,1]$ (finite-dimensional) such that its fibers are the same Kirchberg algebra (UCT) with infinitely generated K-theory.
Local triviality

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A nowhere locally trivial continuous field over \([0, 1]\)
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Example (Dadarlat-’09)

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A nowhere locally trivial continuous field over Hilbert cube (infinite-dimensional) such that its fibers are the same Kirchberg algebra (UCT) with finitely generated $K$-theory.

Theorem

Let $X$ be a finite dimensional compact metric space, and let $D$ be a stable Kirchberg algebra that satisfies the UCT and such that $K_j(D)$ is finitely generated for $j = 0,1$. Let $A$ be a separable continuous field $C^*$-algebra over $X$ such that $A(x) \cong D$ for all $x \in X$. Then there exists a dense open subset $U$ of $X$ such that $A(U)$ is locally trivial.
Corollary

Fix $n \in \mathbb{N} \cup \{\infty\}$. Let $X$ be a finite dimensional compact metrizable space and $A$ be a continuous field over $X$ such that $A(x) \cong O_n \otimes K$ for all $x \in X$. Then there exists a closed subset $V$ of $X$ with nonempty interior such that $A(V) \cong C(V) \otimes O_n \otimes K$. 
Corollary

Fix $n \in \mathbb{N} \cup \{\infty\}$. Let $X$ be a finite dimensional compact metrizable space and $A$ be a continuous field over $X$ such that $A(x) \cong \mathcal{O}_n \otimes \mathcal{K}$ for all $x \in X$. Then there exists a closed subset $V$ of $X$ with nonempty interior such that $A(V) \cong \mathcal{C}(V) \otimes \mathcal{O}_n \otimes \mathcal{K}$.

Example

If $F \subset X$ is a closed nowhere dense set, we provide a continuous field C*-algebra $A$ with all fibers isomorphic to a fixed Cuntz algebra $\mathcal{O}_n \otimes \mathcal{K}$, $3 \leq n \leq \infty$, and such that the set of singular points of $A$ coincides with $F$. 
Corollary

Fix \( n \in \mathbb{N} \cup \{\infty\} \). Let \( X \) be a finite dimensional compact metrizable space and \( A \) be a continuous field over \( X \) such that \( A(x) \cong \mathcal{O}_n \otimes \mathcal{K} \) for all \( x \in X \). Then there exists a closed subset \( V \) of \( X \) with nonempty interior such that \( A(V) \cong C(V) \otimes \mathcal{O}_n \otimes \mathcal{K} \).

Example

If \( F \subset X \) is a closed nowhere dense set, we provide a continuous field C*-algebra \( A \) with all fibers isomorphic to a fixed Cuntz algebra \( \mathcal{O}_n \otimes \mathcal{K} \), \( 3 \leq n \leq \infty \), and such that the set of singular points of \( A \) coincides with \( F \).

Our result is in a certain sense **OPTIMAL!**.
Bibliography


Bibliography


Thanks!