Tropical Resultants for Curves

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Why Resultants?

- Resultants are classical powerful elimination tools
- Related to tropical elimination
- Needed in the context of geometric construction to codify intersection of curves
Notation

- $\mathbb{K}$ is an algebraically closed field of arbitrary characteristic provided by a non trivial valuation $v : \mathbb{K}^* \rightarrow \mathbb{T} \subseteq \mathbb{R}$.
- A valuation $v$ is an homomorphism

$$v : \mathbb{K}^* \longrightarrow \mathbb{T}$$

from the multiplicative group $\mathbb{K}^*$ onto an ordered abelian group such that, if $a + b \neq 0$

$$v(a + b) \geq \min\{v(a), v(b)\}$$

- $\mathbb{R}$ is the valuation ring of $\mathbb{K}$ with maximal ideal $m$.

$$\mathbb{R} = \{x \in \mathbb{K}^* \mid v(x) \geq 0\} \cup \{0\}$$

$$m = \{x \in \mathbb{K}^* \mid v(x) > 0\} \cup \{0\}$$
Notation

$\mathbf{k}$ denotes the residual field with respect to the valuation; $k = R/m$.

Depending on the characteristics of $\mathbb{K}$ and $k$ we will distinguish three cases.

- $\text{char}(\mathbb{K}) = \text{char}(k) = 0$, the equicharacteristic zero case. This is the most common to develop tropical geometry.
  
  Example: The Puiseux series $\mathbb{C}\{t\}$
  
  Residual field $\mathbb{C} \subseteq \mathbb{C}\{t\}$

- $\text{char}(\mathbb{K}) = \text{char}(k) = p > 0$, the equicharacteristic $p$ case.
  
  Example: The algebraic closure of $\mathbb{F}_p(t)$ of rational functions in one variable.
  
  \[ v(t) = 1 \]

  Residual field $\overline{\mathbb{F}}_p \subseteq \overline{\mathbb{F}}_p[t]$
Notation

\begin{itemize}
\item char(\mathbb{K})=0, char(k)=p > 0. The $p$-adic case.
Example $\overline{\mathbb{Q}}_5$ the algebraic closure of the 5-adics.
If $x = 5^{k \frac{p}{q}} \in \mathbb{Q}$, $k \in \mathbb{Z}$, 5 does not divide $p$ and $q$, then

$$v(x) = k.$$ 

This valuation can be extended to the algebraic numbers. Residual field $\overline{\mathbb{F}}_5 \not\subseteq \overline{\mathbb{Q}}_5$. 
\end{itemize}
Notation

- $\pi : R \to k = R/m$ denotes the residual class map.
- The tropicalization $T : \mathbb{K}^* \to \mathbb{T}$ is either $v$ (if working with min) or $-v$ if working with max.
- We suppose that we have a multiplicative subgroup $\{t^\gamma \mid \gamma \in \mathbb{T}\} \subseteq \mathbb{K}^*$ isomorphic to $\mathbb{T}$ by the homomorphism $v$.
- Let $x \in \mathbb{K}^*$, we denote the principal coefficient $Pc(x)$ as the element $\pi(xt^{-v(x)}) \in k$ (this is also called angular component.)
- $x \in \mathbb{K}^*$ is residually generic if $Pc(x)$ is generic in $k$. 
Two Examples of Principal Coefficient

$K = \mathbb{C}\{t\}$ the Puiseux series field with the standard valuation.
$k = \mathbb{C}$. The group $\{t^\gamma \mid \gamma \in \mathbb{T}\}$ is just the set of elements $t^{m/n}$.

\[
x = \frac{2}{t} + 1 + t + t^2,
\]
\[
v(x) = -1, \quad Pc(x) = \pi(tx) = 2 \in \mathbb{C}.
\]
Two Examples of Principal Coefficient

$\mathbb{K} = \overline{\mathbb{Q}}_5$ the algebraic closure of the rationals with a 5–adic valuation. $k = \overline{\mathbb{F}}_5$. We may take the group $t^\gamma$ as an appropriate set of the form $5^{n/m}$.

If $x$ is a root of valuation -1 of

$$\frac{2}{5^5} + 3z + z^5 + 75z^6,$$

then $5x$ is a root of $2 + 3 \cdot 5^4 z + z^5 + 3 \cdot 5z^6$,

$$Pc(x) = \pi(5x)$$

is a root of $2 + z^5 \in \overline{\mathbb{F}}_5$,

$$Pc(x) = 2$$
Tropicalization... (again)

- If $V \subseteq (\mathbb{K}^*)^n$ is an algebraic variety, the tropicalization $T(V)$ is the image of $V$ under the map:

  $$T : (\mathbb{K}^*)^n \rightarrow \mathbb{T}^n$$

  $$(x_1, \ldots, x_n) \mapsto (-v(x_1), \ldots, -v(x_n))$$

If $f = \sum_{i \in I} a_i x^i = \max\{a_i + <i, x>\}$ is a tropical polynomial, its zero set $\mathcal{T}(f)$ is the set of points is attained for at least two different indices $i$.

**Theorem (Kapranov)**

Let $\tilde{f} = \sum_{i \in I} a_i x^i \in \mathbb{K}[x_1, \ldots, x_n]$. Let

$$f = \sum_{i \in I} T(a_i) x^i \in \mathbb{T}[x_1, \ldots, x_n]$$

Then

$$T(V(\tilde{f})) = \mathcal{T}(f)$$
Classical Univariate Resultant

$I, J \subseteq \mathbb{N}$ finite and of cardinality at least 2 such that $0 \in I \cap J$.

$\tilde{f} = \sum_{i \in I} a_i x^i, \quad \tilde{g} = \sum_{j \in J} b_j x^j \in \mathbb{K}[x]$, of support $I$ and $J$.

Let $p$ be the characteristic of $\mathbb{K}$.

There is a unique polynomial in $\mathbb{Z}/(p\mathbb{Z})[a_i, b_j]$, up to a constant factor, called the resultant, such that it vanishes if and only if $\tilde{f}$ and $\tilde{g}$ have a common root.
Definition

We denote by $R(I, J, \mathbb{K}) \in \mathbb{Z}/(p\mathbb{Z})[a_i, b_j]$ this resultant, denote by $R_t(I, J, \mathbb{K})$ the tropicalization of the resultant. This tropical polynomial is called the tropical resultant of supports $I$ and $J$ over $\mathbb{K}$. The tropical variety is denoted by $\mathcal{T}(R_t(I, J, \mathbb{K}))$.

➢ The tropical resultant polynomial depends on the field $\mathbb{K}$ and the valuation!
Example of Different Tropicalizations of the Resultant

Let \( f = a + bx + cx^2 \), \( g = p + qx + rx^2 \).
If \( \text{char}(\mathbb{K}) \neq 2 \) then
\[
R(\{0, 1, 2\}, \{0, 1, 2\}, \mathbb{K}) = r^2a^2 - 2r acp + c^2p^2 - qrbq - qbcq + c^2a + prb^2.
\]
If \( \text{char}(\mathbb{K}) = 2 \) then
\[
R(\{0, 1, 2\}, \{0, 1, 2\}, \mathbb{K}) = r^2a^2 + c^2p^2 - qrbq - qbcq + c^2a + prb^2.
\]

So the tropical resultant polynomial is: Puiseux series:
\[
P_1 = "0r^2a^2 + 0r acp + 0c^2p^2 + 0qrbq + 0qbcq + 0c^2a + 0prb^2".
\]
2-adics:
\[
P_2 = "0r^2a^2 + (\text{red} -1)r acp + 0c^2p^2 + 0qrbq + 0qbcq + 0c^2a + 0prb^2".
\]
char 2:
\[
P_3 = "0r^2a^2 + 0c^2p^2 + 0qrbq + 0qbcq + 0c^2a + 0prb^2".
\]
Geometric Meaning of the Resultant

Theorem
The tropical variety defined by $R_t(I, J, \mathbb{K})$ does not depend on the field $\mathbb{K}$.

Let $f = \sum_{i \in I} a_i x^i$, $g = \sum_{j \in J} b_j x^j$ have a common tropical root if and only if the point $(a_i, b_j)$ belongs to the variety defined by $R_t(I, J, \mathbb{K})$.

Puiseux series:
$P_1 = "0r^2a^2+racp+0c^2p^2+0qrba+0qbcp+0cq^2a+0prb^2".$

2-adics:
$P_2 = "0r^2a^2+(-1)racp+0c^2p^2+0qrba+0qbcp+0cq^2a+0prb^2".$

char 2:
$P_3 = "0r^2a^2+0c^2p^2+0qrba+0qbcp+0cq^2a+0prb^2".$

The tropical variety defined by $P_1$, $P_2$ and $P_3$ is the same.
Idea of the Proof

- Direct: In the resultant, the coefficients of monomials that are vertices of the Newton polytope are always $\pm1$ [Sturmfels, 1994]

- Lifting Proof: Two tropical polynomials with a common tropical root can always be lifted to two algebraic polynomials with a common algebraic root. This happens because they form an acyclic incidence configuration and Kapranov’s theorem.
Bivariate Resultants

Let $\widetilde{f} = \sum \widetilde{a}_{i,j} x^i y^j$, $\widetilde{g} = \sum \widetilde{b}_{k,l} x^k y^l \in \mathbb{K}[x, y]$. 

$\text{Res}(\widetilde{f}, \widetilde{g}, y)$ is a polynomial in $\mathbb{K}[x]$ such that its roots are the $x$-th coordinates of the finite set $\{\widetilde{f} = \widetilde{g} = 0\}$.
Tropical Bivariate Resultants

\[ f = \sum_{i,j} a_{i,j} x^i y^j, \quad g = \sum_{k,l} b_{k,l} x^k y^l \in \mathbb{T}[x,y] \]

We define the resultant of \( f \) and \( g \) with respect to \( y \) as an specialization of the corresponding univariate resultant: \( e_j \):

\[ f = "0 + 2x + 3y", \quad g = "2 + 3x + 3y + 3xy + 2x^2 + 0y^2" \]

- Rewrite them as polynomials in \( \mathbb{T}[y][x] \),

\[ f = (0 + 3y) + (2)x, \quad g = (2 + 3y + 0y^2) + (3 + 3y)x + (2)x^2 \]

- Compute the univariate resultant corresponding to the supports of the polynomials w.r.t. \( x \): \( I = \{0, 1\}, \ J = \{0, 1, 2\}, \)

\[ R(I, J, \mathbb{C}\{t\}) = Res_x(A_0 + A_1x, B_0 + B_1x + B_2x^2) = \]

\[ = A_1^2B_0 - A_0A_1B_1 + B_2A_0^2 \]

\[ R_t(I, J, \mathbb{C}\{t\}) = "0A_1^2B_0 + 0A_0A_1B_1 + 0B_2A_0^2" \]
Tropical Bivariate Resultants

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\]

\[
= A_1^2B_0 - A_0A_1B_1 + B_2A_0^2
\]

\[
R_t(I, J, \mathbb{C}\{t\}, x) = "0A_1^2B_0 + 0A_0A_1B_1 + 0B_2A_0^2"
\]

- Evaluate this tropical resultant in the coefficients of \( f \) and \( g \)

\[
R_t(f, g, x) = "2^2(2+3y+0y^2) + (0+3y)(2)(3+3y) + (2)(0+3y)^2" =
\]

\[
= "6 + 8y + 8y^2"
\]
Tropical Bivariate Resultant

- This resultant polynomial may vary depending essentially on the characteristics of $\mathbb{K}$ and $k$.
- However, the roots of the resultant polynomial do not depend on $\mathbb{K}$ and $v$, only of $f$ and $g$.
- Practical hints to decrease the number of monomials in a resultant polynomial:
  - In the univariate algebraic resultant, we can get rid off every monomial whose coefficient is not $\pm 1$.
  - In the specialization of the coefficients, we can use the equality $“(a + b)^n” = “a^n + b^n”$. 
Valuation Meaning of the Roots of the Resultant

Theorem

Let \( \tilde{f} = \sum_{i,j} \tilde{a}_{i,j} x^i y^j \), \( \tilde{g} = \sum_{k,l} \tilde{b}_{k,l} x^k y^l \in \mathbb{K}[x,y] \).

Let \( f = T(\tilde{f}) = \left\{ \sum_{i,j} T(\tilde{a}_{i,j}) x^i y^j \right\} \), \( g = T(\tilde{g}) = \left\{ \sum_{k,l} T(\tilde{b}_{k,l}) x^k y^l \right\} \).

Let \( h(y) = \text{Res}_x(\tilde{f}, \tilde{g}) \).

If the coefficients of \( \tilde{f}, \tilde{g} \) are residually generic then

\[
T(\{h(y) = 0\}) = \mathcal{T}(\text{Res}_x(f, g, \mathbb{K}))
\]

That is, the tropical resultant encodes the tropicalization of the algebraic resultant, whenever the coefficients are generic enough.
Recall the notion of stable intersection. Given two tropical curves, there is a well defined finite set of intersection points that varies continuously as the curves are perturbed and that verifies Bernstein-Koushnirenko theorem.
Geometric Meaning of the Roots of the Resultant

\[ f = "0 + 2x + 3y", \ g = "2 + 3x + 3y + 3xy + 2x^2 + 0y^2" \]

\[ R_t(f, g, x) = "6 + 8y + 8y^2" \] roots: -2, 0
Geometric Meaning of the Roots of the Resultant

Lemma
Let \( f \) and \( g \) be two tropical polynomials in two variables. Then, for any two lifts \( \tilde{f}, \tilde{g} \) such that their coefficients are residually generic, the intersection of the algebraic curves projects into the stable intersection.

\[
T(\tilde{f} \cap \tilde{g}) \subseteq T(f) \cap_{st} T(g)
\]

Idea: If \( q \) is a non stable intersection point, it belongs to the interior of two parallel edges of \( T(f) \) and \( T(g) \). The residual polynomials \( \tilde{f}_q \) and \( \tilde{g}_q \) over \( q \) can be written as

\[
\tilde{f}_q = \sum_{i=0}^{n} \alpha_i (x^r y^s)^i, \quad \tilde{g}_q = \sum_{j=0}^{m} \beta_i (x^r y^s)^j.
\]

The resultant of the polynomials \( \sum_{i=0}^{n} \alpha_i z^i, \sum_{j=0}^{m} \beta_i z^j \) with respect to \( z \) must vanish. So there is an algebraic dependence among these residual coefficients.
Geometric Meaning of the Roots of the Resultant

**Theorem**
Let $\tilde{f}, \tilde{g} \in \mathbb{K}[x,y]$. Then, it can be computed a finite set of polynomials in the principal coefficients of $\tilde{f}$, $\tilde{g}$ such that, if no one of them vanish, the tropicalization of the intersection of $\tilde{f}$, $\tilde{g}$ is the stable intersection of $f$ and $g$. Moreover, the multiplicities are conserved.

$$
\sum_{\tilde{q} \in \tilde{f} \cap \tilde{g}} \text{mult}(\tilde{q}) = \text{mult}_t(q)
$$

This theorem is proved using the correspondence between the several algebraic and tropical resultants and the previous lemma.
Corollary

Let \( f, g \in \mathbb{T}[x,y] \) be two tropical polynomials. Let

\[
h(y) \in \mathbb{T}[y] = \text{Res}_x(f, g, \mathbb{K})
\]

be a tropical resultant of \( f \) and \( g \) with respect to \( x \). Then, the tropical roots of \( h \) are exactly the \( y \)-th coordinates of the stable intersection of \( f \) and \( g \).

This is an indirect prove that all the polynomial resultants define the same points.
How to Compute the Stable Intersection and the Compatibility with the Algebraic Case

In the tropical setting

$$f \cap_{st} g \subseteq f \cap g \cap \text{Res}_x(f, g) \cap \text{Res}_y(f, g)$$

the right-hand set is finite but may be greater than the stable intersection.

Solution: Let $a$ be such that $x - ay$ is injective in the right-hand set. Let $z = xy^{-a}$.

The resultant

$$\text{Res}_y(\tilde{f}(zy^a, y), \tilde{g}(zy^a, y)) = \tilde{R}(z) = \tilde{R}(xy^{-a})$$

has as roots the values that the function $x - ay$ takes on the stable intersection.
Applications

- Transfer a proof of Berstein-Koushnirenko theorem in the plane to the positive characteristic case. (See Rojas 1999 for an alternative proof of this theorem in positive characteristic in the general context.)

- If \( \tilde{f}, \tilde{g} \in K[x, y], R(x) = \text{Res}_y(\tilde{f}, \tilde{g}), R(y) = \text{Res}_x(\tilde{f}, \tilde{g}), \) Let \( a \) such that \( x - ay \) is injective in \( T(\tilde{f}) \cap T(\tilde{g}) \cap T(R(x)) \cap T(R(y)) \), then

\[
\tilde{f}, \tilde{g}, R(x), R(y), \text{Res}_y(\tilde{f}(zy^a, y), \tilde{f}(zy^a, y))(xy^{-a})
\]

is a tropical basis of the ideal \( (\tilde{f}, \tilde{g}) \) (This result has been generalized independently by Hept and Theobald, 2007)

- Resultants are a tool used in the construction method to prove classical theorems in tropical geometry, ej: converse Pascal theorem or Cayley-Bacharach.
Open Problem

How to compute the tropical resultant directly?

The tropical determinant of the Sylvester matrix should work, this is a problem of deciding if the Newton polytope of the determinant and the permanent of the Sylvester matrix is equal or not.