

Resultants and intersection of two tropical conics

Note: The tropical addition is the minimum of two real numbers.

Let

$$f = "1 + 0x + 0y + 1x^2 + 0xy + 1y^2" \quad g = "0 + 0x + 0y + 1y^2 + 3xy + 8x^2"$$

represent the piecewise affine functions:

$$f(x, y) = \min\{1, x, y, 1 + 2x, x + y, 1 + 2y\} \quad g(x, y) = \min\{0, x, y, 1 + 2y, 3 + x + y, 8 + 2x\}.$$

The tropical conic defined by f is the set of points where f is not differentiable, if we compute these points in a naive way, we may choose two monomials, for example $1x^2$ and $0x$; the edge of the tropical curve associated to these monomials is the set defined by

$$x = 1 + 2x, x \leq 1, x \leq y, x \leq x + y, x \leq 1 + 2y$$

that has as solution the infinite ray $x = -1, y \geq 0$.

However, if we choose the monomials $1x^2, 1y^2$ we get the system of inequalities:

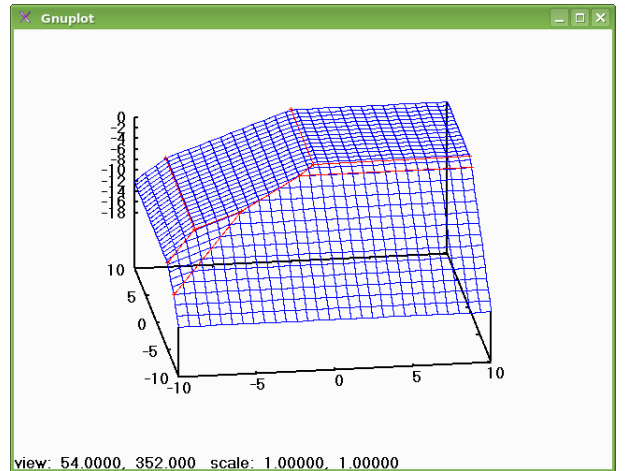
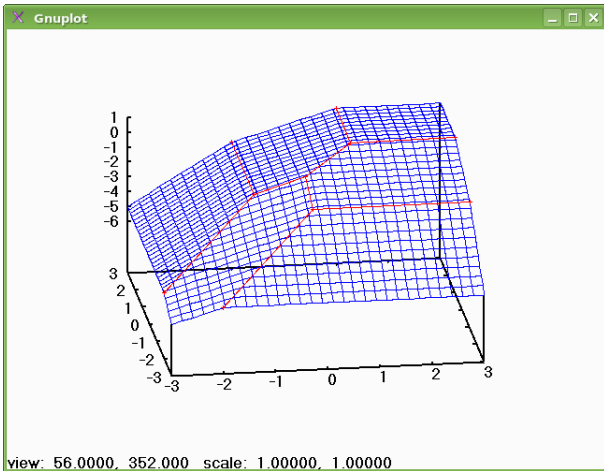
$$1 + 2x = 1 + 2y, 1 + 2x \leq 1, 1 + 2x \leq x, 1 + 2x \leq y, 1 + 2x \leq x + y$$

that has no solution, from the first equation it follows that $x = y$, substituting this in the equation we get $1 \leq 0$. So this pair of monomials does not define any edge of the tropical curve. If we choose the $\binom{6}{2} = 15$ different systems we will compute the segments and the rays of the conic. In particular, we may compute the "vertices" of the conics. They correspond to the points where the minimum of the functions is attained for at least three monomials that do not belong to a common line in the Newton polygon:

vertices of conic f : $(0,0), (1,1), (-1,0), (0,-1)$

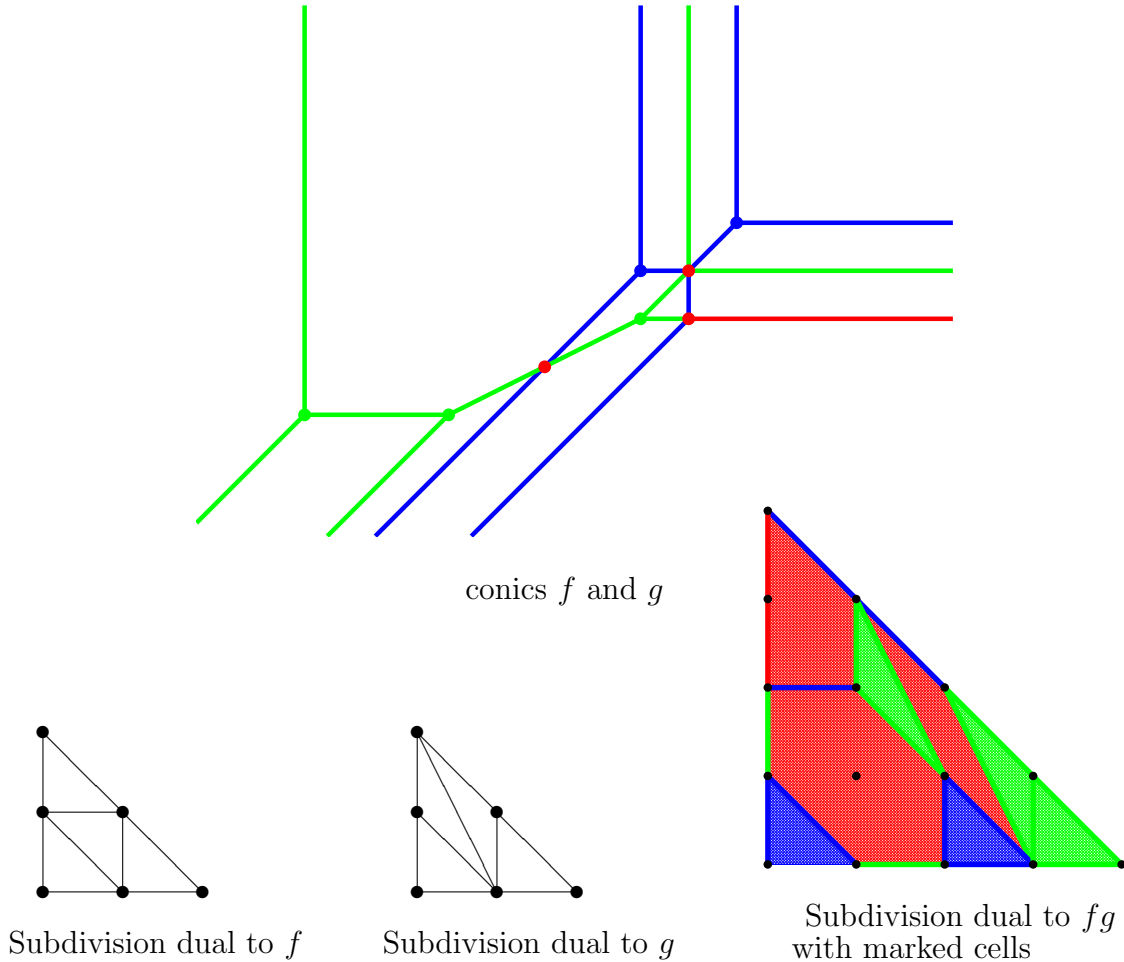
vertices of conic g : $(0,0), (-1,-1), (-5,-3), (-8,-3)$

This is a representation of the graph of f and the graph of g (in blue), the red segments represent the points where the functions are not continuous.



The projection into the plane of the set of points where the functions are not differentiable (red segments) we end up with the tropical conics.

In the next picture, the conic defined by f is displayed in blue, the conic defined by g is displayed in green, their intersection is drawn in red. The intersection consists in the points $(0, 0)$, $(-3, -2)$ and the ray $x \geq 0, y = -1$.



Let us compute the intersection multiplicity of the intersection points. If we perform a generic translation of one of the curves, the point $(0, 0)$ will disappear and two different intersection points will appear. The points $(-3, 2)$ and $(0, -1)$ will just be translated and all the points of the form $(t, -1)$, $t > 0$ will disappear. By Bezout, we know that the sum of the intersection multiplicities should be four, we conclude that $(0, 0)$ has multiplicity 2, the points $(-3, -2)$ and $(0, -1)$ have multiplicity 1 and the points $(t, -1)$, $t > 0$ have multiplicity zero. If for example we want to compute the multiplicity of $(0, 0)$ as intersection point without using perturbations, we can compute mixed volumes instead. The dual cell to $(0, 0)$ as a point of f is the triangle of vertices $(1, 0)$, $(0, 1)$, $(1, 1)$ that correspond with the monomials x, y, xy where the minimum of f is attained in $(0, 0)$. The dual cell of $(0, 0)$ in g is the triangle of vertices $(0, 0)$, $(1, 0)$, $(0, 1)$. By the tropical version of Bernstein-Koushnirenko theorem, the multiplicity of $(0, 0)$ is the mixed volume of these triangles. The Minkowski sum of these triangles in the convex hull of the points

$(1, 1), (1, 0), (0, 1), (2, 1), (2, 0), (1, 2), (0, 2)$. This is the hexagon in the dual subdivision to “ fg ” of area 3. Each triangle has area $1/2$, so the mixed volume is $3 - 1/2 - 1/2 = 2$. $(0, 0)$ is a point of intersection multiplicity 2.

Let us check now that the discussion is coherent with lifts to the Puiseux series field. Let us define the quadratic polynomials:

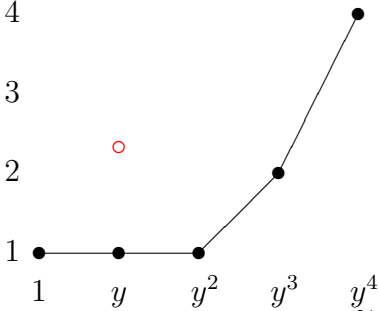
$$\tilde{f} = a_1t + a_x x + a_y y + a_{xy}xy + a_{xx}tx^2 + a_{yy}ty^2$$

$$\tilde{g} = b_1 + b_x x + b_y y + b_{xy}t^3xy + b_{xx}t^8x^2 + b_{yy}ty^2$$

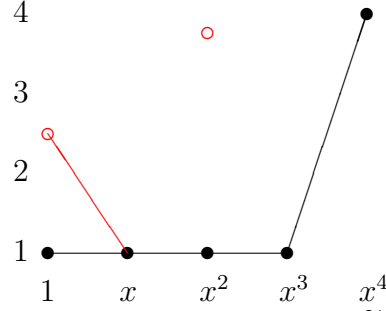
where a_i, b_j are arbitrary Puiseux series of valuation zero, $a_i = \alpha_i + o(t^\epsilon)$, $b_j = \beta_j + o(t^\epsilon)$. Compute the resultant of both curves to compute the intersection points:

$$\begin{aligned} \text{Res}(\tilde{f}, \tilde{g}, x) &= (-a_x a_{xx} b_1 b_x t + \text{h.o.t}) + (-a_x a_{xx} b_x b_y t + a_{xx} a_y b_x^2 t - a_{xx} a_{xy} b_1 b_x t + \text{h.o.t})y + \\ &+ (-a_{xx} a_{xy} b_x b_y t + \text{h.o.t})y^2 + (-a_{xx} a_{xy} b_x b_{yy} t^2 + \text{h.o.t})y^3 + (a_{xx}^2 b_{yy}^2 t^4 + \text{h.o.t})y^4 \end{aligned}$$

$$\begin{aligned} \text{Res}(\tilde{f}, \tilde{g}, y) &= (a_y^2 b_1 b_{yy} t - a_y a_{yy} b_1 b_y t + \text{h.o.t}) + (-a_x a_y b_y b_{yy} t + a_y^2 b_x b_{yy} t + 2a_{xy} a_y b_1 b_{yy} t + a_x a_{yy} b_y^2 t - \\ &a_y a_{yy} b_x b_y t - a_{xy} a_{yy} b_1 b_y t + \text{h.o.t})x + (-a_x a_{xy} b_y b_{yy} t + 2a_{xy} a_y b_x b_{yy} t + a_{xy}^2 b_1 b_{yy} t - a_{xy} a_{yy} b_x b_y t + \text{h.o.t})x^2 + \\ &(a_{xy}^2 b_x b_{yy} t + \text{h.o.t})x^3 + (a_{xx}^2 b_{yy}^2 t^4 + \text{h.o.t})x^4 \end{aligned}$$



Newton Diagram of $\text{Res}(\tilde{f}, \tilde{g}, x)$



Newton Diagram of $\text{Res}(\tilde{f}, \tilde{g}, y)$

The valuations of the coefficients of the resultant with respect to x do not depend on the lift. That is, for every lift of the conics, the algebraic conics will intersect in a finite set of points and the $y - th$ coordinates of the intersection points have valuation $0, 0, -1, -2$. With this information we can confirm that there will be two points (or a double point) projecting onto $(0, 0)$, another intersection point projecting in $(-3, -2)$ and another projecting on the ray $y = -1, x \geq 0$.

If we want to compute the $x - th$ coordinates of the points of intersection, if we look the Newton diagram, we check that the last segment does not depend on the lift, so for every lift, there will be an intersection point whose x coordinate has valuation -3 , this must be a point projecting into $(-2, -3)$. On the other hand, if the coefficients a_i, b_j are generic, the other three points of intersection will have its x coordinate of valuation 0. Note that this happens if and only if $\alpha_y \beta_{yy} - \alpha_{yy} \beta_y \neq 0$ (in general, the residual conditions are just sufficient but not necessary). The valuations of the coefficient in x and x^2 might be greater than one, but this does not affect the Newton diagram, as long as the coefficients in 1 does not change its valuation. The residual conditions for these elements are:

Coefficient 1: $\alpha_y \beta_{yy} - \alpha_{yy} \beta_y$

Coefficient x : $\alpha_x \alpha_y \beta_y \beta_{yy} + \alpha_y^2 \beta_x \beta_{yy} + 2\alpha_{xy} \alpha_y \beta_1 \beta_{yy} + \alpha_x \alpha_{yy} \beta_y^2 - \alpha_y \alpha_{yy} \beta_x \beta_y - \alpha_{xy} \alpha_{yy} \beta_1 \beta_y$

Coefficient x^2 : $\alpha_x \beta_y \beta_{yy} - 2\alpha_y \beta_x \beta_{yy} - \alpha_{xy} \beta_1 \beta_{yy} + \alpha_{yy} \beta_x \beta_y$

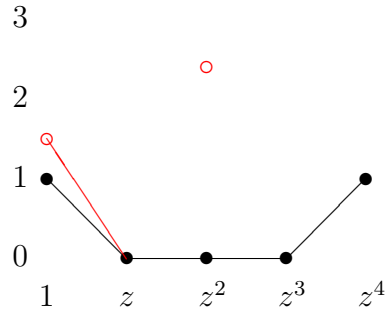
In order to achieve an intersection that does not correspond to the stable one, the residual coefficient in 1 must vanish, in this case $\beta_y = \alpha_y \beta_{yy} / \alpha_{yy}$, if we substitute this value in the rest of the coefficients, we get: Coefficient in x : $\alpha_{xy} \alpha_y \beta_1 \beta_{yy}$, so, even if the valuation in the coefficient of x^2 increases, there will always be two roots of this resultant of valuation zero, whatever the lift is. So, in the residual generic case, we have three roots of multiplicity zero in this resultant, if $\alpha_y \beta_{yy} - \alpha_{yy} \beta_y = 0$, the diagram will change in an undetermined manner (in red, in the diagram).

In general, the previous discussion is too complicated. If we want to compute sufficient residual condition for the intersection being compatible with the stable intersection these resultants are not enough. However, the linear function $x - y$ is injective in the intersection of two conics, so the value that takes the rational function xy^{-1} in the intersection points of the algebraic conics will determine the tropical root. If we call $z = xy^{-1}$ then

$$\tilde{f} = a_1 t + a_x z y + a_y y + a_{xy} z y^2 + a_{xx} z^2 y^2 t + a_{yy} y^2 t$$

$$\tilde{g} = b_1 + b_x z y + b_y y + b_{xy} z y^2 + b_{xx} z^2 y^2 + b_{yy} y^2$$

$$\begin{aligned} \text{Res}(\tilde{f}(zy, y), \tilde{g}(zy, y), y) &= (-a_x a_{xx} b_1 b_x t + \text{h.o.t}) z^4 + (-a_x a_{xy} b_1 b_x + \text{h.o.t}) z^3 + \\ &+ (-a_x a_{xy} b_1 b_y - a_x y a_y b_1 b_x + a_{xy}^2 b_1^2 + \text{h.o.t}) z^2 + (-a_x y a_y b_1 b_y + \text{h.o.t}) z + (a_y^2 b_1 b_{yy} t - a_y a_{yy} b_1 b_y t + \text{h.o.t}) \end{aligned}$$



Newton diagram of $\text{Res}(\tilde{f}(zy, y), \tilde{g}(zy, y), y)$

We check that, whatever the lift is, there is a root such that the valuation of xy^{-1} is -1 , there are two roots such that xy^{-1} has valuation zero. If the residual coefficients verify that $\alpha_y \beta_{yy} - \alpha_{yy} \beta_y = 0$ there will be a root such that the valuation of xy^{-1} in the root will be greater than one. The value of this root will depend on the valuation of the result. Note that the term z^2 of the intersection may also increase its value; this increase will depend on the distance of the two intersection roots (or possibly double root).

The general strategy to face this problem is the following: compute the resultants with respect to x and y . These resultants provide a finite set of possible roots on a lattice. We look for a function $x - ay$ where a is a natural number such that $x - ay$ is injective in the finite set of the lattice. The resultant $\text{Res}(\tilde{f}(zy^a, y), \tilde{g}(zy^a, y), y)$ allows us to distinguish among these points and provide conditions for the compatibility.

How to compute two conics with a double intersection point compatible with tropicalization? We specialize one conic to a specific one and work on the other conic. Moreover, we know that the behaviour of the points that project into $(0,0)$ depends on the coefficients on the dual cells, so we may take:

$$\tilde{f} = t + x + y + xy + tx^2 + ty^2, \tilde{g} = 1 + 1x + b_y y + t^3 xy + t^8 x^2 + ty^2.$$

If we have a double intersection point, the discriminant of the resultant must vanish

$$\text{disc}(\text{Res}(\tilde{f}, \tilde{g}, x)) = \ll \text{¡Expresión excesivamente larga para ser mostrada!} \gg$$

$$\text{disc}(\text{Res}(\tilde{f}, \tilde{g}, y)) = \ll \text{¡Expresión excesivamente larga para ser mostrada!} \gg$$

$$\text{disc}(\text{Res}(\tilde{f}(zy, y), \tilde{g}(zy, y), y)) = \ll \text{¡Expresión excesivamente larga para ser mostrada!} \gg$$

The greatest common divisor of these polynomials is: $64b_y^2t^{36} + 64t^{38} + 64b_y^2t^{35} - 64b_yt^{36} - 64t^{37} + 16b_y^2t^{34} - 128b_yt^{35} - 48t^{36} - 80b_yt^{34} + 64t^{35} + 80t^{34} + 16b_yt^{32} + 40t^{33} - 96b_y^2t^{30} - 28b_yt^{31} - 68t^{32} - 192b_y^2t^{29} + 64b_yt^{30} + 52t^{31} + 64b_y^3t^{27} - 264b_y^2t^{28} + 344b_yt^{29} - 3t^{30} + 112b_y^3t^{26} - 256b_y^2t^{27} + 448b_yt^{28} + 56t^{29} + 8b_y^3t^{25} - 120b_y^2t^{26} + 320b_yt^{27} - 32t^{28} - 36b_y^2t^{25} + 238b_yt^{26} - 496t^{27} + 64b_y^2t^{24} - 52b_yt^{25} - 234t^{26} - 80b_y^3t^{22} + 26b_y^2t^{23} - 120b_yt^{24} - 114t^{25} - 32b_y^4t^{20} - 72b_y^3t^{21} + 169b_y^2t^{22} + 150b_yt^{23} + 218t^{24} - 32b_y^4t^{19} - 92b_y^3t^{20} + 632b_y^2t^{21} - 96b_yt^{22} + 222t^{23} + 72b_y^4t^{18} - 138b_y^3t^{19} + 832b_y^2t^{20} - 816b_yt^{21} - 117t^{22} + 40b_y^4t^{17} - 180b_y^3t^{18} + 194b_y^2t^{19} - 1146b_yt^{20} - 194t^{21} + b_y^4t^{16} - 160b_y^3t^{17} + 86b_y^2t^{18} - 724b_yt^{19} + 587t^{20} + 48b_y^3t^{16} + 178b_y^2t^{17} + 88b_yt^{18} + 840t^{19} - 40b_y^4t^{14} + 196b_y^3t^{15} - 61b_y^2t^{16} + 294b_yt^{17} - 59t^{18} + 8b_y^4t^{13} + 232b_y^3t^{14} - 28b_y^2t^{15} - 58b_yt^{16} - 100t^{17} - 16b_y^5t^{11} + 40b_y^4t^{12} + 26b_y^3t^{13} - 547b_y^2t^{14} - 450b_yt^{15} - 303t^{16} + 4b_y^5t^{10} - 10b_y^4t^{11} + 48b_y^3t^{12} - 928b_y^2t^{13} + 434b_yt^{14} + 14t^{15} + 14b_y^5t^9 - 30b_y^4t^{10} + 58b_y^3t^{11} - 301b_y^2t^{12} + 1220b_yt^{13} + 593t^{14} + 4b_y^5t^8 - 50b_y^4t^9 + 132b_y^3t^{10} - 50b_y^2t^{11} + 1016b_yt^{12} - 592t^{13} - 7b_y^4t^8 + 126b_y^3t^9 - 104b_y^2t^{10} + 190b_yt^{11} - 826t^{12} - 12b_y^5t^6 + 4b_y^4t^7 - 134b_y^3t^8 - 96b_y^2t^9 - 362b_yt^{10} - 206t^{11} + 4b_y^5t^4 - 10b_y^4t^5 + 55b_y^3t^6 - 116b_y^2t^7 + 218b_yt^8 - 146b_yt^9 + 306t^{10} + 4b_y^6t^3 - 6b_y^5t^4 + 30b_y^4t^5 - 76b_y^3t^6 + 154b_y^2t^7 + 38b_yt^8 + 166t^9 + b_y^6t^2 - 18b_y^5t^3 + 9b_y^4t^4 - 90b_y^3t^5 + 301b_y^2t^6 - 42b_yt^7 + 86t^8 - 12b_y^5t^2 + 34b_y^4t^3 + 24b_y^3t^4 + 210b_y^2t^5 - 236b_yt^6 - 144t^7 - 2b_y^5t + 30b_y^4t^2 - 42b_y^3t^3 - 102b_y^2t^4 - 336b_yt^5 - 148t^6 + 12b_y^4t - 58b_y^3t^2 + 66b_y^2t^3 - 66b_yt^4 + 396t^5 + b_y^4 - 12b_y^3t + 24b_y^2t^2 + 36b_yt^3 - 4b_y^3 + 18b_y^2t - 27t^2$

If b_y is a root of order zero, looking at the residual conditions over the root of valuation zero ($t = 0$) we have $\beta_y^4 - 4\beta_y^3 = 0$. The unique nonzero solution is $\beta_y = 4$. Note that, in particular, $\alpha_y\beta_{yy} - \alpha_{yy}\beta_y = -3 \neq 0$. So the intersection of the two conics intersects onto the stable intersection. An approximation up to order five of this root is:

$$b_y \sim 4 - \frac{544}{64}t + \frac{1079}{64}t^2 - \frac{4253}{128}t^3 + \frac{1043385}{16384}t^4 - \frac{4234233}{32768}t^5 + o(t^6)$$