Linear systems of surfaces with double points: Terracini revisited

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Abstract

In this paper we study the linear systems of degree m hypersurfaces in $\mathbb{P}^n$, with d fixed points of multiplicity e in general position, focusing on the case $n = 3$, $e = 2$, i.e., linear systems of surfaces in projective 3-space with d double points in general position. The goal is to compute the dimension of such systems. We present to modern readers a method due to Terracini, showing its similarities and differences to recent approaches.

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Introduction

We work over an algebraically closed field K of characteristic zero. Given a scheme $X \subset \mathbb{P}^n$, we denote by $\mathcal{I}_X$ its ideal sheaf, and by $I_X \subset K[\mathbb{P}^n] \cong K[x_0, \dots, x_n]$ its (saturated) homogeneous ideal. If $I \subset K[\mathbb{P}^n]$ is a homogeneous ideal, I_m will be its component in degree m.

Given a finite set of points $X \subset \mathbb{P}^n$, let X^e be the scheme consisting of these points taken with multiplicity e (i.e., if $\mathcal{I}$ is the ideal sheaf defining X as a reduced scheme, then X^e is the subscheme of $\mathbb{P}^n$ defined by $\mathcal{I}^e$). Thus, the

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linear system of degree m hypersurfaces going through the points of X with multiplicity at least e is $\mathcal{L}_{X^e}(m) := \mathbb{P}((I_{X^e})_m) = \mathbb{P}(H^0(\mathcal{I}^e(m)))$. When X is a general set of d points, the *virtual* and *expected* dimensions of $\mathcal{L}_{X^e}(m)$ are given by

$$\begin{aligned} \operatorname{vd}\mathcal{L}_{X^e}(m) &= \operatorname{vd}_{n;d,e}(m) = \binom{m+n}{n} - d\binom{e+n-1}{n} - 1, \\ \operatorname{ed}\mathcal{L}_{X^e}(m) &= \operatorname{ed}_{n;d,e}(m) = \max\{\operatorname{vd}_{n;d,e}(m), -1\}. \end{aligned}$$

When $e = 2$ and $n = 3$ we shall write simply $\operatorname{vd}_d(m) = \operatorname{vd}_{3;d,2}(m)$ and $\operatorname{ed}_d(m) = \operatorname{ed}_{3;d,2}(m)$. Remark that $\dim\mathcal{L}_{X^e}(m) = \operatorname{vd}_{n;s,e}(m)$ if and only if $h^1(\mathcal{I}_{X^e}(m)) = 0$. Following Terracini's paper [6], we shall prove

Theorem 1. *Let $q_m = \min\{d \in \mathbb{Z} | \operatorname{vd}_d(m) < 0\}$. Assume $m \geq 5$ and let X be a general set of q_m points in $\mathbb{P}^3$. Then $\dim\mathcal{L}_{X^2}(m) = \operatorname{ed}_{q_m}(m) = -1$.*

This is essentially the same as the main result of [1], [2] or [3], except that the latter apply to $\mathbb{P}^n, \forall n \geq 2$. To prove Theorem 1, Terracini uses semicontinuity by specializing X to $G = A \cup B$, where B is a general set of points in a plane π, and then implicitly applies the long exact sequence in cohomology of

$$0 \longrightarrow \mathcal{I}_{A^2 \cup B}(m-1) \longrightarrow \mathcal{I}_{G^2}(m) \longrightarrow \mathcal{I}_{B^2 \cap \pi | \pi}(m) \longrightarrow 0.$$

This is a very well known procedure nowadays, used in fact in all the above mentioned papers, which allows to prove the result by induction on the degree, if $|B|$ is chosen so that $H^0(\mathcal{I}_{A^2 \cup B}(m)) = H^0(\mathcal{I}_{B^2 \cap \pi | \pi}(m)) = 0$ or $H^1(\mathcal{I}_{A^2 \cup B}(m)) = H^1(\mathcal{I}_{B^2 \cap \pi | \pi}(m)) = 0$ (which is done by assuming the analogous result known in $\mathbb{P}^2$). However, since the degree of $B^2 \cap \pi$ is necessarily a multiple of 3, there are cases where it is impossible to choose $|B|$ in such a way. For these cases, some further technique is needed, and at this point Terracini's method diverges from that of Alexander-Hirschowitz or Chandler. Indeed, modern authors use sophisticated tools such as the differential Horace method. The approach presented here uses only the characterization of linear systems which admit a double point in general position in terms of their Jacobian matrix.

The paper is divided into two parts, in which the two cases $m \notin (3)$, $m \in (3)$ are separately dealt with. Proposition 1.1 gives the departure point for the induction, which is done in Propositions 1.4 and 2.2, thus proving Theorem 1.

1 The "easy" case: $m \notin (3)$

Assume that the dimension of linear systems with double points in $\mathbb{P}^2$ is known (as was known by Terracini, see [5], or use the result by Hirschowitz, [4]), that is, assume we know that for a general set X of d points

$$\dim\mathcal{L}_{X^2}(m) = \operatorname{ed}_{2;d,2}(m)$$

except for $d = 2$, $m = 2$ and for $d = 5$, $m = 4$, in which cases $\dim\mathcal{L}_{X^2}(m) = 0$.

Let $R_n = K[x_0, \ldots, x_n]$ be the polynomial ring in $n+1$ indeterminates, and write $R_{n,m}$ for the vector space of the forms of degree m. Set

$$r_{n,m} := \dim R_{n,m} = \binom{n+m}{n};$$

recall that, by definition, $q_m = \min\{d \in \mathbb{Z} | \operatorname{vd}_d(m) < 0\}$, so

$$q_m = \left\lceil \frac{r_{3,m}}{4} \right\rceil; \text{ define also } \eta_m := 4q_m - r_{3,m}.$$

Note that $0 \leq \eta_m \leq 3$ and $r_{n,m} - r_{n,m-1} = r_{n-1,m}$ for every m, n. The first step to prove Theorem 1 is given by the following proposition.

Proposition 1.1. *Let X be a general set of q_5 points in $\mathbb{P}^3$. Then $\dim \mathcal{L}_{X^2}(5) = -1$.*

Proof. Suppose that there is a surface, of degree 5 in $\mathbb{P}^3$ with $q_5 = 14$ double points in general position. Then by semicontinuity there is a surface $S^5 \subset \mathbb{P}^3$ of degree 5 having a set A of 7 double points general in $\mathbb{P}^3$ and a set B of other 7 double points general on a general plane π. In this settlement S^5 must contain π. In fact if $\pi \not\subset S^5$, then $\pi \cap S^5$ would be a quintic of π with 7 general double points. But in $\mathbb{P}^2$, $\dim \mathcal{L}_{B^2}(5) = \operatorname{ed}_{2;7,2}(5) = -1$ so such a quintic does not exist. Then $S^5 = \pi \cup S^4$ where S^4 is a surface of degree 4 passing through $A^2 \cup B$. But, by using similar ideas one sees that $\dim \mathcal{L}_{A^2}(4) = \operatorname{ed}_7(4) = 6$, and, by the genericity of π, the surfaces of degree 4 having double points in A cut on π a linear system of quartics of the same dimension, therefore none of them passes through all the points of B (since they are 7 general points), a contradiction. ☐

The following lemma of linear algebra is surely not new. However, as we have not been able to find a suitable reference, we include a brief proof.

Lemma 1.2. *Let E be a vector space, and let $F_1, \dots, F_k \subset E$ be a finite set of linear subspaces. Then*

$$\sum_{i=1}^{k} \dim F_i \leq \dim \left(\bigcap_{i=1}^{k} F_i \right) + (k-1) \dim \left(\sum_{i=1}^{k} F_i \right).$$

Proof. We proceed by induction on k. The case $k = 1$ is trivial and the case $k = 2$ is the well known Grassmann formula. If $k > 2$ we have

$$\begin{aligned} \sum_{i=1}^{k} \dim F_i &= \sum_{i=1}^{k-1} \dim F_i + \dim F_k \\ &\leq \dim \left(\bigcap_{i=1}^{k-1} F_i \right) + (k-2) \dim \left(\sum_{i=1}^{k-1} F_i \right) + \dim F_k \\ &= \dim \left(\bigcap_{i=1}^{k} F_i \right) + \dim \left(\bigcap_{i=1}^{k-1} F_i + F_k \right) + (k-2) \dim \left(\sum_{i=1}^{k-1} F_i \right) \\ &\leq \dim \left(\bigcap_{i=1}^{k} F_i \right) + \dim \left(\sum_{i=1}^{k} F_i \right) + (k-2) \dim \left(\sum_{i=1}^{k} F_i \right), \end{aligned}$$

as wanted. ☐

Lemma 1.3. *Suppose that there are no hypersurfaces of degree $m > 0$ with d points of multiplicity $\geq e$ in general position. Then $d' = d + \operatorname{vd}_{n;d,e}(m) + 1$ points of multiplicity e in general position impose independent conditions on hypersurfaces of degree m. In other words, denoting by $\mathcal{I}_s$ the ideal sheaf of a general set of s points in $\mathbb{P}^n$, $h^0(\mathcal{I}_d^e(m)) = 0$ implies $h^1(\mathcal{I}_{d'}^e(m)) = 0$.*

Note that $h^0(\mathcal{I}_d^e(m)) = 0$ implies $\mathrm{vd}_{n;d,e}(m) < 0$, so $d' \leq d$. Terracini proved this simple lemma in the case $n = e = 2$, and used it also when $n = 3$, $e = 2$. We give here a general proof, which follows his and uses linear algebra only.

Proof. Obviously we can assume $d' > 0$. We shall prove that $\forall s \leq d'$, and for a general set Y of s points of $\mathbb{P}^n$, $\dim \mathcal{L}_{Y^e}(m) = \mathrm{vd}_{n;s,e}(m)$, using induction on s. For $s = 0$ the claim is obvious.

The induction step goes as follows. The hypothesis of the lemma says that for a general set X of d points, $\dim \mathcal{L}_{X^e}(m) = -1$; i.e., $I_{X^e}(m) = 0$. Let $A \subset X$ be a subset of $s-1$ points of X (which are therefore general). The induction hypothesis is that $\dim \mathcal{L}_{A^e}(m) = \mathrm{vd}_{n;s-1,e}(m)$, and we shall prove that for every point $p \in X \setminus A$, $\dim \mathcal{L}_{A^e \cup p^e}(m) = \mathrm{vd}_{n;s,e}(m)$. Indeed, for each p of the $d-s+1$ points in $X \setminus A$ there is an inclusion

$$I_{A^e \cup p^e}(m) \subset I_{A^e}(m),$$

and on the other hand

$$\bigcap_{p \in X \setminus A} I_{A^e \cup p^e}(m) = I_{X^e}(m) = 0.$$

Applying lemma 1.2 we obtain

$$\sum_{p \in X \setminus A} \dim I_{A^e \cup p^e}(m) \leq (d-s) \dim \left(\sum_{p \in X \setminus A} I_{A^e \cup p^e}(m) \right), \text{ so}$$
$$(d-s+1) \dim I_{A^e \cup p^e}(m) \leq (d-s) \dim I_{A^e}(m) = (d-s)(\mathrm{vd}_{n;s-1,e}(m)+1).$$

Now an elementary computation shows that

$$(d-s)(\mathrm{vd}_{n;s-1,e}(m)+1) = (d-s+1)(\mathrm{vd}_{n;s,e}+1) - \mathrm{vd}_{n;d,e} - 1,$$

and the hypothesis that $s \leq d'$ implies $-\mathrm{vd}_{n;d,e} - 1 \leq d-s$, so putting everything together we obtain

$$\dim \mathcal{L}_{A^e \cup p^e}(m) \leq \mathrm{vd}_{n;s,e} + 1 + \frac{d-s}{d-s+1} - 1 < \mathrm{vd}_{n;s,e} + 1;$$

as we know that $\dim \mathcal{L}_{A^e \cup p^e}(m) \geq \mathrm{vd}_{n;s,e}$, the claim follows. □

Proposition 1.4. *If $m > 5$ is not multiple of 3 and, for a general set of q_{m-1} points $Y \subset \mathbb{P}^3$, $\dim \mathcal{L}_Y(m-1) = -1$ then for a general set of q_m points $X \subset \mathbb{P}^3$, $\dim \mathcal{L}_X(m) = -1$.*

Proof. Note that if m is not multiple of 3 then $r_{2,m}$ is multiple of 3.

Let $G = A \cup B$ be a set of points such that $|A| = q_m - \frac{r_{2,m}}{3}$ and $|B| = \frac{r_{2,m}}{3}$, and such that the points in A are general in $\mathbb{P}^3$ and the points in B are general in a plane $\pi \subset \mathbb{P}^3$. By semicontinuity, it is enough to prove that there is no surface S^m, $\deg S^m = m$, such that $G^2 \subset S^m$. If $\pi \not\subset S^m$ then the plane curve $S^m \cap \pi$ should contain $B^2_{|\pi}$ i.e. it should contain $\frac{r_{2,m}}{3}$ general double points of $\mathbb{P}^2$. But in $\mathbb{P}^2$, $\dim \mathcal{L}_{B^2_{|\pi}}(m) = \mathrm{ed}_{2;\frac{r_{2,m}}{3},2}(m) = -1$, so necessarily $\pi \subset S^m$. So $S^m = S^{m-1} \cup \pi$ with $\deg S^{m-1} = m-1$. Moreover S^{m-1} contains the schemes A^2 and B.

Since by hypothesis there is no surface of degree $m-1$ with q_{m-1} double points in general position, then, by Lemma 1.3, $q_{m-1} + \mathrm{vd}_{q_{m-1}}(m-1) + 1$ points of multiplicity 2 in general position impose independent conditions on surfaces of degree $m-1$. Since $\mathrm{vd}_{q_{m-1}}(m-1)+1 = r_{3,m-1} - 4q_{m-1} = r_{3,m-1} - 4\frac{r_{3,m-1}+\eta_{m-1}}{4} = -\eta_{m-1} \geq -3$ we have that $d \leq q_{m-1} - 3$ points of multiplicity 2 in general position impose independent conditions on surfaces of degree $m-1$.

Now we have

$$\begin{gathered} x_m := q_{m-1} - |A| = \left\lceil \frac{r_{3,m-1}}{4} \right\rceil - \left\lceil \frac{r_{3,m}}{4} \right\rceil + \frac{r_{2,m}}{3} = \\ \frac{r_{3,m-1}+\eta_{m-1}}{4} - \frac{r_{3,m}+\eta_m}{4} + \frac{r_{2,m}}{3} = \\ \frac{-r_{2,m}+\eta_{m-1}-\eta_m}{4} + \frac{r_{2,m}}{3} = \\ \frac{r_{2,m}}{12} + \frac{\eta_{m-1}-\eta_m}{4} \geq \frac{15}{4} - \frac{3}{4} = 3, \end{gathered}$$

for $m \geq 8$. Moreover $x_7 = 21 - 30 + 12 = 3$, therefore $x_m \geq 3$ for every $m > 5$ and m not multiple of 3. Since $|A| = q_{m-1} - x_m \leq q_{m-1} - 3$, A^2 imposes independent conditions on surfaces of degree $m-1$. Let $\mathcal{L} = \mathcal{L}_{A^2}(m-1)$ be the linear system of surfaces of degree $m-1$ containing A^2. Then

$$\begin{gathered} \dim \mathcal{L} = r_{3,m-1} - 4|A| - 1 = r_{3,m-1} - 4\left\lceil \frac{r_{3,m}}{4} \right\rceil + 4\frac{r_{2,m}}{3} - 1 = \\ r_{3,m-1} - r_{3,m} - \eta_m + 4\frac{r_{2,m}}{3} - 1 = \frac{r_{2,m}}{3} - \eta_m - 1. \end{gathered}$$

$\mathcal{L}$ cuts on π a linear system of curves, of degree $m-1$, $\mathcal{L}_{|\pi}$ whose dimension is $\leq \frac{r_{2,m}}{3} - \eta_m - 1$. Since the points in B are in general position, they impose $|B| = \frac{r_{2,m}}{3}$ independent conditions on $\mathcal{L}_{|\pi}$; so the subsystem of $\mathcal{L}_{|\pi}$ of the curves through B has dimension $\leq -\eta_m - 1 < 0$, a contradiction. □

2 The "hard" case: $m \in (3)$

As already mentioned in the introduction, if m is a multiple of 3 the method of the previous section does not work, because then $r_{2,m}$ is not a multiple of 3, so one needs some extra subtlety. Terracini obtains the needed information via a nice lemma on Jacobians of linear systems. As was the case for Lemma 1.3, the following lemma is valid in $\mathbb{P}^n$ for arbitrary n with the same proof Terracini gave for particular cases in $\mathbb{P}^3$.

Let now $R = K[x_0, \ldots, x_n]$ be the polynomial ring in $n+1$ indeterminates, and write R_m for the vector space of the forms of degree m. If $G_0, \ldots, G_s \in R_m$ are linearly independent forms spanning a linear system $\mathcal{L}$, we denote by $\mathrm{Jac}(G_0, \ldots, G_s)$ their Jacobian matrix. Note that the rank of the Jacobian matrix evaluated at a given point does not depend on the set of generators $G_0, \ldots, G_s \in R_m$, but only on the linear system $\mathcal{L}$, and that it is maximal for a general point (i.e., on a dense open set of $\mathbb{P}^n$); thus we define the rank of the Jacobian of $\mathcal{L}$ (at a general point) as $\mathrm{rank}_{\mathrm{J}}\,\mathcal{L} := \mathrm{rank}\,\mathrm{Jac}(G_0, \ldots, G_s)$. It is not hard to see that $\mathrm{rank}_{\mathrm{J}}\,\mathcal{L} \leq \dim \mathcal{L}$ if and only if for every point $p \in \mathbb{P}^n$ there are hypersurfaces in $\mathcal{L}$ with multiplicity at least 2 at p.

Lemma 2.1. *Let $\mathcal{L}$ be a linear system in $\mathbb{P}^n$ with $\dim \mathcal{L} \leq n$, and let $\pi \subset \mathbb{P}^n$ be a hyperplane. Assume*

1. $\operatorname{rank}_{\mathrm{J}} \mathcal{L} \leq \dim \mathcal{L}$*,*

2. $\dim(\mathcal{L} - \pi) = \dim \mathcal{L} - 1$.

Then $\operatorname{rank}_{\mathrm{J}} (\mathcal{L} - \pi)_{|\pi} \leq \dim (\mathcal{L} - \pi)_{|\pi}$.

Proof. Take coordinates such that the hyperplane π is defined by $x_0 = 0$, and let $s = \dim \mathcal{L}$. By the second assumption, the linear system $\mathcal{L}$ is spanned by $F, x_0 G_1, \ldots, x_0 G_s$ for some $F \in R_m$, $G_i \in R_{m-1}$, such that x_0 does not divide F. Therefore by the first assumption the matrix

$$\begin{pmatrix} F_{x_0} & G_1 + x_0 (G_1)_{x_0} & \cdots & G_s + x_0 (G_s)_{x_0} \\ F_{x_1} & x_0 (G_1)_{x_1} & \cdots & x_0 (G_s)_{x_1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{x_n} & x_0 (G_1)_{x_n} & \cdots & x_0 (G_s)_{x_n} \end{pmatrix}$$

has not maximal rank, i.e., its maximal minors vanish. Expanding these minors according to powers of x_0 and collecting terms with x_0^{s-1}, one sees that the maximal minors of

$$\begin{pmatrix} 0 & G'_1 & \cdots & G'_s \\ F'_{x_1} & (G'_1)_{x_1} & \cdots & (G'_s)_{x_1} \\ \vdots & \vdots & \ddots & \vdots \\ F'_{x_n} & (G'_1)_{x_n} & \cdots & (G'_s)_{x_n} \end{pmatrix}$$

must vanish, where for every form $P(x_0, \ldots, x_n)$, we set $P' = P(0, x_1, \ldots, x_n)$. Now applying the Euler identity, also the maximal minors of

$$\begin{pmatrix} -mF' & 0 & \cdots & 0 \\ F'_{x_1} & (G'_1)_{x_1} & \cdots & (G'_s)_{x_1} \\ \vdots & \vdots & \ddots & \vdots \\ F'_{x_n} & (G'_1)_{x_n} & \cdots & (G'_s)_{x_n} \end{pmatrix}$$

must vanish. But $F' \neq 0$, because x_0 does not divide F, so the maximal minors of

$$\begin{pmatrix} (G'_1)_{x_1} & \cdots & (G'_s)_{x_1} \\ \vdots & \ddots & \vdots \\ (G'_1)_{x_n} & \cdots & (G'_s)_{x_n} \end{pmatrix}$$

vanish, i.e., $\operatorname{Jac}(G'_1, \ldots G'_s)$ does not have maximal rank. As $G'_1, \ldots G'_s$ obviously span $(\mathcal{L} - \pi)_{|\pi}$, we are done. ☐

Proposition 2.2. *Let* m *be multiple of* 3 *and assume that either* $m = 6$ *or for a general set of* q_{m-1} *points* $Y \subset \mathbb{P}^3$, $\dim \mathcal{L}_Y(m-1) = -1$ *and, for a general set of* q_{m-2} *points* $Z \subset \mathbb{P}^3$, $\dim \mathcal{L}_Z(m-2) = -1$*, then, for a general set of* q_m *points* $X \subset \mathbb{P}^3$, $\dim \mathcal{L}_X(m) = -1$.

Proof. Let $G = A \cup B$ be a set of points such that $|A| = q_m - \frac{r_{2,m-1}}{3} - 1$ and $|B| = \frac{r_{2,m-1}}{3}$, and such that the points in A are general in $\mathbb{P}^3$ and the points in B are general on a plane π.

First we would like to compute the dimension of the linear system $\mathcal{L} = \mathcal{L}_{G^2}(m)$. Of course we have that $\dim \mathcal{L} \geq r_{3,m} - 1 - 4|G| = 3 - \eta_m$. We shall show that the equality holds. Let

$$\begin{aligned} x_m := q_{m-1} - |A| = \left\lceil \frac{r_{3,m-1}}{4} \right\rceil - \left\lceil \frac{r_{3,m}}{4} \right\rceil + \frac{r_{2,m} - 1}{3} + 1 = \\ \frac{r_{3,m-1} + \eta_{m-1}}{4} - \frac{r_{3,m} + \eta_m}{4} + \frac{r_{2,m} - 1}{3} + 1 = \\ \frac{r_{2,m} - 1}{12} + \frac{\eta_{m-1} - \eta_m + 3}{4}. \end{aligned} \tag{1}$$

Note that $\frac{r_{2,9}-1}{12} = \frac{9}{2}$ and that $\frac{\eta_{m-1}-\eta_m+3}{4} \geq 0$ so $x_m > 4$ for every $m \geq 9$, m multiple of 3. Moreover $x_6 = 3$, so $x_m \geq 3$ for every $m \geq 6$, m multiple of 3. Then, since by hypothesis there is no surface of degree $m-1$ with q_{m-1} double points in general position, using Lemma 1.3 we get that A^2 imposes $4|A|$ independent conditions on the linear system of the surfaces of degree $m-1$. So, writing $\mathcal{L}_1 = \mathcal{L}_{A^2}(m-1)$ we have

$$\dim \mathcal{L}_1 = r_{3,m-1} - 1 - 4|A| = \frac{r_{3,m} - 1}{3} + 2 - \eta_m$$

Moreover

$$\begin{aligned} |A| - q_{m-2} = \frac{r_{3,m} + \eta_m}{4} - \frac{r_{2,m} - 1}{3} - 1 - \frac{r_{3,m-2} + \eta_{m-2}}{4} = \\ \frac{3r_{3,m} - 3r_{3,m-1} + 3r_{3,m-1} - 3r_{3,m-2} - 4r_{2,m} + 3(\eta_m - \eta_{m-2}) - 8}{12} = \\ \frac{3r_{2,m-1} - r_{2,m} + 3(\eta_m - \eta_{m-2}) - 8}{12} = \frac{m^2 - 1 + 3(\eta_m - \eta_{m-2}) - 8}{12} = \\ \frac{m^2}{12} + \frac{\eta_m - \eta_{m-2} - 3}{4} \geq \frac{m^2}{12} - \frac{6}{4} > \frac{m^2}{12} - 2 \geq 1, \end{aligned}$$

i.e. $|A| > q_{m-2}$ for every m. On the other hand, for $m \geq 9$, m multiple of 3, the assumptions say that there is no surface of degree $m-2$ with q_{m-2} double points in general position, so $\dim \mathcal{L}_{A^2}(m-2)) = -1$. For $m = 6$ we have $|A| = 21 - 9 - 1 = 11$ and there is no quartic surface with 11 double points in general position, so $\dim \mathcal{L}_{A^2}(m-2) = -1$, for every $m > 5$, m multiple of 3. Now let $\mathcal{L}'_1$ be the linear system of plane curves of degree $m-1$ cut by $\mathcal{L}_1$ on π. If $\dim \mathcal{L}'_1 < \dim \mathcal{L}_1$ then there is in $\mathcal{L}_1$ a surface containing π and consequently there should be a surface of degree $m-2$ containing A^2, a contradiction. So we have that $\dim \mathcal{L}'_1 = \dim \mathcal{L}_1$. Moreover, since the points in B are in general position, the curves of $\mathcal{L}'_1$ through B form a linear system $\mathcal{L}'_1(B)$ whose dimension is $\dim \mathcal{L}'_1 - |B| = 2 - \eta_m$ and the surfaces of $\mathcal{L}_1$ through B form a linear system $\mathcal{L}(B)$ of the same dimension. But

$$\dim \mathcal{L}_{B^2}(m) = \mathrm{ed}_{2;|B|,2}(m) = \max\{r_{2,m} - 3|B| - 1, -1\} = 0$$

so we have only one curve in π through B^2 and consequently $\dim \mathcal{L} \leq 2 - \eta_m + 1 = 3 - \eta_m$ i.e. $\dim \mathcal{L} = 3 - \eta_m$. Moreover $\mathcal{L}$ is generated by $\pi\mathcal{L}_1(B)$, whose dimension is $2 - \eta_m$ and by one surface not containing π.

To complete the proof it is enough to show that there are no surfaces in $\mathcal{L}$ having a further double point in general position or equivalently that the jacobian matrix of $\mathcal{L}$ has maximal rank.

If $\eta_m = 3$ or $\eta_m = 2$ it is trivial. Let us suppose that $\eta_m = 1$ and that the jacobian matrix of $\mathcal{L}$ has not maximal rank. Then using Lemma 2.1 we obtain that the jacobian matrix of the linear system $\mathcal{L}'_1(B)$ has not maximal rank. But $\dim \mathcal{L}'_1(B) = 2 - \eta_m = 1$, a contradiction.

Finally let us suppose that $\eta_m = 0$ and that the jacobian matrix of $\mathcal{L}$ has not maximal rank. Then $\dim \mathcal{L}'_1(B) = 2 - \eta_m = 2$, and by Lemma 2.1, the jacobian matrix of $\mathcal{L}'_1(B)$ has not maximal rank, so all the curves in $\mathcal{L}'_1(B)$ are reducible. Then the surfaces in $\mathcal{L}_1$ cut on every general plane π a linear system $\mathcal{L}'_1$, $\dim \mathcal{L}'_1 = |B| + 2$, such that all the jacobian matrices of the 2-dimensional linear systems determined from it by fixing $|B|$ general points have not maximal rank. Consequently the curve of $\mathcal{L}'_1$ obtained by fixing $|B| + 2$ general points is reducible, therefore every curve in $\mathcal{L}'_1$ is reducible. Then all the surfaces in $\mathcal{L}_1$ have reducible plane section, so they are reducible. It follows that either $\mathcal{L}_1$ has a fixed component or it is a pencil involution.

more detailed explanation. If $\mathcal{L}_1$ has a fixed component S^d, $\deg S^d = d$, then by the genericity, the points in A should be either all double, or all simple, or S^d does not pass through A. They cannot be double for S^d because does not exist any surface of degree less than $m-1$ through A^2. If they were simple then the surfaces in the movable part of $\mathcal{L}_1$, of degree $m-d-1$, should pass through A also. So we should have simultaneously

$$r_{3,d} - 1 \geq |A| \quad \text{and} \quad r_{3,m-d-1} - 1 \geq |A| + |B| + 2$$

i.e.

$$r_{3,d} \geq q_m - \frac{r_{2,m} - 1}{3} \quad \text{and} \quad r_{3,m-d-1} \geq q_m + 2; \tag{2}$$

but these inequalities are incompatible. In fact, since $r_{3,m} < \frac{(m+2)^3}{6}$ and $q_m \geq \frac{r_{3,m}}{4}$ for every m, they imply that

$$\frac{(d+2)^3}{6} > \frac{r_{3,m}}{4} - \frac{r_{2,m} - 1}{3} \quad \text{and} \quad \frac{(m-d+1)^3}{6} > \frac{r_{3,m}}{4} + 2;$$

i.e.

$$(d+2)^3 > \frac{m^3 + 2m^2 - m + 6}{4} \quad \text{and} \quad (m-d+1)^3 > \frac{(m+1)(m+2)(m+3)}{4} + 12,$$

from which we get

$$(d+2)^3 > \frac{m^3}{4} \quad \text{and} \quad (m-d+1)^3 > \frac{m^3}{4};$$

so we have

$$\frac{m}{\sqrt[3]{4}} - 1 < m - d < m - \frac{m}{\sqrt[3]{4}} + 2$$

and comparing the first with the last term we obtain

$$m < \frac{3}{\sqrt[3]{2} - 1} < 12;$$

moreover if $m = 6$, $q_6 = 21$ and the inequalities (2) imply

$$(d+2)^3 > 6(21 - 9) = 72 \quad \text{and} \quad (7 - d)^3 > 6(21 + 2) = 138,$$

i.e. simultaneously $d > 2$ and $d < 2$; if $m = 9$, $q_9 = 55$ and we get

$$(d+2)^3 > 6(55-18) = 222 \quad \text{and} \quad (10-d)^3 > 6(55+2) = 342$$

i.e. simultaneously $d > 4$ and $d < 4$.

If S^d does not pass through A then there should be a surface of degree $n-d-1 < n-1$ passing through A^2, again a contradiction.

If $\mathcal{L}_1$ were a pencil involution then its jacobian matrix would not have maximal rank, so $|A|+1$ double points in general position should impose on $|\mathcal{O}_{\mathbb{P}^3}(m-1)|$ less than $4(|A|+1)$ independent conditions. But using (1) and recalling that $\eta_m = 0$ we have

$$q_{m-1} - (|A|+1) = \frac{r_{2,m}-1}{12} + \frac{\eta_{m-1}+3}{4} - 1 \geq \frac{27}{12} + \frac{3}{4} - 1 = 2.$$

Then if $0 \leq \eta_{m-1} \leq 2$, $q_{m-1} - (|A|+1) \geq \eta_{m-1}$; if $\eta_{m-1} = 3$ then $m \geq 9$ and

$$q_{m-1} - (|A|+1) \geq \frac{54}{12} + \frac{6}{4} - 1 \geq 5 > \eta_{m-1},$$

so in any case $q_{m-1} - (|A|+1) \geq \eta_{m-1} = -\operatorname{vd}_{3;q_{m-1},2}(m-1)$; then we can apply Lemma 1.3 and we obtain that $|A|+1$ general double points impose $4(|A|+1)$ independent conditions on surfaces of degree $m-1$, a contradiction. ☐

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