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MULTIPOINT SESHADRI CONSTANTS ON $\mathbb{P}^2$

Abstract. Working over **C** and formalizing and sharpening approaches introduced in [12], [9] and [8], we give a method for verifying when a divisor on a blow up of $\mathbb{P}^2$ at general points is nef. The method is useful both theoretically and when doing computer computations. The main application is to obtaining lower bounds on multipoint Seshadri constants on $\mathbb{P}^2$. In combination with methods developed in [4], significantly improved explicit lower bounds are obtained.

Given a positive integer n, the Seshadri constant for points $p_1, \dots, p_n$ of $\mathbb{P}^2$ is the real number

$$\varepsilon(N, p_1, \dots, p_n) = \inf\left\{\frac{\deg(C)}{\Sigma_{i=1}^n \mathrm{mult}_{p_i} C}\right\},$$

where the infimum is taken with respect to all curves C, through at least one of the points. We also take $\varepsilon(n)$ to be defined as $\sup\{\varepsilon(p_1, \dots, p_n)\}$, where the supremum is taken with respect to all choices of n distinct points p_i of $\mathbb{P}^2$ (see [6], [2] and [11]). It is well known and not difficult to prove that $\varepsilon(n) \leq 1/\sqrt{n}$, with equality if n is a square. Also, by results of Nagata [6], $\varepsilon(n)$ is known for $n < 10$, and, when $n \geq 10$ is not a square, Nagata [7] conjectured that $\varepsilon(n) = 1/\sqrt{n}$. Although this conjecture has not yet been verified for any $n \geq 10$ not a square, the general belief is that it is correct, hence the attention paid here and elsewhere to obtaining lower bounds for $\varepsilon(n)$, focusing in the case $n \geq 10$.

Here, refining an approach of [9] and [10] (see also Tutaj-Gasińska's contribution to the present volume) which in turn refine and extend the method used in [12], we give a method that provides a basis for obtaining arbitrarily accurate estimates of $\varepsilon(n)$, which we apply to obtain lower bounds for $\varepsilon(n)$ which for almost all n improve on the bounds cited above. Let us denote by $\alpha(m, p_1, \dots, p_n)$ (respectively, $\alpha_0(m, p_1, \dots, p_n)$) the least degree of a curve (respectively, irreducible curve) passing with multiplicity at least m (respectively, exactly m) through each point p_i. If the points are in general position in $\mathbb{P}^2$, we write simply $\alpha(m^{[n]})$ and $\alpha_0(m^{[n]})$. Our method involves two steps. The first step shows how to convert estimates of values of α to bounds on $\varepsilon(n)$. The second step, based on our work in [4], concerns actually making the estimates of the values of α.

To provide a basis for making comparisons of different lower bounds on $\varepsilon(n)$, it is convenient to write them in the form $\varepsilon(n) \geq (1/\sqrt{n})(\sqrt{1 - 1/f(n)})$, where f is a function of n. Note that the larger $f(n)$ is, the better is the bound.

THEOREM 1. *Let $n \geq 10$ be an integer, and $\mu \geq 1$ a real number.*

1. *If $\alpha(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ for every integer $1 \leq m < \mu$, then $\varepsilon(n) > \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{(n-2)\mu}}$.*

2. *If* $\alpha_0(m^{[n]}) \geq m\sqrt{n - \frac{1}{\mu}}$ *for every integer* $1 \leq m < \mu$, *and if* $\mu \leq 6(n-1)$, *then* $\varepsilon(n) \geq \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{n\mu}}$.

The basic tool for the proof of Theorem 1 (to be found in [5]) is the study of *abnormal* curves, i.e., irreducible counterexamples to Nagata's conjecture. From the properties of the intersection product on general blowups of $\mathbb{P}^2$ we obtain restrictions on abnormal curves; using these, for every $t < 1/\sqrt{n}$ we can produce an explicit finite list of tuples $(d, m_1, \ldots, m_n)$ such that if $\varepsilon(n) < t$ then for some degree d and multiplicities m_i on the list there exists an abnormal curve $C(d, m_1, \ldots, m_n)$ whose degree and multiplicities are one of the entries of the list, and thus $\varepsilon(n) = d/(m_1 + \cdots + m_n)$. So to conclude that $\varepsilon(n) \geq t$ it is enough to show that each tuple on the list does *not* correspond to an irreducible plane curve. For any specific n, our best lower bound on $\varepsilon(n)$ is obtained by direct application of this method. For each nonsquare $10 \leq n \leq 58$, Table 1 gives the best value we know for $f(n)$ (truncated to two decimals), along with a possible abnormal curve $C(d, m^{[n]})$ which we are unable to rule out but which would have to be ruled out in order to verify a larger value for $f(n)$.

This direct approach is algorithmic; by analyzing the algorithm, based on our work in [4], we are also able to give weaker but explicit lower bounds in terms of n.

COROLLARY 1. *Let* $n > 16$ *be a nonsquare integer, let* $d = \lfloor\sqrt{n}\rfloor$ *and consider* $\Delta = n - d^2 > 0$. *Let us define*

$$f(n) = \begin{cases} n(n-1) & \text{if } \Delta = 2, \\ n(n - 3\sqrt{n} - 4)/2 & \text{if } \Delta > 2 \text{ is even}, \\ n(n - 3\sqrt{n} - 2) & \text{if } \Delta \text{ is odd and } \Delta < 4\sqrt[4]{n} + 1, \\ n^2 & \text{if } \Delta \text{ is odd and } 2d - 1 > \Delta \geq 4\sqrt[4]{n} + 1, \\ n(n\sqrt{n} - 5n + 5\sqrt{n} - 1)/2 & \text{if } \Delta = 2d - 1; \end{cases}$$

then $\varepsilon(n) \geq \frac{1}{\sqrt{n}}\sqrt{1 - \frac{1}{f(n)}}$.

Perhaps the best previous general bound for $n \geq 10$ is given in [10], for which $f(n) = 12n + 1$. As Corollary 1 shows, for our bounds $f(n)$ is at least quadratic in n, so for n large enough (indeed, for $n \geq 59$), our bounds involve larger values of $f(n)$. For special values of n, [1] also gives bounds better than those of [10], and these bounds are also quadratic in n. However, except when $n - 1$ is a square, our bounds are better, for n large enough. Bounds are also given in [3], which are almost always better than any bound for which $f(n)$ is linear in n. Although these bounds are not simple enough to make comparisons easy, computations for specific values of n show in almost all cases that the bounds we obtain here are better than those of [3]. The results shown in Table 1 for $n - 1$ a square and for $n = 19, 22$ are given by [1] and are as good or better than what we obtain; the result for $n = 41$ comes from [3]; all other results shown in Table 1 are better than what was known previously.

n	f	C(d,m[n])	n	f	C(d,m[n])	n	f	C(d,m[n])
10	886.62	C(256,81)	27	997.96	C(161,31)	43	1741.5	C(236,36)
11	402.28	C(106,32)	28	1304.25	C(201,38)	44	1985.5	C(252,38)
12	300.52	C(83,24)	29	639.45	C(113,21)	45	3782.25	C(275,41)
13	325	C(90,25)	30	1230.76	C(219,40)	46	3140.26	C(217,32)
14	740.6	C(86,23)	31	1093.26	C(128,23)	47	7109.17	C(994,145)
15	566.78	C(89,23)	32	940.52	C(147,26)	48	1521.39	C(187,27)
17	1089	C(136,33)	33	1093.55	C(178,31)	50	9801	C(700,99)
18	466.94	C(89,21)	34	1731.93	C(239,41)	51	3313.98	C(407,57)
19	28900	C(5928,1360)	35	974.47	C(136,23)	52	6257.33	C(274,38)
20	660.64	C(143,32)	37	5329	C(444,73)	53	3499.89	C(313,43)
21	1187.1	C(142,31)	38	1898.97	C(265,43)	54	5713.2	C(338,46)
22	38809	C(7392,1576)	39	1779.7	C(231,37)	55	2370.64	C(304,41)
23	576	C(115,24)	40	1601.66	C(196,31)	56	3193.01	C(419,56)
24	1009.2	C(142,29)	41	1025	C(160,25)	57	2608.42	C(234,31)
26	2601	C(260,51)	42	1306.94	C(149,23)	58	9802	C(396,52)

Table 1: Current best known values of $f(n)$ for nonsquares $10 \leq n \leq 58$

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