# Workshop on Categorical Groups

Barcelona, June 16th–20th, 2008

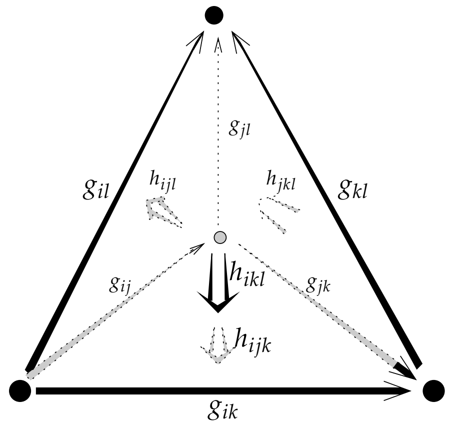
— an event within the CRM special year on
Homotopy Theory and Higher Categories
`http://www.crm.cat/HOCAT/`

## Participants

- Aldrovandi, Ettore
- Anel, Mathieu
- Baez, John
- Bakovic, Igor
- Bartlett, Bruce
- Bokor, Imre
- Broto, Carles
- Cantero, Federico
- Carrasco, M. Pilar
- Casacuberta, Carles
- del Río, Aurora
- El Kaoutit Zerri, Laiachi
- Elgueta, Josep
- Fernandez Boix, Alberto
- Hernández Paricio, L. Javier
- Hoffnung, Alexander
- Joyal, André
- Kock, Anders
- Kock, Joachim
- Marciniak, Dorota
- Martínez Moreno, Juan
- Muro, Fernando
- Neumann, Frank
- Noohi, Behrang
- Pan, Weiwei
- Pitsch, Wolfgang
- Polesello, Pietro
- Porter, Timothy
- Raventós, Oriol
- Rodríguez Blancas, J. Luis
- Rodríguez Garzón, Antonio
- Rousseau, Alain
- Tonks, Andrew
- Vitale, Enrico
- Wise, Derek
- Zambon, Marco

**John Baez**

## Topological 2-groups and their classifying spaces




Abstract: Categorifying the concept of topological group, one obtains the notion of a topological 2-group. This in turn allows a theory of "principal 2-bundles" generalizing the usual theory of principal bundles. It is well-known that under mild conditions on a topological group $G$ and a space $M$, principal $G$-bundles over $M$ are classified by either the Cech cohomology $H^1(M, G)$ or the set of homotopy classes $[M, BG]$, where $BG$ is the classifying space of $G$. Here we review work by Bartels, Jurco, Baas-Bökstedt-Kro, and others generalizing this result to topological 2-groups. We explain various viewpoints on topological 2-groups and the Cech cohomology $H^1(M, \mathbf{G})$ with coefficients in a topological 2-group $\mathbf{G}$, also known as "nonabelian cohomology". Then we sketch a proof that under mild conditions on $M$ and $\mathbf{G}$ there is a bijection between $H^1(M, \mathbf{G})$ and $[M, B\,|\mathbf{G}|]$, where $B\,|\mathbf{G}|$ is the classifying space of the geometric realization of the nerve of $\mathbf{G}$. Applying this result to the "string 2-group" $\mathrm{String}(G)$ of a simply-connected compact simple Lie group $G$, we obtain a theory of characteristic classes for principal $\mathrm{String}(G)$-2-bundles.

## Bruce Bartlett

## Geometry of $2$-representations of groups and their $2$-characters

Abstract: In general, a 2-representation of a group is a group action on some sort of linear category, and can be regarded as the "categorification" of the idea of an ordinary representation of a group. One can also introduce the "2-character" of a 2-representation, which produces an equivariant vector bundle over the group. In this talk I will explain the geometric picture of unitary 2-representations of groups on 2-Hilbert spaces and their 2-characters in terms of equivariant gerbes, and I will show how this is the categorification of the classical geometric quantization idea of ordinary representations in terms of equivariant line bundles. I will also explain how the 2-character can be made functorial with respect to morphisms of 2-representations. Indeed it becomes a unitarily faithful functor, which is the categorification of the fact that the ordinary character is a unitary map from the representation ring to the space of class functions.

**Laiachi El Kaoutit Zerri**

## On coendomorphism bialgebroids

(Joint work with C. Menini and A. Ardizzoni.)

Abstract: In the paper [D. Tambara, *The coendomorphism bialgebra of an algebra*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **37** (1990), 425–456] it was shown that to any finite dimensional algebra $A$ one can associate an endofunctor $\mathscr{L}$ in the category of algebras over a fixed field, which is right adjoint to the functor $A \otimes -$. This leads obviously to a structure of bialgebra on $\mathscr{L}(A)$. Tambara claimed that this result holds also if the base field is replaced by a not necessarily commutative ring $R$ and the tensor product by the Sweedler-Takeuchi product $- \times_R -$. Unfortunately, he did not give any proof of this fact. In this talk we discuss the following aspects. Given $R \rightarrow A$ any extension of unitary associative rings such that $A$ is finitely generated and projective as left $R$-module, the functor $- \times_R A$ is well defined from the category of $(R \otimes R^o)$-rings to the category of $R$-rings. This functor has a right adjoint. The image of $A$ by this latter functor admits a structure of (left) $R$-bialgebroid (called coendomorphism bialgebroid). This will give a procedure of constructing new examples of bialgebroids over noncommutative rings.

## Enrico Vitale

## Basic homological algebra in 2-categories

Abstract: In the first part of my talk, I will describe some fundamental constructions in the 2-category of (symmetric) categorical groups: kernel and cokernel, relative kernel and relative cokernel, pip and root. These constructions, which are all instances of bilimits, will be used to define efficient notions of exactness for categorical groups, extension, homology. Finally, I will put these notions at work to prove basic homological lemmas. Main references for this part are [1] and [2].

In the second part, following recent works of my students Mathieu Dupont [3] and Beppe Metere [4], I will develop one of the following topics (maybe both, depending on time and on the interest of participants):

– A careful analysis of the previous constructions leads to an axiomatic 2-categorical environment for homological algebra, i.e. to a possible definition of abelian 2-category.

– A higher dimensional version of kernel and exactness can be defined, so to get a "ziqqurath" of exact sequences of $n$-groupoids from a morphism of $n$-groupoids.

[1] Del Río, Martínez-Moreno, Vitale: *Chain complexes of symmetric categorical groups*, JPAA **196** (2005).
[2] Carrasco, Garzón, Vitale: *On categorical crossed modules*, TAC **16** (2006).
[3] Dupont: *Catégories abéliennes en dimension* 2, Ph. D. Thesis, Université catholique de Louvain (2008).
[4] Metere: *The Ziqqurath of exact sequences of n-groupoids*, Ph. D. Thesis, Università di Milano (2008).

**Aurora del Río**

## Algebraic $K$-theory for categorical groups

Abstract: Homotopy categorical groups of any pointed space are defined via the fundamental groupoid of iterated loop spaces. This notion allows, paralleling the group case, to introduce the notion of $K$-categorical groups $K_iR$ of any ring $R$. We also show the existence of a fundamental categorical crossed module associated to any fibre homotopy sequence, and then $K_1R$ and $K_2R$ are characterized, respectively, as the homotopy cokernel and kernel of the fundamental categorical crossed module associated to the fibre homotopy sequence

$$F(R) \xrightarrow{d_R} BGL(R) \xrightarrow{q_R} BGL(R)^+ .$$

## Luis Javier Hernández

## Categorical groups and $[n, n+1]$-types of exterior spaces

(Joint work with A. del Río and M. T. Rivas.)

Abstract: The objective of this talk is to present a program to develop some algebraic models for the study of exterior $\mathbb{R}_+$-$[n, n+1]$-types and exterior $\mathbb{N}$-$[n, n+1]$-types.

Firstly, we introduce a small category $\mathcal{C}$ with a finite number of objects which admits a "presentation" with finitely many maps and relations that induces a category of presheaves $\boldsymbol{Set}^{\mathcal{C}^{op}}$ which contains algebraic properties of categorical groups and the combinatorial information to give its topological realization. (It plays a role similar to 1-reduced Kan simplicial sets.) Using this small category we give an alternative description of the classifying space of a categorical group. Similar small categories are also given for braided and symmetric categorical groups.

On one hand, one has that categorical groups, braided categorical groups and symmetric categorical groups are models for exterior $\mathbb{R}_+$-$[1, 2]$-types, $\mathbb{R}_+$-$[2, 3]$-types and $\mathbb{R}_+$-$[n, n+1]$-types, $n \geq 3$, respectively. Nevertheless, on the other hand, one can see that using the induced canonical functors, these categories are not good algebraic models for $\mathbb{N}$-$[n, n+1]$-types. We think that an adequate extension of the small category $\mathcal{C}$ will induce an enriched notion of categorical group that will give a better algebraic model for $\mathbb{N}$-$[n, n+1]$-types.

## Derek Wise

## Representations of 2-groups on higher Hilbert spaces

Abstract: The general theory of 2-group representations is a straightforward categorification of group representation theory: representations are 2-functors, intertwining operators are pseudonatural transformations, and 2-intertwiners between these are modifications. However, while the category of vector spaces is the standard target category for group representations, one of the major challenges of 2-group representation theory has been to find a suitable target 2-category. For example, one would like a 2-category in which Lie 2-groups have an interesting variety of representations. One promising proposal is the 2-category ***Meas*** of "measurable categories", which first appeared in the work of Crane, Sheppeard, and Yetter. I will explain this 2-category, whose objects can be thought of as infinite-dimensional "higher Hilbert spaces". I will then describe representations of 2-groups on these higher Hilbert spaces, together with the intertwiners and 2-intertwiners between them. Examples will clarify their geometric meaning. I will also briefly discuss how 2-group representations in ***Meas*** have already shown up in physics, particularly in the state sum model of Baratin and Freidel.

(Joint work with John Baez, Aristide Baratin, and Laurent Freidel.)

## Behrang Noohi

## Butterflies and morphisms between weak 2-groups

Abstract: A lax monoidal functor between two 2-groups can be presented by a certain canonical diagram of groups called a butterfly. Butterflies form the 1-morphisms of a bicategory which is a model for the 2-category of pointed connected 2-types. The cohomological data (e.g., cocycles) of a lax functor can be very easily recovered from its butterfly. The canonicity of the butterfly (and the fact that it is cocycle-free), however, makes it particularly useful in geometric applications. We will briefly discuss three applications to illustrate this:

1. Classification of group actions on stacks (and calculation of their quotients).
2. Functorial study of principal 2-bundles (e.g., explicit description of 'extension of the structure 2-group via a lax monoidal functor').
3. Nonabelian generalization of Deligne's result relating additive functors between Picard stacks to the derived category of abelian sheaves. Time permitting, we will also discuss higher butterflies (which encode morphisms between higher homotopy types).

Part of this work is joint with E. Aldrovandi and will be discussed in more detail in his talk.

**Ettore Aldrovandi**

## Butterflies, morphisms between gr-stacks, and non-abelian cohomology

Abstract: Weak morphisms of cat-groups can be encoded by certain diagrams called butterflies. Cat-groups, equipped with weak morphisms and an appropriate notion of 2-morphism, form a fibered bicategory. We describe this bicategory, and argue that it is, in an appropriate sense, a bi-stack. From another viewpoint, weak morphisms of cat-groups can be characterized (actually, defined) in terms of additive functors between the corresponding (associated) gr-stacks. It turns out one can construct a functor sending weak morphisms between cat-groups to additive functors between their associated gr-stacks. We show that this functor is an equivalence of stacks, and in addition conclude that the bicategory of cat-groups and the 2-category of gr-stacks are bi-equivalent. (In fact they are bi-equivalent as bi-stacks.) This dual description extends to higher degree non-abelian cohomology: The first non-abelian cohomology set with values in a cat-group can be characterized either in terms of gerbes bound by it, or in terms of torsors over the associated gr-stack. Given two cat-groups $H$ and $G$, we show how the push-forward of torsors over the gr-stack associated to $H$ can be explicitly described as a push-forward of a gerbe bound by $H$ along a butterfly from $H$ to $GH$. If time permits, we will sketch how these ideas are extended to 2-crossed modules and 2-gerbes.

**Pietro Polesello**

## Character for locally constant stacks

Abstract: Locally constant stacks are the higher analogue of locally constant sheaves, and they are classified by the representations of the fundamental 2-group of a topological space (i.e. the fundamental groupoid of its loop space). In this talk we show how to associate to a locally constant stack its character, that is, a locally constant sheaf on the loop space which has character-like properties. This is done by using the notion of character of a representation of a 2-group, which was introduced by Ganter and Kapranov for the discrete 2-groups. If times allows, we concentrate on the case of $BG$, for a given topological group $G$.

## Tim Porter

### Classifying spaces of categorical groups, and relations with non-Abelian cohomology

Abstract: (We will adopt a simplicial viewpoint throughout the background section at least. The key reference will be Breen's 1990 Bitorsors paper. Many of the arguments there are becoming increasingly relevant to today's problems in categorical groups, yet they have not all been absorbed.)

Historically categorical groups first occurred in the linked contexts of modelling homotopy 2-types and interpreting cohomology. We will trace the route from categorical groups to simplicial groups to classifying spaces via the Wbar construction. Some of the properties of this will be looked at in some detail from the point of view of simplicial T-complexes and complicial sets. Turning to cohomology, we will recall $G$-torsors, and then link $G$-bitorsors with the Cech approach to cocycles. From bitorsors it is an easy step to $M$-torsors for $M$ a sheaf of categorical groups and these link back to the classifying space of $M$ as a simplicial sheaf. Back in homotopy theory we look at the classical Puppe sequence, and interpret it for a fibration of classifying spaces of categorical groups. This leaves some terms we know and some which need interpreting. We will discuss briefly the approach given by Breen and another due to Debremaker, whose work in 1975 is related to other talks here.

After the break we will visit TQFT land before looking at their variant the HQFTs of Turaev. These come with a base as coefficients. Looking at the simplest case of the base, if it is a classifying space of a group, we're fine, but what if it is a classifying space of a categorical group. Some results will be discussed and the link with non-Abelian cohomology will be examined. This leads to questions! We really need to extend the Puppe sequence a bit more, but it is not clear how to. I will discuss some ideas for doing this, if time permits. This will involve 2-crossed modules, 3-types, 3-groups and some slightly speculative ideas. The other main question is to see what the change of base functors do on HQFTs, especially using Puppe type arguments and fibration sequences of categorical groups.

## Fernando Muro

## Categorical groups in brave new algebra

(Based on joint work with H.-J. Baues.)

Abstract: Categorical groups play an important role in homotopy theory as algebraic models for homotopy types with only two non-vanishing homotopy groups in consecutive dimensions. Nerve constructions yield spaces and spectra out of categorical groups. One can go back by taking homotopy groupoids.

However it seems that categorical groups have not been exploited enough in connection with the new monoidal structures in stable homotopy theory, a.k.a. brave new algebra.

Elmendorf and Mandell (2006) modified Segal's construction to obtain brave new rings (i.e. ring spectra) out of categories with ring structure. Similarly for modules, etc.

The aim of this talk is to present a functor going in thr opposite direction. We will introduce categorical (commutative) rings, categorical algebras, categorical modules... together with their graded versions, and we will show how to associate such 2-dimensional algebraic structures to a symmetric spectrum. This will be done by means of a theory of homotopy 2-groups for spectra.

Everything relies on the computation of an algebraic model for the symmetric monoidal 2-category of one-point unions of copies of the sphere spectrum. This 2-category lifts the symmetric monoidal category of free abelian groups. The model we compute is as small as possible, it is built upon the category of free groups of nilpotency class 2. In the process we use a new Hopf invariant for tracks which can be computed by using Clifford algebras.

## Andy Tonks

## Categorical groups in $K$-theory and number theory

Abstract: Generalising the Knudsen-Mumford notion of determinant modules to not necessarily commutative rings $R$, Deligne remarked that the first Postnikov piece of the $K$-theory spectrum of $R$ is in fact classified by a categorical group $V(R)$ of so-called virtual objects. This monoidal category $V(R)$ is the target of the universal determinant functor on the category of perfect complexes of $R$-modules.

In joint work with Fernando Muro we extended this construction to Waldhausen categories $W$ and give a functorial, 2-nilpotent, strict symmetric categorical group, or Picard category, $DW$ which models the 1-type of the $K$-theory of $W$ and has a straightforward presentation with generators given by the basic data of $W$: the objects, weak equivalences and cofibration sequences. This object $DW$ is also the target of the universal determinant functor on $W$.

Our method of proof uses bisimplicial sets and a Cartier-Eilenberg-Zilber theorem for crossed complexes. It also suggests the existence of a higher-dimensional theory, perhaps using cat$^n$-groups. This would be especially useful for Witte's work on the Fukaya-Kato conjecture in non-commutative Iwasawa theory, but much of the necessary machinery, such as a monoidal structure analogous to that of crossed complexes, is still to be built.

## Schedule

Workshop on Categorical Groups — Barcelona, 16th–20th, 2008

| | Monday | Tuesday | Wednesday | Thursday | Friday |
|---|---|---|---|---|---|
| 10:00–11:00 | Baez | Vitale | Wise | Noohi | Porter |
| 11:00–11:30 | coffee break | coffee break | coffee break | coffee break | coffee break |
| 11:30–13:00 | Baez | Vitale | Wise | Noohi | Porter |
| 13:00–15:00 | lunch | lunch | | lunch | lunch |
| 15:00–16:00 | Bartlett | del Río | | Aldrovandi | Muro |
| 16:00–16:30 | coffee break | coffee break | | coffee break | coffee break |
| 16:30–17:30 | El Kaoutit | Hernández | | Polesello | Tonks |

Morning talks: Aula B6

Afternoon talks: Aula 03 (aulari Josep Carner, entrance from carrer Aribau)