On coendomorphism bialgebroids

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In 1990, D. Tambara, proved that if A is a finite dimensional algebra over field $\Bbbk$, then the functor

$$A \otimes_{\Bbbk} - : \Bbbk - \mathtt{algebras} \longrightarrow \Bbbk - \mathtt{algebras}$$

has a left adjoint

$$\mathcal{A}(A, -) : \Bbbk - \mathtt{algebras} \longrightarrow \Bbbk - \mathtt{algebras}.$$

The algebra $\mathcal{A}(A, A)$ has a natural structure of bialgebra (*coendomorphism bialgebra*) and coacts universally on the algebra A. It turns out that, if $\dim(A) > 1$, then the category of right $\mathcal{A}(A, A)$-comodules is equivalent as monoidal category to the category of chain complexes of $\Bbbk$-modules.

This extended Manin's works on quadratic bialgebras, and those of Pareigis on certain Hopf algebras.

Tambara claimed (without any indication on the proof) that the above adjunction is a special case of the following one. Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal categories and $\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$ an adjunction between them with $\mathcal{R}$ a monoidal functor. If $\mathcal{N}$ has inductive limits and the multiplication commutes with this limits, then this adjunction induces an adjunction between the associated categories of monoids.

In case of the noncommutative base-ring R, he claimed that an application of this monoidal result by taking an R-ring A with finitely generated projective underlying left module, leads to the construction of an $\times_R$-bialgebra in the sense of Takeuchi.

By a result of T. Brzeziński and G. Militaru, there a bijection between $\times_R$-bialgebras and bialgebroids. If we know with a detailed proof Tambara's claims, then we can offer a proceses of constructing a new examples of noncommutative bialgebroid.

Bialgebroids are in some sense a generalization of bialgebras in the framework of monoidal categories of bimodules. Here are some examples of this object:

•) In algebraic topology, any Hopf algebroid in the sense of D. Ravenel, is a commutative bialgebroid with a commutative base-ring.

•) The Heisenberg double, i.e. the smash product $A^*\sharp A$, is a bialgebroid over a finite dimensional Hopf algebra A [Jaing-Hua Lu, 1995]. (In 2002, T. Brzeziński and G. Militaru gave a more general construction of this.)

•) Let H be a bialgebra and B an algebra together with a measuring $\rightharpoonup: H \otimes B \to B$ and convolution invertible cocycle $\sigma : H \otimes H \to B$. If $\rightharpoonup$ is a σ-twisted H-module structure, the $B \otimes H \otimes B^o$ is a bialgebroid over B [Schauenburg, 1998].

•) Any depth-2 ring extension leads to a bialgebroid [L. Kadison and K. Szlachányi, 2002].

Let $(\mathcal{M}, \otimes, \mathfrak{a}, \mathfrak{l}, \mathfrak{r}, \mathbb{I})$ be a monoidal category. We say that the multiplication $\otimes$ *preserve coequalizers* (if they exist) provided that, for every object $Y \in \mathcal{M}$, the functors $- \otimes Y$ and $Y \otimes -$ preserve them.
Let (A, μ_A, η_A) be a monoid in $\mathcal{M}$, for each morphism $\alpha : X \to A$ in $\mathcal{M}$, we associated the morphism

$$\begin{tikzcd}
(A \otimes X) \otimes A \arrow[r, "\mathfrak{a}"] \arrow[drrr, dashed, "\Lambda_\alpha"'] & A \otimes (X \otimes A) \arrow[r, "1(\alpha 1)"] & A \otimes (A \otimes A) \arrow[r, "1\mu_A"] & A \otimes A \arrow[d, "\mu_A"] \\
& & & A
\end{tikzcd}$$

So we have the following commutative diagram

$$\begin{tikzcd}
(\mathbb{I} \otimes X) \otimes \mathbb{I} \arrow[r, "(\eta_A X)\eta_A"] & (A \otimes X) \otimes A \arrow[r, "\Lambda_\alpha"] & A \\
X \arrow[u, "\cong"] \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

Lemma

Let $(\mathcal{M}, \otimes, \mathfrak{a}, \mathfrak{l}, \mathfrak{r}, \mathbb{I})$ be a monoidal category with coequalizers. Assume that $\otimes$ preserve coequalizers. Then the category $\mathcal{M}_m$ of monoids in $\mathcal{M}$ has coequalizers too.

Explicitly, let $\alpha, \beta : E \to A$ be homomorphisms of monoids in the category $\mathcal{M}$. Then the coequalizer

$$
\begin{tikzcd}
(A \otimes E) \otimes A \arrow[rr, shift left, "\Lambda_\alpha"] \arrow[rr, shift right, "\Lambda_\beta"'] & & A \arrow[r, "\pi"] & B \\
(\mathbb{I} \otimes E) \otimes \mathbb{I} \arrow[u, "(\eta_A E)\eta_A"] & & & \\
E \arrow[u, "\cong"] \arrow[uurr, bend right, shift left, "\alpha"] \arrow[uurr, bend right, shift right, "\beta"'] & & &
\end{tikzcd}
$$

(B, π) of $(\Lambda_\alpha, \Lambda_\beta)$ in $\mathcal{M}$ carries a unique monoid structure such that (B, π) is the coequalizer of (α, β) in the category $\mathcal{M}_m$.

Let $(\mathcal{M}, \otimes_{\mathcal{M}}, \mathbb{I}_{\mathcal{M}})$ and $(\mathcal{N}, \otimes_{\mathcal{N}}, \mathbb{I}_{\mathcal{N}})$ be a monoidal categories. A *monoidal functor* from $\mathcal{M}$ to $\mathcal{N}$ is a triple (F, Φ^2, Φ^0) where $F : \mathcal{N} \to \mathcal{M}$ is a functor, $\Phi^0 : \mathbb{I}_{\mathcal{M}} \to F(\mathbb{I}_{\mathcal{N}})$ is a morphism and

$$\Phi^2_{(-,-)} : F(-) \otimes_{\mathcal{M}} F(-) \longrightarrow F(- \otimes_{\mathcal{N}} -)$$

a natural transformation such that

$$\begin{tikzcd}
 & F(U) \otimes_{\mathcal{M}} (F(V) \otimes_{\mathcal{M}} F(W)) \arrow[r, "F(U)\otimes \Phi^2_{(V,\, W)}"] & F(U) \otimes_{\mathcal{M}} F(V \otimes_{\mathcal{N}} W) \arrow[d, "\Phi^2_{(U,\, V\otimes W)}"] \\
(F(U) \otimes_{\mathcal{M}} F(V)) \otimes_{\mathcal{M}} F(W) \arrow[ur, "\cong"] \arrow[d, "\Phi^2_{(U,\, V)}\otimes F(W)"'] & & F(U \otimes_{\mathcal{N}} (V \otimes_{\mathcal{N}} W)) \\
F(U \otimes_{\mathcal{N}} V) \otimes_{\mathcal{M}} F(W) \arrow[r, "\Phi^2_{(U\otimes V,\, W)}"'] & F((U \otimes_{\mathcal{N}} V) \otimes_{\mathcal{N}} W) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$$\begin{tikzcd} \mathbb{I}_{\mathcal{M}} \otimes_{\mathcal{M}} F(U) \arrow[r, "\Phi^0 \otimes F(U)"] \arrow[d, "\cong"'] & F(\mathbb{I}_{\mathcal{N}}) \otimes_{\mathcal{M}} F(U) \arrow[d, "\Phi^2_{(\mathbb{I}_{\mathcal{N}},\, U)}"] \\ F(U) \arrow[r, "\cong"'] & F(\mathbb{I}_{\mathcal{N}} \otimes_{\mathcal{N}} U) \end{tikzcd} \qquad \begin{tikzcd} F(U) \otimes_{\mathcal{M}} \mathbb{I}_{\mathcal{M}} \arrow[r, "F(U) \otimes \Phi^0"] \arrow[d, "\cong"'] & F(U) \otimes_{\mathcal{M}} F(\mathbb{I}_{\mathcal{N}}) \arrow[d, "\Phi^2_{(U,\, \mathbb{I}_{\mathcal{N}})}"] \\ F(U) \arrow[r, "\cong"'] & F(U \otimes_{\mathcal{N}} \mathbb{I}_{\mathcal{N}}) \end{tikzcd}$$

A *comonoidal functor* from $\mathcal{M}$ to $\mathcal{N}$ is a monoidal functor from $\mathcal{M}$ to $\mathcal{N}^o$ (the dual category of $\mathcal{N}$).
Consider an adjunction

$$\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$$

between two monoidal categories with $\mathcal{L}$ left adjoint to $\mathcal{R}$.
Denote by

$$\theta_{-} : id_{\mathcal{M}} \longrightarrow \mathcal{R}\mathcal{L}, \quad \xi_{-} : \mathcal{L}\mathcal{R} \longrightarrow id_{\mathcal{N}}$$

the unit and the counit of this adjunction.

Assume that $\mathcal{R}$ is a monoidal functor with structure morphisms $\Phi^2_{(-,-)} : \mathcal{R}(-) \otimes \mathcal{R}(-) \longrightarrow \mathcal{R}(- \otimes -)$ and $\Phi^0 : \mathbb{I}_{\mathcal{M}} \longrightarrow \mathcal{R}(\mathbb{I}_{\mathcal{N}})$. For every pair of objects X and Y in $\mathcal{B}$, we set

$$\begin{tikzcd}
\mathcal{L}(X \otimes Y) \arrow[rr, dashed, "\Psi^2_{(X,Y)}"] \arrow[d, "\mathcal{L}(\theta_X \otimes \theta_Y)"'] & & \mathcal{L}(X) \otimes \mathcal{L}(Y) \\
\mathcal{L}\left(\mathcal{R}\mathcal{L}(X) \otimes \mathcal{R}\mathcal{L}(Y)\right) \arrow[rr, "\mathcal{L}(\Phi^2_{(\mathcal{L}(X),\mathcal{L}(Y))})"'] & & \mathcal{L}\mathcal{R}\left(\mathcal{L}(X) \otimes \mathcal{L}(Y)\right) \arrow[u, "\xi_{\mathcal{L}(X) \otimes \mathcal{L}(Y)}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{L}(\mathbb{I}_{\mathcal{M}}) \arrow[r, dashed, "\Psi^0"] \arrow[d, "\mathcal{L}(\Phi^0)"'] & \mathbb{I}_{\mathcal{N}} \\
\mathcal{L}\mathcal{R}(\mathbb{I}_{\mathcal{N}}) \arrow[ur, "\xi_{\mathbb{I}_{\mathcal{N}}}"'] &
\end{tikzcd}$$

One can show that the three-tuple $(\mathcal{L}, \Psi^2_{(-,-)}, \Psi^0)$ is in fact a comonoidal functor. The converse is also true. That is, if $\mathcal{L}$ is a comonoidal functor, then $\mathcal{R}$ is a monoidal functor. We thus arrive to the following well known fact in monoidal categories:

Lemma

Let $(\mathcal{M}, \otimes_{\mathcal{M}}, \mathbb{I}_{\mathcal{M}})$ and $(\mathcal{N}, \otimes_{\mathcal{N}}, \mathbb{I}_{\mathcal{N}})$ be monoidal categories. Consider an adjunction

$$\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{R}$$

with $\mathcal{L}$ is a left adjoint to $\mathcal{R}$. Then

$\mathcal{L}$ is comonoidal if and only if $\mathcal{R}$ is monoidal.

It is clear that any monoidal functor $\mathcal{R} : \mathcal{N} \to \mathcal{M}$ induces a functor $\mathcal{R}_m : \mathcal{N}_m \to \mathcal{M}_m$ between the corresponding categories of monoids such that the following diagram

$$\begin{tikzcd}
\mathcal{M}_m \arrow[r, "\mathcal{R}_m"] \arrow[d, "\mathcal{H}"'] & \mathcal{N}_m \arrow[d, "\mathcal{H}'"] \\
\mathcal{M} \arrow[r, "\mathcal{R}"'] & \mathcal{N}
\end{tikzcd}$$

is commutative, where $\mathcal{H}$ and $\mathcal{H}'$ are the forgetful functors.

In what follows, we want to construct a left adjoint functor $\mathcal{L}_m : \mathcal{M}_m \to \mathcal{N}_m$, when $\mathcal{N}$ has an inductive limits and $\mathcal{R}$ is a monoidal functor with left adjoint $\mathcal{L} : \mathcal{M} \to \mathcal{N}$.

Assume that the handled monoidal category $\mathcal{N}$ has inductive limits. Following Mac Lane the forgetful functor $\mathcal{H}' : \mathcal{N}_m \to \mathcal{N}$ admits a left adjoint $\mathcal{T} : \mathcal{N} \to \mathcal{N}_m$, where for every object X in $\mathcal{N}$,

$$\mathcal{T}(X) \;=\; \mathbb{I} \oplus X \oplus (X \otimes X) \oplus \cdots \oplus X^{\otimes^n} \oplus \cdots$$

is the tensor monoid of X. Here the objects $X^{\otimes^n}$ stand for $X^{\otimes^n} := X^{\otimes^{n-1}} \otimes X$, for $n \geq 2$ where $X^{\otimes^1} := X$ and by convention $X^{\otimes^0} := \mathbb{I}$. We denote by $\iota_n^X : X^{\otimes^n} \to \mathcal{T}(X)$, $n \geq 0$, the canonical injections.
Assume that $\mathcal{R}$ is a monoidal functor with structure morphisms $\Phi^2_{(-,-)}$ and Φ^0, and with a left adjoint functor $\mathcal{L}$. As in the previous Lemma, we consider the structure morphisms $\Psi^2_{(-,-)}$ and Ψ^0 of the comonoidal functor $\mathcal{L}$.

Given a monoid (B, μ_B, η_B) in $\mathcal{M}$, we have four morphisms

$$\begin{tikzcd} \mathcal{L}(\mathbb{I}_{\mathcal{M}}) \arrow[rr, dashed, "\alpha_1"] \arrow[dr, "\Psi^0"'] & & \mathcal{T}\mathcal{L}(B) \\ & \mathbb{I}_{\mathcal{N}} \arrow[ur, "\iota_0^{\mathcal{L}(B)}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(\mathbb{I}_{\mathcal{M}}) \arrow[rr, dashed, "\beta_1"] \arrow[dr, "\mathcal{L}(\eta_B)"'] & & \mathcal{T}\mathcal{L}(B) \\ & \mathcal{L}(B) \arrow[ur, "\iota_1^{\mathcal{L}(B)}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{L}(B \otimes B) \arrow[rr, dashed, "\alpha_2"] \arrow[dr, "\Psi^2_{(B,B)}"'] & & \mathcal{T}\mathcal{L}(B) \\ & \mathcal{L}(B) \otimes \mathcal{L}(B) \arrow[ur, "\iota_2^{\mathcal{L}(B)}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{L}(B \otimes B) \arrow[rr, dashed, "\beta_2"] \arrow[dr, "\mathcal{L}(\mu_B)"'] & & \mathcal{T}\mathcal{L}(B) \\ & \mathcal{L}(B) \arrow[ur, "\iota_1^{\mathcal{L}(B)}"'] & \end{tikzcd}$$

By the universal property of the tensor monoid there are unique homomorphisms of monoids

$$f_1, g_1 : \mathcal{TL}(\mathbb{I}_{\mathcal{M}}) \to \mathcal{TL}(B) \qquad \text{and} \qquad f_2, g_2 : \mathcal{TL}(B \otimes B) \to \mathcal{TL}(B)$$

such that

$$\begin{tikzcd} \mathcal{TL}(\mathbb{I}_{\mathcal{M}}) \arrow[r, dashed, "f_1"] & \mathcal{TL}(B) \\ \mathcal{L}(\mathbb{I}_{\mathcal{M}}) \arrow[u, "\iota_1^{\mathcal{L}(\mathbb{I}_{\mathcal{M}})}"] \arrow[ur, "\alpha_1"'] & \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{TL}(\mathbb{I}_{\mathcal{M}}) \arrow[r, dashed, "g_1"] & \mathcal{TL}(B) \\ \mathcal{L}(\mathbb{I}_{\mathcal{M}}) \arrow[u, "\iota_1^{\mathcal{L}(\mathbb{I}_{\mathcal{M}})}"] \arrow[ur, "\beta_1"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{TL}(B \otimes B) \arrow[r, dashed, "f_2"] & \mathcal{TL}(B) \\ \mathcal{L}(B \otimes B) \arrow[u, "\iota_1^{\mathcal{L}(B \otimes B)}"] \arrow[ur, "\alpha_2"'] & \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{TL}(B \otimes B) \arrow[r, dashed, "g_2"] & \mathcal{TL}(B) \\ \mathcal{L}(B \otimes B) \arrow[u, "\iota_1^{\mathcal{L}(B \otimes B)}"] \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

are commutative diagrams.

So we have the following commutative diagram of coequalizers in the category of monoids $\mathcal{N}_m$

$$\begin{tikzcd}
\mathcal{T}\mathcal{L}(\mathbb{I}_{\mathcal{M}}) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] & \mathcal{T}\mathcal{L}(B) \arrow[r, "\gamma_1"] \arrow[rr, bend left, "\pi_B"] & E_1 \arrow[r, "\gamma_2"] & E_2 := E_B \\
& \mathcal{T}\mathcal{L}(B \otimes B) \arrow[u, shift left, "f_2"] \arrow[u, shift right, "g_2"'] \arrow[ur, shift left, "\gamma_1 \circ f_2"] \arrow[ur, shift right, "\gamma_1 \circ g_2"'] & &
\end{tikzcd}$$

The pair (E_B, π_B) is shown to be the universal object equalizing in $\mathcal{N}$ at the same times both the pairs $(\Lambda_{\alpha_1}, \Lambda_{\beta_1})$ and $(\Lambda_{\alpha_2}, \Lambda_{\beta_2})$.

Over objects, the functor $\mathcal{L}_m : \mathcal{M}_m \to \mathcal{N}_m$ is then defined by

$$\mathcal{L}_m(B, \mu_B, \eta_B) := E_B.$$

Now, for every morphism $h : (B, \mu_B, \eta_B) \to (B', \mu_{B'}, \eta_{B'})$ in $\mathcal{M}_m$, one can show that $\pi_{B'} \circ \mathcal{TL}\,(h)$ equalizes both the pairs $\left(\Lambda_{\alpha_1^B}, \Lambda_{\beta_1^B}\right)$ and $\left(\Lambda_{\alpha_2^B}, \Lambda_{\beta_2^B}\right)$. Therefore there exists a unique homomorphism of monoids

$$\mathfrak{L}_m(h) : \mathcal{L}_m(B, \mu_B, \eta_B) \to \mathcal{L}_m(B', \mu_{B'}, \eta_{B'})$$

rendering commutative the following diagram

$$\begin{tikzcd}
(\mathcal{TL}(B) \otimes \mathcal{L}(\mathbb{I}_{\mathcal{M}})) \otimes \mathcal{TL}(B) \arrow[dr, shift left, "\Lambda_{\alpha_1}"] \arrow[dr, shift right, "\Lambda_{\beta_1}"'] & & \\
& \mathcal{TL}(B) \arrow[r, "\pi_B"] \arrow[d, "\mathcal{TL}(h)"] & E_B \arrow[d, dashed, "\mathcal{L}_m(h)"] \\
(\mathcal{TL}(B) \otimes \mathcal{L}(B \otimes B)) \otimes \mathcal{TL}(B) \arrow[ur, shift left, "\Lambda_{\alpha_2}"] \arrow[ur, shift right, "\Lambda_{\beta_2}"'] & \mathcal{TL}(B') \arrow[r, "\pi_{B'}"'] & E_{B'}
\end{tikzcd}$$

This complets the construction of the functor $\mathcal{L}_m : \mathcal{N}_m \to \mathcal{M}_m$.

Consider an object (C, μ_C, η_C) in $\mathcal{N}_m$, and let $\phi_C : \mathcal{T}(C) \to C$ be the unique monoid homomorphism that restricted to C gives the identity.
It turns out that ϕ_C satisfies the following two equalities:

$$\begin{aligned} \varphi_C \circ \mathcal{T}(\xi_C) \circ \Lambda_{\alpha_1^{\mathcal{R}(C)}} &= \varphi_C \circ \mathcal{T}(\xi_C) \circ \Lambda_{\beta_1^{\mathcal{R}(C)}} \qquad \text{and} \\ \varphi_C \circ \mathcal{T}(\xi_C) \circ \Lambda_{\alpha_2^{\mathcal{R}(C)}} &= \varphi_C \circ \mathcal{T}(\xi_C) \circ \Lambda_{\beta_2^{\mathcal{R}(C)}}. \end{aligned}$$

where ξ_{-} is the counit of the adjunction $\mathcal{L} \dashv \mathcal{R}$.
By the universal property of $\left(E_{\mathcal{R}(C)}, \pi_{\mathcal{R}(C)}\right)$, there is a unique homomorphism of monoids

$$\xi_C^m : E_{\mathcal{R}(C)} = \mathcal{L}_m \mathcal{R}_m(C) \longrightarrow C \ \text{ such that } \ \xi_C^m \circ \pi_{\mathcal{R}(C)} = \varphi_C \circ \mathcal{T}(\xi_C).$$

This leads to a natural transformation

$$\xi_{-}^m : \mathcal{L}_m \mathcal{R}_m \to id_{\mathcal{N}_m}.$$

On the other hand, for every object $(B, \mu_B, \eta_B) \in \mathcal{M}_m$ we define

$$\theta_B^m : B \to \mathcal{R}_m \mathcal{L}_m (B) = \mathcal{R}(E_B)$$

by

$$\theta_B^m := \mathcal{R}(\pi_B) \circ \mathcal{R}\left(i_1^{\mathcal{L}(B)}\right) \circ \theta_B,$$

where θ_{-} is the unit of the adjunction $\mathcal{L} \dashv \mathcal{R}$. This is morphisms of monoids, which leads to a natural transformation

$$\theta_{-}^m : id_{\mathcal{M}_m} \to \mathcal{R}_m \mathcal{L}_m.$$

Theorem

Let $(\mathcal{M}, \otimes_{\mathcal{M}}, \mathbb{I}_{\mathcal{M}})$ *and* $(\mathcal{N}, \otimes_{\mathcal{N}}, \mathbb{I}_{\mathcal{N}})$ *be a monoidal categories. Let* $\mathcal{L} \dashv \mathcal{R}$ *be an adjunction with unit* θ *and counit* ξ*, and where* $\mathcal{R} : \mathcal{N} \to \mathcal{M}$ *is a monoidal functor with structure morphisms* $\Phi^2_{(-,-)}$ *and* Φ^0*. Then* $\mathcal{R}$ *induces a functor*

$$\mathcal{R}_m : \mathcal{N}_m \longrightarrow \mathcal{M}_m.$$

Assume that $\mathcal{N}$ *has inductive limits and that the tensor product preserves them. Then* $\mathcal{R}_m$ *has a left adjoint* $\mathcal{L}_m : \mathcal{M}_m \longrightarrow \mathcal{N}_m$ *with unit* θ^m *and counit* ξ^m *above defined.*

Example

Let $\mathcal{A}$ and $\mathcal{B}$ two Grothendieck categories. We denote by $\overline{\mathrm{Funct}}\,(\mathcal{A},\,\mathcal{B})$ the set-category of continuous functors from $\mathcal{A}$ to $\mathcal{B}$ (i.e. functors which commute with inductive limits, or equivalently, they are right exact and commute with direct sums). In this way, we consider $\overline{\mathrm{Funct}}\,(\mathcal{A},\,\mathcal{A})$, and $\overline{\mathrm{Funct}}\,(\mathcal{B},\,\mathcal{B})$ as a strict monoidal categories.
Assume that there is an adjunction $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ with $F \dashv G$, and $F \in \overline{\mathrm{Funct}}\,(\mathcal{A},\,\mathcal{B})$, $G \in \overline{\mathrm{Funct}}\,(\mathcal{B},\,\mathcal{A})$. Let $\xi : FG \to id_{\mathcal{B}}$ and $\theta : id_{\mathcal{A}} \to GF$ are, respectively, the counit and unit of this adjunction. One can easily check that that the following functor

$$\begin{array}{ccc}
\overline{\mathrm{Funct}}\,(\mathcal{B},\,\mathcal{B}) & \xrightarrow{\quad \mathcal{R} \quad} & \overline{\mathrm{Funct}}\,(\mathcal{A},\,\mathcal{A}) \\
H & \longrightarrow & GHF \\
\left[\sigma : H \to H'\right] & \longrightarrow & \left[G\sigma_F : GHF \to GH'F\right]
\end{array}$$

is a monoidal functor with the following structure morphisms

Example (follows)

$$\Phi^2_{H,\,H'} \,:\, \mathcal{R}(H)\mathcal{R}(H') \xrightarrow{\;GH\eta_{H'F}\;} \mathcal{R}(HH')\ ,$$

$$\Phi^0 \,:\, id_{\mathcal{A}} \xrightarrow{\;\theta\;} GF = \mathcal{R}(id_{\mathcal{B}})$$

It is not difficult to check that the functor

$$\begin{array}{ccc} \overline{\mathrm{Funct}}\,(\mathcal{A},\,\mathcal{A}) & \xrightarrow{\quad\mathcal{L}\quad} & \overline{\mathrm{Funct}}\,(\mathcal{B},\,\mathcal{B}) \\ T & \longrightarrow & FTG \\ \left[\alpha : T \to T'\right] & \longrightarrow & \left[F\alpha_G : FTG \to FT'G\right] \end{array}$$

is left adjoint to $\mathcal{R}$. Since $\overline{\mathrm{Funct}}\,(\mathcal{B},\,\mathcal{B})$ has cokernels and direct sums its has an inductive limits. Therefore, we can assert using the previous Theorem that the adjunction $\mathcal{L} \dashv \mathcal{R}$ can be restricted to the categories of monoids $\overline{\mathrm{Funct}}\,(\mathcal{B},\,\mathcal{B})_m$ and $\overline{\mathrm{Funct}}\,(\mathcal{A},\,\mathcal{A})_m$.

We work over a ground commutative ring with 1 denoted by $\Bbbk$. All rings are $\Bbbk$-algebras, and bimodules are assumed to be central $\Bbbk$-bimodules. Given a ring R, we denote by $- \otimes_R -$ the tensor product over R. The unadorned symbol $\otimes$ stands for the tensor product over $\Bbbk$. As usual, we use the symbols $\mathrm{Hom}_{R-}(-,-)$, $\mathrm{Hom}_{-R}(-,-)$ and $\mathrm{Hom}_{R-R}(-,-)$ to denote the Hom-functor of left R-linear maps, right R-linear maps and R-bilinear maps, respectively.
Given an R-bimodule X, we denote especially by ${}^*X = \mathrm{Hom}_{R-}\Big(X,\, R\Big)$ it left dual and will be considered as an R-bimodule via the canonical actions

$$r\varphi : x \mapsto \varphi(xr), \;\; \text{and} \;\; \varphi s : x \mapsto \varphi(x)s,$$

for every $\varphi \in {}^*X$, $r, s \in R$, and $x \in X$.

Let R be a ring, we use the notation r^o, for $r \in R$, to denote the elements of the opposite ring R^o. By $R^{\mathrm{e}} := R \otimes R^o$ we denote the enveloping ring of R.

Given an R^{e}-bimodule M, there is a structure of an R-multimodule on M, and so the underlying $\Bbbk$-module M admits a several structures of R-bimodule. Among them, we will select the following two ones.
The first structure is that of the opposite bimodule ${}_{1\otimes R^o}M_{1\otimes R^o}$ which we denote by M^o. That is, the R-biaction on M^o is given by

$$r\, m^o \;=\; \Big(m\,(1\otimes r^o)\Big)^o, \quad m^o\, s \;=\; \Big((1\otimes s^o)\, m\Big)^o,$$

for every $m^o \in M^o$ and $r, s \in R$. This construction defines in fact a functor

$$(-)^o : {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \longrightarrow {}_R\mathrm{Mod}_R.$$

The second structure is defined by the left R^{e}-module ${}_{R^{\mathrm{e}}}M$. That is, the R-bimodule $M^l = {}_{R\otimes 1^o}M_R$ whose R-biaction is defined by

$$r \cdot m^l = \Big((r \otimes 1^o)m\Big)^l, \quad m^l \cdot s = \Big((1 \otimes s^o)m\Big)^l,$$

for every $m^l \in M^l$ and $r, s \in R$. This also defines a functor, namely, the right R^{e}-actions forgetful functor

$$(-)^l : {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \longrightarrow {}_R\mathrm{Mod}_R.$$

We have a commutative diagram:

$$\begin{tikzcd} {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \arrow[r, "(-)^l"] \arrow[d, "(-)^o"'] & {}_R\mathrm{Mod}_R \arrow[d, "(-)_R"] \\ {}_R\mathrm{Mod}_R \arrow[r, "(-)_R"'] & \mathrm{Mod}_R, \end{tikzcd}$$

where $(-)_R$ is the left R-actions forgetful functor.

Another R^e-bimodule derived from M which will be used in the sequel is $M^\dagger$. The underlying $\Bbbk$-module of $M^\dagger$ is M, and an element $m \in M$ is denoted by $m^\dagger$ when it is viewed in $M^\dagger$. The R^e-biaction on $M^\dagger$ is given by

$$(p \otimes q^o)\, m^\dagger\, (r \otimes s^o) \;=\; \Big((p \otimes r^o)\, m\, (q \otimes s^o)\Big)^\dagger,$$

for every $m^\dagger \in M^\dagger$, $p, r \in R$ and $q^o, s^o \in R^o$.

Here also we have a functor $(-)^\dagger : {}_{R^e}\mathrm{Mod}_{R^e} \longrightarrow {}_{R^e}\mathrm{Mod}_{R^e}$ which is an idempotent faithful functor, in the sense that, we have

$${}_{R^e}(M^\dagger)^\dagger{}_{R^e} \;=\; {}_{R^e}M_{R^e}$$

and

$$\mathrm{Hom}_{R^e-R^e}\Big(M^\dagger,\, U^\dagger\Big) \;=\; \mathrm{Hom}_{R^e-R^e}\Big(M,\, U\Big),$$

for every pair of R^e-bimodules U and M.

Furthermore, there is a commutative diagram

$$\begin{tikzcd}
{}_{R^e}\mathrm{Mod}_{R^e} \arrow[rr, "(-)^o"] \arrow[d, "(-)^\dagger"'] & & {}_R\mathrm{Mod}_R \arrow[d, equal] \\
{}_{R^e}\mathrm{Mod}_{R^e} \arrow[rr, "(-)_{R^e}"'] & & \mathrm{Mod}_{R^e}.
\end{tikzcd}$$

The left module ${}_{R^e}M^\dagger$ induces the already existing R-bimodule structure of ${}_{R\otimes 1^o}M_{R\otimes 1^o}$.
Now, let N be another R-bimodule, and consider the tensor product $M^o \otimes_R N$. The additive $\mathbb{k}$-submodule of invariant elements

$$(M^o \otimes_R N)^R = \left\{ \sum_i m_i^o \otimes_R n_i \,\middle|\, \sum_i r m_i^o \otimes_R n_i = \sum_i m_i^o \otimes_R n_i r, \text{ for all } r \in R \right\}$$

admits a structure of an R-bimodule given by the following actions:

$$r \rightharpoonup \left(\sum_i m_i^o \otimes_R n_i \right) = \sum_i \left((r \otimes 1^o)\, m_i \right)^o \otimes_R n_i,$$

$$\left(\sum_i m_i^o \otimes_R n_i \right) \leftharpoonup s = \sum_i \left(m_i\, (s \otimes 1^o) \right)^o \otimes_R n_i,$$

for every elements $\sum_i m_i^o \otimes_R n_i \in M^o \otimes_R N$ and $r, s \in R$.
In this way, to each R-bimodule N one can associate to it two functors:

$${}_{R^e}\mathrm{Mod}_{R^e} \xrightarrow{\left((-)^o \otimes_R N\right)^R} {}_R\mathrm{Mod}_R, \quad {}_R\mathrm{Mod}_R \xrightarrow{\left(- \otimes {}^*N\right)^{\dagger}} {}_{R^e}\mathrm{Mod}_{R^e},$$

where for each R-bimodule X, we consider $X \otimes {}^*N$ as an R^e-bimodule with the following actions

$$(p \otimes q^o) \left(\sum_i x_i \otimes \varphi_i \right) (r \otimes s^o) = \sum_i (p\, x_i\, q) \otimes (s\, \varphi_i\, r),$$

for every element $\sum_i x_i \otimes \varphi_i \in X \otimes {}^*N$, $p, q, r, s \in R$.

Lemma

Let N be an R-bimodule such that ${}_RN$ is finitely generated and projective module with left dual basis $\{(e_j, {}^*e_j)\}_{1\leq j\leq m} \subset N \times {}^*N$. There is a natural isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{R-R}\left(X, \, (M^o \otimes_R N)^R\right) & \longrightarrow & \mathrm{Hom}_{R^e-R^e}\left((X \otimes {}^*N)^\dagger, \, M\right) \\ \sigma & \longmapsto & \left[(x \otimes \varphi)^\dagger \longmapsto \left((M^o \otimes_R \varphi) \circ \sigma(x)\right)\right] \\ \left[x \longmapsto \sum_j \alpha\left(\left(x \otimes {}^*e_j\right)^\dagger\right)^o \otimes_R e_j\right] & \longleftarrow\!\!| & \alpha \end{array}$$

for every R-bimodule X and R^e-bimodule M. That is, the functor $(- \otimes {}^*N)^\dagger$ is left adjoint to the functor $((-)^o \otimes_R N)^R$.

As we have seen before there is a bi-functor

$$- \times_R - := \left((-)^o \otimes_R - \right)^R : {}_{R^e}\mathrm{Mod}_{R^e} \times {}_R\mathrm{Mod}_R \longrightarrow {}_R\mathrm{Mod}_R.$$

This is a Sweedler-Takeuchi's product of bimodules, which can be also redefined using Mac Lane's notion of ends (limits) and coends (colimits).
Given an R^e-bimodule M and an R-bimodule N, an element $\sum_i m_i^o \otimes_R n_i$ which belongs to $M \times_R N$ will be denoted by $\sum_i m_i \times_R n_i$.
If N is an R^e-bimodule, then there are several structures of R-bimodules on N over which one can construct $M \times_R N$. Here we define $M \times_R N$ by using the R-bimodule ${}_{R \otimes 1^o}N_{R \otimes 1^o}$. In this way, $M \times_R N$ admits a structure of R^e-bimodule:

Given by the following rule

$$(r \otimes s^o)\left(\sum_i m_i \times_R n_i\right)(p \otimes q^o) = \sum_i \Big((r \otimes 1^o)\, m_i\, (s \otimes 1^o)\Big) \times_R \Big((1 \otimes p^o)\, n_i\, (1 \otimes q^o)\Big),$$

for every elements $\sum_i m_i \times_R n_i \in M \times_R N$ and $r, s, p, q \in R$.
Whence the R^{e}-biaction on $(M \times_R N)^{\dagger}$ is given by the formula:

$$(r \otimes s^o)\left(\sum_i m_i \times_R n_i\right)^{\dagger}(p \otimes q^o) = \left(\sum_i \Big((r \otimes 1^o)\, m_i\, (p \otimes 1^o)\Big) \times_R \Big((1 \otimes s^o)\, n_i\, (1 \otimes q^o)\Big)\right)^{\dagger}.$$

In this way, the functor $- \times_R -$ is restricted to the category ${}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \times {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}}$.

Here we consider this restriction as the following compositions of functors:

$$\begin{tikzcd}
{}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \times {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \arrow[d, "\left((-)^o \underset{R}{\otimes} {}_{R\otimes 1^o}(-)_{R\otimes 1^o}\right)^R"'] \arrow[drr, dashed, "-\times_R-"] & & \\
{}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}} \arrow[rr, "(-)^\dagger"'] & & {}_{R^{\mathrm{e}}}\mathrm{Mod}_{R^{\mathrm{e}}}
\end{tikzcd}$$

Given another R^{e}-bimodule W, there are three R^{e}-bimodule under consideration. Namely, $M \times_R (N \times_R U)$, $(M \times_R N) \times_R U$, and $M \times_R N \times_R W$. The later is constructed as follows: First we consider the underlying left R^{e}-module of N, that is, $N^l = {}_{R^{\mathrm{e}}}N$ which we consider obviously as an R-bimodule.

Secondly, we construct the $\Bbbk$-module $M^o \otimes_R N^l \otimes_R W$ using the left R-module ${}_{R\otimes 1^o}W$. This is an R^e-bimodule within the actions

$$(r\otimes t^o)\left(\sum_i m_i^o \otimes_R n_i^l \otimes_R w_i\right)(p\otimes q^o) \;=\; \sum_i rm_i^o \otimes_R (n_i(p\otimes q^o))^l \otimes_R w_i(t\otimes 1^o),$$

for every elements $\sum_i m_i^o \otimes_R n_i^l \otimes_R w_i \in M^o \otimes_R N^l \otimes_R W$ and $p, q, r, t \in R$. Lastly, we take $M \times_R N \times_R W$ as the R^e-invariant subbimodule, that is,

$$\begin{aligned} M \times_R N \times_R W \;&=\; \left(M^o \otimes_R N^l \otimes_R W\right)^{R^e} \;=\\ &\left\{ \sum_i m_i^o \otimes_R n_i^l \otimes_R w_i \middle|\; \sum_i rm_i^o \otimes_R n_i^l \otimes_R w(s\otimes 1^o) \right.\\ &\left. \qquad = \sum_i m_i^o \otimes_R (n_i(r\otimes u^o))^l \otimes_R w, \text{ for all } r, s \in R \right\}. \end{aligned}$$

The $\Bbbk$-module $M \times_R N \times_R W$ admits a structure of an R^e-bimodule given by

$$(r \otimes s^o) \left(\sum_i m_i \times_R n_i \times_R w_i \right) (p \otimes q^o)$$
$$= \sum_i \Big((r \otimes 1^o) m_i (p \otimes 1^o) \Big) \times_R n_i \times_R \Big((1 \otimes s^o) w_i (1 \otimes q^o) \Big),$$

for every elements $\sum_i m_i \times_R n_i \times_R w_i \in M \times_R N \times_R W$ and $r, s, p, q \in R$.

The bi-functor $- \times_R -$ is not associative. However, the are an natural R^e-bilinear maps

$$\alpha_l : (M \times_R N) \times_R W \longrightarrow M \times_R N \times_R W,$$

and

$$\alpha_r : M \times_R (N \times_R W) \longrightarrow M \times_R N \times_R W$$

Another useful natural transformation of R^{e}-bimodules is given as follows: For every R^{e}-bimodules M, M', N, N', we have an R^{e}-bilinear map:

$$
\begin{aligned}
(M \times_R M') \otimes_{R^{\mathrm{e}}} (N \times_R N') &\xrightarrow{\quad\tau\quad} (M \otimes_{R^{\mathrm{e}}} N) \times_R (M' \otimes_{R^{\mathrm{e}}} N') \\
\left(\sum_i m_i \times_R m_i'\right) \otimes_{R^{\mathrm{e}}} \left(\sum_j n_j \times_R n_j'\right) &\longmapsto \sum_{i,j} (m_i \otimes_{R^{\mathrm{e}}} n_j) \times_R (m_i' \otimes_{R^{\mathrm{e}}} n_j').
\end{aligned}
$$

In this way, $S \times_R T$ is an R^{e}-ring whenever S and T they are. Precisely, the multiplication of $S \times_R T$ is defined using τ and explicitly given by

$$
\left(\sum_i x_i \times_R y_i\right) \left(\sum_j u_j \times_R v_j\right) = \sum_{i,j} x_i u_j \times_R y_i v_j,
$$

for every pair of elements $\sum_i x_i \times_R y_i$ and $\sum_j u_j \times_R v_j$ in $S \times_R T$.
The unit is the map $R^{\mathrm{e}} \longrightarrow S \times_R T$ which sends $p \otimes q^o \longmapsto ((p \otimes 1^o) 1_S) \times_R (1_T (1 \otimes q^o))$.

Let A be an R-ring, and construct a functor $-\times_R A : {}_{R^{e}}\mathrm{Mod}_{R^{e}} \to {}_R\mathrm{Mod}_R$. For every pair of R^{e}-bimodules M and N, we have a well defined and R-bilinear maps:

$$\begin{array}{ccc} (M\times_R A)\otimes_R(N\times_R A) & \xrightarrow{\Phi^2_{(M,\,N)}} & (M\otimes_{R^{e}} N)\times_R A, \\ (m\times_R a)\otimes_R(n\times_R a') & \longmapsto & (m\otimes_{R^{e}} n)\times_R aa' \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\Phi^0} & R^{e}\times_R A \\ r & \longmapsto & (r\otimes 1^{o})\times_R 1_A \end{array}$$

where $\Phi^2_{(-,-)}$ is obviously a natural transformation. An easy verification shows that $-\times_R A : {}_{R^{e}}\mathrm{Mod}_{R^{e}} \to {}_R\mathrm{Mod}_R$ is in fact a monoidal functor.

Assume that A is finitely generated and projective as left R-module, and fix a left dual basis $\{({}^*e_j, e_j)\}_{1\leq j\leq n} \subset {}^*A\times A$. By the previous Lemma

$$\mathcal{R} = -\times_R A : {}_{R^{e}}\mathrm{Mod}_{R^{e}} \longrightarrow {}_R\mathrm{Mod}_R$$

is a right adjoint to the functor

$$\mathcal{L} = (-\otimes {}^*A)^{\dagger} : {}_R\mathrm{Mod}_R \longrightarrow {}_{R^{e}}\mathrm{Mod}_{R^{e}}.$$

By the Theorem of the monoidal case, the adjunction $\mathcal{L} \dashv \mathcal{R}$ is restricted to the categories of ring extension. That is, we have an adjunction

$$\mathcal{L}_m : R - \mathtt{Rings} \rightleftarrows R^{\mathrm{e}} - \mathtt{Rings} : \mathcal{R}_m$$

For a given R-ring C, the R^{e}-ring $\mathcal{L}_m(C)$ is defined by the quotient algebra

$$\mathcal{L}_m(C) \;=\; \mathcal{T}_{R^{\mathrm{e}}}\left(\mathcal{L}(C)\right)/\mathcal{J}_{\mathcal{L}(C)}$$

where $\mathcal{T}_{R^{\mathrm{e}}}\left(\mathcal{L}(C)\right)$ is the tensor algebra of the R^{e}-bimodule $\mathcal{L}(C) \;=\; (C \otimes {}^*A)^{\dagger}$ and wherein $\mathcal{J}_{\mathcal{L}(C)}$ is the two-sided ideal generated by the set

$$\left\{ \sum_i \left((c \otimes e_i\varphi)^{\dagger} \otimes_{R^{\mathrm{e}}} (c' \otimes {}^*e_i)^{\dagger} \right) - (cc' \otimes \varphi)^{\dagger};\; 1_R \otimes \varphi(1_A)^o - (1_C \otimes \varphi)^{\dagger} \right\}$$

where $c, c' \in C$ and $\varphi \in {}^*A$

The $\Bbbk$-linear endomorphisms ring $\mathrm{End}_{\Bbbk}(R)$ is an R^{e}-ring via the map $\varrho : R^{\mathrm{e}} \to \mathrm{End}_{\Bbbk}(R)$ which sends $p \otimes q^o \mapsto [r \mapsto p\, r\, q]$.

Given a pair of R^{e}-bimodules M and N, there are two R^{e}-bilinear maps

$$\begin{array}{rcl} \theta_r : M \times_R \mathrm{End}_{\Bbbk}(R) & \longrightarrow & M, \\ \sum_i m_i \times_R f_i & \longmapsto & \sum_i (1 \otimes f_i(1)^o)\; m_i \end{array}$$

$$\begin{array}{rcl} \theta_l : \mathrm{End}_{\Bbbk}(R) \times_R N & \longrightarrow & N \\ \sum_i f_i \times_R n_i & \longmapsto & \sum_i (f_i(1) \otimes 1^o)\; n_i. \end{array}$$

Following Takeuchi, a $\times_R$-coalgebra is an R^{e}-bimodule C together with two R^{e}-bilinear maps $\Delta : \mathrm{C} \to \mathrm{C} \times_R \mathrm{C}$ (comultiplication) and $\varepsilon : \mathrm{C} \to \mathrm{End}_{\Bbbk}(R)$ (counit) such that the diagrams

$$
\begin{tikzcd}
 & C \times_R C \arrow[r, "\Delta \times_R C"] & (C \times_R C) \times_R C \arrow[dr, "\alpha_l"] & \\
C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & & C \times_R C \times_R C \\
 & C \times_R C \arrow[r, "C \times_R \Delta"] & C \times_R (C \times_R C) \arrow[ur, "\alpha_r"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \times_R C \arrow[d, "\varepsilon \times_R C"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & C \times_R C \arrow[d, "C \times_R \varepsilon"] \\
\mathrm{End}_{\Bbbk}(R) \times_R C \arrow[r, "\theta_l"] & C & C \times_R \mathrm{End}_{\Bbbk}(R) \arrow[l, "\theta_r"']
\end{tikzcd}
$$

are commutative. A $\times_R$-coalgebra C is said to be an $\times_R$-*bialgebra* provided that comultiplication and counit are morphisms of an R^e-rings.

Proposition

Let A be an R-ring which is finitely generated and projective as left R-module with dual basis $\{({}^*e_i, e_i)\}_i$. Then $\mathcal{L}_m(A)$ is an $\times_R$-bialgebra with structure maps

$$\Delta : \mathcal{L}_m(A) \longrightarrow \mathcal{L}_m(A) \times_R \mathcal{L}_m(A), \left(\pi_A(a \otimes \varphi) \mapsto \sum_j \pi_A(a \otimes {}^*e_j) \times_R \pi_A(e_j \otimes \varphi) \right)$$

$$\varepsilon : \mathcal{L}_m(A) \longrightarrow \mathrm{End}_{\Bbbk}(R), \left(\pi_A(a \otimes \varphi) \longmapsto \left[r \mapsto \varphi(ar) \right] \right)$$

Moreover, A is a left $\mathcal{L}_m(A)$-comodule R-ring with a structure map the unit of the adjunction $\mathcal{L}_m \dashv \mathcal{R}_m$ at A:

$$\eta_A^m : A \longrightarrow \mathcal{L}_m(A) \times_R A, \quad \left(a \longmapsto \sum_i \pi_A(a \otimes {}^*e_i) \times_R e_i \right)$$

Following the terminology of D. Tambara, the $\times_R$-bialgebra $\mathfrak{L}_m(A)$ is refereed to as *coendomorphism bialgebroid*. By a result of T. Brzeziński and G. Militaru, $\mathfrak{L}_m(A)$ is in fact a (left) bialgebroid whose structure of R^{e}-ring is the map $\pi_A \circ \iota_0 : R^{\mathrm{e}} \to \mathfrak{L}_m(A)$ and its structure of R-coring is given by

$$\Delta : \mathfrak{L}_m(A) \longrightarrow \mathfrak{L}_m(A) \otimes_R \mathfrak{L}_m(A),$$

$$\left(\pi_A(a \otimes \varphi) \longmapsto \sum_i \pi_A(a \otimes {}^*e_i) \otimes_R \pi_A(e_i \otimes \varphi)\right),$$

$$\varepsilon : \mathfrak{L}_m(A) \longrightarrow R, \quad \Big(\pi_A(a \otimes \varphi) \longmapsto \varphi(a)\Big).$$

Example

Assume that $A = R^n$, the obvious R-ring attached to the free R-module of rank n. So $\mathcal{L}_m(A)$ is an R-bialgebroid generated as a ring by R^e and a set of R^e-invariant elements $\{x_{ij}\}_{1 \leq i,j \leq n}$ with relation

$$\begin{aligned} x_{ii}^2 &= x_{ii}, \quad \text{for all } i = 1, 2, \cdots, n. \\ x_{ji}\, x_{ki} &= 0, \quad \text{for all } j \neq k, \text{ and } i, j, k = 1, 2, \cdots, n. \\ \sum_{i=1}^{n} x_{ij} &= 1, \quad \text{for all } j = 1, 2, \cdots, n. \end{aligned}$$

Its structure of R-coring is given by the following comultiplication and counit

$$\begin{aligned} \Delta(x_{ij}) &= \sum_{k=1}^{n} x_{ik} \otimes_R x_{kj}, \quad \text{for all } i, j = 1, 2, \cdots, n; \\ \varepsilon(x_{ij}) &= \delta_{ij}, \quad \text{(Kronecker delta) for all } i, j = 1, 2, \cdots, n. \end{aligned}$$

Example

Let A be the trivial crossed product of R by a cyclic group of order n denoted by $\mathcal{G}_n$. We know that ${}_R A$ is left free module with basis $\mathcal{G}_n$. If $n = 2$, then $\mathcal{L}_m(A)$ is an R-bialgebroid generated as an R^{e}-ring by two R^{e}-invariant elements x, y subject to the relations $xy + yx = 0$ and $1 = x^2 + y^2$. The comultiplication and counit of the underlying R-coring structure are given by

$$\Delta(x) = x \otimes_R 1 + y \otimes_R x, \quad \Delta(y) = y \otimes_R y, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 1.$$

For $n > 2$. Then $\mathcal{L}_m(A)$ is an R^{e}-ring generated by an R^{e}-invariant elements $x_{(k,\, l)}$ with $(k, l) \in (\mathbb{Z}_n \setminus \{0\}) \times \mathbb{Z}_n$ subject to the following relations:

Example (follows)

$$x_{(k,\,l)} = \sum_{s=0}^{n-1} x_{(t,\,l-s)}\, x_{(k-t,\,s)},\ \forall\,(k,l) \in (\mathbb{Z}_n \backslash \{0,1\}) \times \mathbb{Z}_n,\ \forall\, t \in \mathbb{Z}_n \backslash \{0\} \text{ with } t < k,$$

$$x_{(1,\,l)} = \sum_{s=0}^{n-1} x_{(n-t,\,l-s)}\, x_{(n-t',\,s)},\ \ \forall\, l \in \mathbb{Z}_n,\ \forall\, t,\ t' \in \mathbb{Z}_n \setminus \{0\},\ \text{with } t + t' = n-1,$$

and

$$1 = \sum_{s=0}^{n-1} x_{(t,\,n-s)}\, x_{(t',\,s)},\ \ \forall\, t,\ t' \in \mathbb{Z}_n \setminus \{0\},\ \text{with } t + t' = 0,$$

wherein the ring $\mathbb{Z}_n$ is given the canonical ordering $0 < 1 < \cdots < (n-1)$. The comultiplication and counit of its underlying R-coring structure are given by

$$\Delta(x_{(k,\,l)}) = \sum_{s=0}^{n-1} x_{(k,\,s)} \otimes_R x_{(s,\,l)}, \quad \varepsilon(x_{(k,\,l)}) = \delta_{k,l}, \quad \forall\,(k,l) \in (\mathbb{Z}_n \setminus \{0\}) \times \mathbb{Z}_n.$$