Homotopy Theory and Higher Categories

WORKSHOP ON CATEGORICAL GROUPS

Categorical groups and $[n, n+1]$-types of exterior spaces

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1. Introduction

Proper homotopy theory

Classification of non compact surfaces

B. Kerékjártó, *Vorlesungen uber Topologie* , vol.1, Springer-Verlag (1923).

Ideal point

H. Freudenthal, *Über die Enden topologisher Räume und Gruppen* , Math. Zeith. 53 (1931) 692-713. *End of a space*

L.C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five* , Tesis, 1965.

Proper homotopy invariants at one end represented by a base ray

H.J. Baues, A. Quintero, *Infinite Homotopy Theory*, K-Monographs in Mathematics, 6. Kluwer Publishers, 2001.

Invariants associated at a base tree

One of the main problems of the proper category is that there are few limits and colimits.

Pro-spaces

J.W. Grossman, *A homotopy theory of pro-spaces* , Trans. Amer. Math. Soc.,201 (1975) 161-176.

T. Porter, *Abstract homotopy theory in procategories* , Cahiers de topologie et geometrie differentielle, vol 17 (1976) 113-124.

A. Edwards, H.M. Hastings, *Every weak proper homotopy equivalence is weakly properly homotopic to a proper homotopy equivalence* , Trans. Amer. Math. Soc. 221 (1976), no. 1, 239–248.

Exterior spaces

J. García Calcines, M. García Pinillos, L.J. Hernández Paricio, *A closed model category for proper homotopy and shape theories*, Bull. Aust. Math. Soc. 57 (1998) 221-242.

J. García Calcines, M. García Pinillos, L.J. Hernández Paricio, *Closed Simplicial Model Structures for Exterior and Proper Homotopy Theory*, Applied Categorical Structures, 12, (2004) , pp. 225-243.

J. I. Extremiana, L.J. Hernández, M.T. Rivas , *Postnikov factorizations at infinity*, Top and its Appl. 153 (2005) 370-393.

n-types

J.H.C. Whitehead, *Combinatorial homotopy. I , II* , Bull. Amer. Math. Soc., 55 (1949) 213-245, 453-496.

Crossed complexes and crossed modules

proper n-types

L. J. Hernández and T. Porter, *An embedding theorem for proper n-types*, Top. and its Appl. , 48 n°3 (1992) 215-235.

L. J. Hernández y T. Porter, *Categorical models for the n-types of pro-crossed complexes and $\mathcal{J}_n$-prospaces*, Lect. Notes in Math., n° 1509, (1992) 146-186

2. Proper maps, exterior spaces and categories of proper and exterior [n,n+1]-types

A continuous map $f : X \to Y$ is said to be *proper* if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X.

Top topological spaces and continuous maps

P spaces and proper maps

P does not have enough limits and colimits

Definition 2.1 *Let (X, τ) be a topological space. An* externology *on (X, τ) is a non empty collection ε of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$. An* exterior space *$(X, \varepsilon \subset \tau)$ consists of a space (X, τ) together with an externology ε. A map $f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be* exterior *if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

The category of exterior spaces and maps will be denoted by $\mathbf{E}$.

$\mathbb{N}$ non negative integers, usual topology, cocompact externology
$\mathbb{R}_+$ $[0, \infty)$, usual topology, cocompact externology
$\mathbf{E}^{\mathbb{N}}$ exterior spaces under $\mathbb{N}$
$\mathbf{E}^{\mathbb{R}_+}$ exterior spaces under $\mathbb{R}_+$
(X, λ) object in $\mathbf{E}^{\mathbb{R}_+}$, $\lambda\colon \mathbb{R}_+ \longrightarrow X$ a *base ray* in X
The natural restriction $\lambda|_{\mathbb{N}}\colon \mathbb{N} \longrightarrow X$ is a *base sequence* in X

$$\mathbf{E}^{\mathbb{R}_+} \longrightarrow \mathbf{E}^{\mathbb{N}} \qquad \text{forgetful functor}$$

X, Z exterior spaces, Y topological space
$X \bar{\times} Y$, Z^Y exterior spaces
Z^X topological space (box $\supset$ topology $Z^X \supset$ compact-open)

S^q q-dimensional (pointed) sphere:

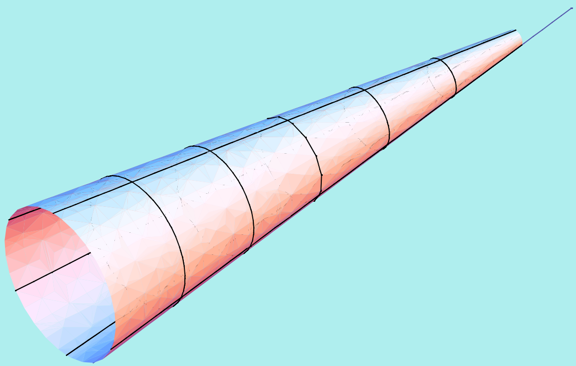
$$Hom_{\mathbf{E}}(\mathbb{N} \bar{\times} S^q, X) \cong Hom_{\mathbf{Top}}(S^q, X^{\mathbb{N}})$$
$$Hom_{\mathbf{E}}(\mathbb{R}_+ \bar{\times} S^q, X) \cong Hom_{\mathbf{Top}}(S^q, X^{\mathbb{R}_+})$$

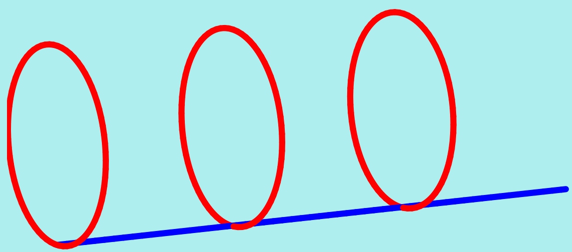
Definition 2.2 *Let* (X,λ) *be in* $\mathbf{E}^{\mathbb{R}_+}$ *and an integer* $q \geq 0$ *.*
The q-th $\mathbb{R}_+$-exterior homotopy group *of* (X,λ)*:*

$$\pi_q^{\mathbb{R}_+}(X,\lambda) = \pi_q(X^{\mathbb{R}_+},\lambda)$$

The q-th $\mathbb{N}$-exterior homotopy group *of* (X,λ)*:*

$$\pi_q^{\mathbb{N}}(X,\lambda|_{\mathbb{N}}) = \pi_q(X^{\mathbb{N}},\lambda|_{\mathbb{N}})$$





Definition 2.3 *An exterior map* $f\colon (X,\lambda) \to (X',\lambda')$ *is said to be* a weak $[n,n+1]$-$\mathbb{R}_+$-equivalence *(* weak $[n,n+1]$-$\mathbb{N}$-equivalence *) if* $\pi_n^{\mathbb{R}_+}(f), \pi_{n+1}^{\mathbb{R}_+}(f)$ *(*$\pi_n^{\mathbb{N}}(f), \pi_{n+1}^{\mathbb{N}}(f)$ *) are isomorphisms.*

$\Sigma_{\mathbb{R}_+}^{[n,n+1]}$ class of weak $[n,n+1]$-$\mathbb{R}_+$-equivalences
$\Sigma_{\mathbb{N}}^{[n,n+1]}$ class of weak $[n,n+1]$-$\mathbb{N}$-equivalences

The category of *exterior* $\mathbb{R}_+$*-[n,n+1]-types* is the category of fractions

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{R}_+}^{[n,n+1]}]^{-1},$$

the category of *exterior* $\mathbb{N}$*-[n,n+1]-types*

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{[n,n+1]}]^{-1}$$

and the corresponding subcategories of *proper [n,n+1]-types*

$$\mathbf{P}^{\mathbb{R}_+}[\Sigma_{\mathbb{R}_+}^{[n,n+1]}]^{-1}, \quad \mathbf{P}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{[n,n+1]}]^{-1}.$$

Two objects X, Y have the same type if they are isomorphic in the corresponding category of fractions

$$\text{type}(X) = \text{type}(Y) .$$

Example 2.1 $X = \mathbb{R}^2$, $Y = \mathbb{R}^3$:

$$\textit{[1,2]-type}(X) = \textit{[1,2]-type}\ (Y)$$

$$\mathbb{N}\text{-}\textit{[1,2]-type}(X) \neq \mathbb{N}\text{-}\textit{[1,2]-type}(Y), \quad \mathbb{R}_+\text{-}\textit{[1,2]-type}(X) \neq \mathbb{R}_+\text{-}\textit{[1,2]-type}(Y)$$

Example 2.2 $X = \mathbb{R}_+ \sqcup (\sqcup_0^\infty S^3))/n \sim *_n$, $Y = \mathbb{R}_+$:

$$\textit{[1,2]-type}(X) = \textit{[1,2]-type}\ (Y)$$

$$\mathbb{N}\text{-}\textit{[1,2]-type}(X) = \mathbb{N}\text{-}\textit{[1,2]-type}(Y), \quad \mathbb{R}_+\text{-}\textit{[1,2]-type}(X) \neq \mathbb{R}_+\text{-}\textit{[1,2]-type}(Y)$$

Example 2.3 $X = \mathbb{R}_+ \sqcup (\sqcup_0^\infty S^1))/n \sim *_n$, $Y = \mathbb{R}_+$:

$$\textit{[1,2]-type}(X) \neq \textit{[1,2]-type}\ (Y)$$

$$\mathbb{N}\text{-}\textit{[1,2]-type}(X) \neq \mathbb{N}\text{-}\textit{[1,2]-type}(Y), \quad \mathbb{R}_+\text{-}\textit{[1,2]-type}(X) = \mathbb{R}_+\text{-}\textit{[1,2]-type}(Y)$$

3. Categorical groups

A *monoidal category* $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ consists of a category $\mathbb{G}$, a functor (tensor product) $\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, an object I (unit) and natural isomorphisms called, respectively, the associativity, left-unit and right-unit constraints

$$a = a_{\alpha,\beta,\omega} : (\alpha \otimes \beta) \otimes \omega \xrightarrow{\sim} \alpha \otimes (\beta \otimes \omega) \ ,$$

$$l = l_{\alpha} : I \otimes \alpha \xrightarrow{\sim} \alpha \quad , \quad r = r_{\alpha} : \alpha \otimes I \xrightarrow{\sim} \alpha \ ,$$

which satisfy that the following diagrams are commutative

$$\begin{tikzcd}
((\alpha \otimes \beta) \otimes \omega) \otimes \tau \arrow[rr, "a \otimes 1"] \arrow[d, "a"'] & & (\alpha \otimes (\beta \otimes \omega)) \otimes \tau \arrow[d, "a"] \\
(\alpha \otimes \beta) \otimes (\omega \otimes \tau) \arrow[dr, "a"'] & & \alpha \otimes ((\beta \otimes \omega) \otimes \tau) \arrow[dl, "1 \otimes a"] \\
& \alpha \otimes (\beta \otimes (\omega \otimes \tau)) &
\end{tikzcd} \ ,$$

$$
\begin{tikzcd}
(\alpha \otimes I) \otimes \beta \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & \alpha \otimes (I \otimes \beta) \arrow[dl, "1\otimes l"] \\
& \alpha \otimes \beta\ . &
\end{tikzcd}
$$

A *categorical group* is a monoidal groupoid, where every object has an inverse with respect to the tensor product in the following sense:
For each object α there is an inverse object α^* and canonical isomorphisms

$$(\gamma_r)_\alpha \colon \alpha \otimes \alpha^* \longrightarrow I$$

$$(\gamma_l)_\alpha \colon \alpha^* \otimes \alpha \longrightarrow I$$

$\mathbf{CG}$ categorical groups

A categorical group $\mathbb{G}$ is said to be a *braided categorical group* if it is also equipped with a family of natural isomorphisms $c = c_{X,Y} : X \otimes Y \to Y \otimes X$ (the braiding) that interacts with a, r and l such that, for any $X, Y, Z \in \mathbb{G}$, the following diagrams are commutative:

$$\begin{tikzcd}
& (\beta \otimes \alpha) \otimes \omega \arrow[r, "a"] & \beta \otimes (\alpha \otimes \omega) \arrow[dr, "1\otimes c"] & \\
(\alpha \otimes \beta) \otimes \omega \arrow[ur, "c\otimes 1"] \arrow[dr, "a"'] & & & \beta \otimes (\omega \otimes \alpha) \\
& \alpha \otimes (\beta \otimes \omega) \arrow[r, "c"] & (\beta \otimes \omega) \otimes \alpha \arrow[ur, "a"'] &
\end{tikzcd},$$

$$
\begin{tikzcd}
 & \alpha \otimes (\omega \otimes \beta) & (\alpha \otimes \omega) \otimes \beta \arrow[l, "a"'] \arrow[dr, "c\otimes 1"] & \\
\alpha \otimes (\beta \otimes \omega) \arrow[ur, "1\otimes c"] & & & (\omega \otimes \alpha) \otimes \beta \arrow[dl, "a"] \\
 & (\alpha \otimes \beta) \otimes \omega \arrow[ul, "a"] \arrow[r, "c"] & \omega \otimes (\alpha \otimes \beta) & .
\end{tikzcd}
$$

BCG braided categorical groups

A braided categorical group $(\mathbb{G}, c)$ is called a *symmetric categorical group* if the condition $c^2 = 1$ is satisfied.

SCG symmetric categorical groups

4. The small categories $\mathcal{C} = E(E(\bar{4}) \times EC(\Delta/2))$, $\mathcal{BC}$ and $\mathcal{SC}$

Objectives:

-To give a more geometric version of the well known equivalences between $[1,2]$-types and categorical groups up to weak equivalences, and similarly for $[2,3]$-types, $[n, n+1]$-types $(n \geq 3)$ and braided categorical groups, symmetric categorical groups, respectively

-To obtain an adapted version for exterior $[n, n+1]$-types

(exterior spaces)

(pointed spaces) adjunction (presheaves) adjuntion (categorical groups)

The category $\mathcal{C} = E(E(\bar{4}) \times EC(\Delta/2))$:

$\Delta/2$ is the 2-truncation of the usual category Δ whose objects are ordered sets $[q] = \{0 < 1 \cdots < q\}$ and monotone maps.

Now we can construct the pushouts

$$\begin{tikzcd}
{[0]} \arrow[r, "\delta_1"] \arrow[d, "\delta_0"'] & {[1]} \arrow[d, "\mathsf{in}_r"] \\
{[1]} \arrow[r, "\mathsf{in}_l"'] & {[1] +_{[0]} [1]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]} \arrow[r, "\mathsf{in}_l"] \arrow[d, "\mathsf{in}_r"'] & {[1] +_{[0]} [1]} \arrow[d] \\
{[1] +_{[0]} [1]} \arrow[r] & {[1] +_{[0]} [1] +_{[0]} [1]}
\end{tikzcd}$$

$C(\Delta/2)$ is the extension of the category $\Delta/2$ given by the objects

$$[1] +_{[0]} [1], \quad [1] +_{[0]} [1] +_{[0]} [1]$$

and all the natural maps induced by these pushouts.

In order to have vertical composition and inverses up to homotopy we extend this category with some additional maps and relations:
$V\colon [2] \longrightarrow [1]$, $V\delta_2 = \mathrm{id}$, $V\delta_1 = \delta_1\epsilon_0$, $(V\delta_0)^2 = \mathrm{id}$,
$K\colon [2] \longrightarrow [1] +_{[0]} [1]$, $K\delta_2 = \mathsf{in}_l$, $K\delta_0 = \mathsf{in}_r$,
$A\colon [2] \longrightarrow [1] +_{[0]} [1] +_{[0]} [1]$, $A\delta_2 = (K\delta_1 + \mathrm{id})K\delta_1$, $A\delta_1 = (\mathrm{id} + K\delta_1)K\delta_1$,
$A\delta_0 = A\delta_1\delta_0\epsilon_0$.
The new extended category will be denoted by $EC(\Delta/2)$.

With the objective of obtaining a tensor product with a unit object and inverses, we take the small category $\bar{4}$ generated by the object 1 and the induced coproducts $0, 1, 2, 3, 4$, all the natural maps induced by coproducts and three additonal maps:
$e_0\colon 1 \longrightarrow 0$, $\nu\colon 1 \longrightarrow 1$ and $\mu\colon 1 \longrightarrow 2$.
This gives a category $E(\bar{4})$.

Consider the product category $E(\bar{4}) \times EC(\Delta/2)$.

The object $(i,[j])$, and morphisms $\mathrm{id}_i \times g$, $f \times \mathrm{id}_{[j]}$ will be denoted by $i[j]$ and g , f, respectively.

We extend again this category by adding new maps:
$a\colon 1[1] \longrightarrow 3[0]$, $r\colon 1[1] \longrightarrow 1[0]$, $l\colon 1[1] \longrightarrow 1[0]$, $\gamma_r\colon 1[1] \longrightarrow 1[0]$, $\gamma_l\colon 1[1] \longrightarrow 1[0]$, $t\colon 1[2] \longrightarrow 2[0]$, $p\colon 1[2] \longrightarrow 4[0]$,
satisfying adequate relations to induce asociativity, identity and inverse isomorphisms for the associated categorical group structure. The commutativity of the pentagonal and triangular diagrams of a categorical group will be a consequence of the maps and properties of p and t.
The new extended category will be denoted by

$$\mathcal{C} = E(E(\bar{4}) \times EC(\Delta/2))$$

The small-braided category $\mathcal{BC}$:
The small category $\mathcal{C}$ above can be extended with a new map
$c\colon 1[1] \longrightarrow 2[0]$ such that $c\delta_0 = \mu$ and $c\delta_1 = \tau\mu$,
where if $i_l, i_r\colon 1 \longrightarrow 2$ are the canonical inclusions, then $\tau = i_r + i_l$ $(\mathrm{id}_2 = i_l + i_r)$.
In order to have the properties of the braided structure we also need two maps
$h_l : 1[2] \longrightarrow 3[0]$, $h_r : 1[2] \longrightarrow 3[0]$ satisfying adequate relations to induce the commutativity of the usual hexagonal diagrams of the braided structure.

The small-symmetric category $\mathcal{SC}$:
Finally a new extension of $\mathcal{BC}$ can be considered by taking a map
$s : 1[2] \longrightarrow 2[0]$ such that $s\delta_2 = \mu\epsilon_0$, $s\delta_1 = (\tau c + c)K\delta_1$, $s\delta_0 = s\delta_1\delta_0\epsilon_0$.

5. The functors $S \wedge \Delta^+ \colon \mathcal{C} \to \mathbf{Top}^*$, $S^2 \wedge \Delta^+ \colon \mathcal{BC} \to \mathbf{Top}^*$ and $S^n \wedge \Delta^+ \colon \mathcal{SC} \to \mathbf{Top}^*$ $(n \geq 3)$

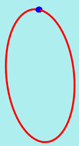
Now we take the covariant functors:
$S \colon E(\bar{4}) \to \mathbf{Top}^*$, preserving coproducts and such that $S(1) = S^1$, $S(\mu) \colon S^1 \to S^1 \vee S^1$ is the co-multiplication and $S(\nu) \colon S^1 \to S^1$ gives the inverse loop.
$\Delta \colon \Delta/2 \to \mathbf{Top}$ is given by $\Delta[p] = \Delta_p$ and extends to $C(\Delta/2)$ preserving pushouts, $\Delta([1] +_{[0]} [1]) = \Delta_1 \cup_{\Delta_0} \Delta_1$, et cetera.
We also consider adequate maps: $\Delta(V)$, $\Delta(K)$, $\Delta(A)$ that will give vertical inverses, vertical composition and associativity properties. Then, one has an induced functor $\Delta \colon EC(\Delta/2) \to \mathbf{Top}$.
Taking the functors $()^+ \colon \mathbf{Top} \to \mathbf{Top}^*$, $X^+ = X \sqcup \{*\}$, and the smash $\wedge \colon \mathbf{Top}^* \times \mathbf{Top}^* \to \mathbf{Top}^*$, we construct an induced functor

$$S \wedge \Delta^+ \colon E(\bar{4}) \times EC(\Delta/2)) \to \mathbf{Top}^*.$$

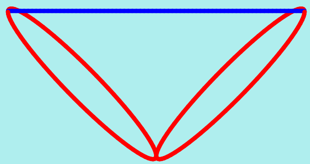
Finally, we can give maps $(S \wedge \Delta^+)(a)$, $(S \wedge \Delta^+)(r)$, $(S \wedge \Delta^+)(l)$, $(S \wedge \Delta^+)(\gamma_r)$, $(S \wedge \Delta^+)(\gamma_l)$, $(S \wedge \Delta^+)(p)$, $(S \wedge \Delta^+)(t)$ to obtain the desired functor

$$S \wedge \Delta^+ : \mathcal{C} = E(E(\bar{4}) \times EC(\Delta/2)) \to \mathbf{Top}^*.$$

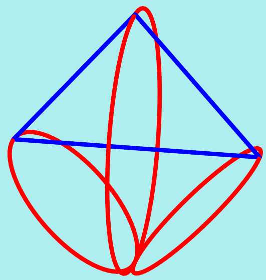
$S \wedge \Delta^+(1[0])$



$S \wedge \Delta^+(1[1])$



$S \wedge \Delta^+(1[2])$



We note that it is not possible to find a map $\tilde{c}\colon S^1 \wedge \Delta_1^+ \longrightarrow S^1 \vee S^1$ such that $\tilde{c}\tilde{\tilde{\delta}}_0 = \tilde{\mu}$ and $\tilde{c}\tilde{\tilde{\delta}}_1 = \tilde{\tau}\tilde{\mu}$ since the canonical commutator $aba^{-1}b^{-1}$ is not trivial in $\pi_1(S^1 \vee S^1)$ where a and b denote the canonical generators. However one can choose a canonical map $\tilde{c}\colon S^2 \wedge \Delta_1^+ \longrightarrow S^2 \vee S^2$ such that $\tilde{c}\tilde{\tilde{\delta}}_0 = \tilde{\mu}$ and $\tilde{c}\tilde{\tilde{\delta}}_1 = \tilde{\tau}\tilde{\mu}$, since $\pi_2(S^2 \vee S^2)$ is abelian and now the canonical commutator $aba^{-1}b^{-1}$ is trivial. Therefore one can define a functor

$$S^2 \wedge \Delta^+\colon \mathcal{BC} \longrightarrow \mathbf{Top}^*, \quad S^2 \wedge \Delta^+(1[q]) = S^2 \wedge \Delta_q^+$$

such that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "S^1\wedge\Delta^+"] & \mathcal{BC} \arrow[d, "S^2\wedge\Delta^+"] \\
\mathbf{Top}^* \arrow[r, "S"'] & \mathbf{Top}^*
\end{tikzcd}$$

Similarly for $n \geq 3$, we have a canonical map $\tilde{s}\colon S^n \wedge \Delta_2^+ \to S^n \vee S^n$ and the induced functors

$$S^n \wedge \Delta^+\colon \mathcal{SC} \to \mathbf{Top}^*, \quad S^n \wedge \Delta^+(1[q]) = S^n \wedge \Delta_q^+$$

such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{BC} \arrow[r] \arrow[d, "S^2\wedge\Delta^+"] & \mathcal{SC} \arrow[d, "S^n\wedge\Delta^+"] \\ \mathbf{Top}^* \arrow[r, "S^{n-2}"'] & \mathbf{Top}^* \end{tikzcd}$$

Remark 5.1 *Given an object X in $\mathbf{Top}^*$ the existence of functors from $\mathcal{C}$, $\mathcal{BC}$, $\mathcal{SC}$ to $\mathbf{Top}^*$ such that $1[0]$ is carried into X depends if this object admits the structure of an (braided, symmetric) categorical cogroup object in the Gpd-category $\mathbf{Top}^*$.*

A. R. Garzón, J. G. Miranda, A. Del Río, Tensor structures on homotopy groupoids of topological spaces, *International Mathematical Journal 2, 2002, pp. 407-431.*

6. Singular and realization functors. The categorical group of a presheaf

$S \wedge \Delta^+ \colon \mathcal{C} = E(E(\bar{4}) \times EC(\Delta/2)) \to \mathbf{Top}^*$ induces a pair of adjoint functors

$$\mathrm{Sing}\colon \mathbf{Top}^* \to \mathbf{Set}^{\mathcal{C}^{op}}$$

$$|\cdot|\colon \mathbf{Set}^{\mathcal{C}^{op}} \to \mathbf{Top}^*$$

We will denote by

$$\mathbf{Set}_{pp}^{\mathcal{C}^{op}}$$

the category of presheaves $X\colon \mathcal{C} = (E(E(\bar{4}) \times EC(\Delta/2)))^{op} \to \mathbf{Set}$ such that $X(i,-)$ transforms the pushouts of $C(\Delta/2)$ in pullbacks and $X(-,[j])$ transforms the coproducts of $\bar{4}$ in products.

Given a presheaf X in $\mathbf{Set}_{pp}^{\mathcal{C}^{op}}$ one can define its fundamental categorical group $G(X)$ as a quotient object. This gives a functor

$$G\colon \mathbf{Set}_{pp}^{\mathcal{C}^{op}} \to \mathbf{CG}$$

Proposition 6.1 *The functor $G\colon \mathbf{Set}_{pp}^{\mathcal{C}^{op}} \to \mathbf{CG}$ is left adjoint to the forgetful functor $U\colon \mathbf{CG} \to \mathbf{Set}_{pp}^{\mathcal{C}^{op}}$.*

The composites $\rho_2 = G\,\mathsf{Sing}$, $B = |\cdot|\,U$

$$\rho_2\colon \mathbf{Top}^* \to \mathbf{CG}$$

$$B\colon \mathbf{CG} \to \mathbf{Top}^*$$

will be called the *fundamental categorical group* and *classifying* functors.

Theorem 6.1 *The realization functor $|\cdot|\colon \mathbf{Set}_{pp}^{\mathcal{C}^{op}} \to \mathbf{Top}^*$ satisfies that $\pi_0(X) \cong \pi_1(|X|)$ and $\pi_1(X) \cong \pi_2(|X|)$*

Theorem 6.2 *The functors ρ_2, ρ_3 and ρ_n of the following diagrams:*

$$
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[dr, shift left, "\rho_2"] & \mathbf{Set}_{pp}^{\mathcal{C}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
& \mathbf{CG} \arrow[ul, shift left, "B"] \arrow[u, shift left, "U"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[dr, shift left, "\rho_3"] & \mathbf{Set}_{pp}^{\mathcal{BC}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
& \mathbf{BCG} \arrow[ul, shift left, "B"] \arrow[u, shift left, "U"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[dr, shift left, "\rho_n"] & \mathbf{Set}_{pp}^{\mathcal{SC}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
& \mathbf{SCG} \arrow[ul, shift left, "B"] \arrow[u, shift left, "U"]
\end{tikzcd}
$$

induce equivalence of categories of $[1, 2]$-types, $[2, 3]$-types and $[n, n+1]$-types ($n \geq 3$) of pointed spaces and the categories of categorical groups, braided categorical groups and symmetric categorical group ut to weak equivalences, respectively.

Remark 6.1 *For other descriptions of the functors ρ_n for pointed spaces or Kan simplicial sets, you can see some papers of Carrasco, Cegarra, Garzón, etc. For example, see:*
Carrasco, P., Cegarra, A.M., Garzón A.R. The homotopy categorical crossed module of a CW-complex, Topology and its Applications 154 (2007) 834–847.

Remark 6.2 *Note that $\rho_{q+2}(X) \cong \rho_2(\Omega^q(X))$.*

7. The categorical groups $\rho_2, \rho_2^{\mathbb{N}}, \rho_2^{\mathbb{R}_+}$ and long exact sequences

For a given pointed topological space X, we can consider its fundamental categorical group

$$\rho_2(X) = G\,\mathsf{Sing}(X)$$

An alternative description of its higher dimensional analogues is given by

$$\rho_{q+2}(X) = \rho_2(\Omega^q(X)),$$

where Ω is the loop functor.
Given an object (X, λ) in the category $\mathbf{E}^{\mathbb{R}_+}$, one has the pointed spaces $(X^{\mathbb{R}_+}, \lambda)$, $(X^{\mathbb{N}}, \lambda|_{\mathbb{N}})$ and the restriction fibration $\mathrm{res}\colon \mathrm{X}^{\mathbb{R}_+} \longrightarrow \mathrm{X}^{\mathbb{N}}$, $\mathrm{res}(\mu) = \mu|_{\mathbb{N}}$. The fibre is the space

$$F_{\mathrm{res}} = \{\mu \in X^{\mathbb{R}_+} |\ \mu|_{\mathbb{N}} = \lambda|_{\mathbb{N}}\}$$

Denote $\mu_i = \mu|_{[i,i+1]}$. The maps $\varphi\colon (F_{\mathrm{res}}, \lambda) \to \Omega(X^{\mathbb{N}}, \lambda)$, $\phi\colon \Omega(X^{\mathbb{N}}, \lambda) \to (F_{\mathrm{res}}, \lambda)$, given by $\varphi(\mu) = (\mu_0\lambda_0^{-1}, \mu_1\lambda_1^{-1}, \cdots)$ for $\mu \in F_{\mathrm{res}}$ and $\phi(\alpha) = (\alpha_0\lambda_0, \alpha_1\lambda_1, \cdots)$ for $\alpha \in \Omega(X^{\mathbb{N}}, \lambda)$, determine a pointed homotopy equivalence.

Therefore, the pointed map $\mathrm{res}\colon \mathrm{X}^{\mathbb{R}_+} \to \mathrm{X}^{\mathbb{N}}$ induces the fibre sequence

$$\cdots \to \Omega^2(X^{\mathbb{N}}) \to \Omega^2(X^{\mathbb{N}}) \to \Omega(X^{\mathbb{R}_+}) \to \Omega(X^{\mathbb{N}}) \to \Omega(X^{\mathbb{N}}) \to X^{\mathbb{R}_+} \to X^{\mathbb{N}}$$

We define the $\mathbb{R}_+$-*fundamental exterior categorical group* by

$$\rho_2^{\mathbb{R}_+}(X) = \rho_2(X^{\mathbb{R}_+})$$

and the $\mathbb{N}$-*fundamental exterior categorical group* by

$$\rho_2^{\mathbb{N}}(X) = \rho_2(X^{\mathbb{N}}) \ .$$

In the obvious way we have the higher analogues and we can consider fundamental groupoids for the one dimensional cases

$$\rho_1^{\mathbb{R}_+}(X) = \rho_1(X^{\mathbb{R}_+}), \quad \rho_1^{\mathbb{R}_+}(X) = \rho_1(X^{\mathbb{R}_+}).$$

All these exterior homotopy invariants are related as follows:

Theorem 7.1 *Given an exterior space X with a base ray $\lambda\colon \mathbb{R}_+ \to X$ there is a long exact sequence*

$$\cdots \to \rho_q^{\mathbb{R}_+}(X) \to \rho_q^{\mathbb{N}}(X) \to \rho_q^{\mathbb{N}}(X) \to \rho_{q-1}^{\mathbb{R}_+}(X) \to$$

$$\cdots \to \rho_3^{\mathbb{R}_+}(X) \to \rho_3^{\mathbb{N}}(X) \to \rho_3^{\mathbb{N}}(X) \to \rho_2^{\mathbb{R}_+}(X) \to \rho_2^{\mathbb{N}}(X) \to \rho_2^{\mathbb{N}}(X) \to$$
$$\rho_1^{\mathbb{R}_+}(X) \to \rho_1^{\mathbb{N}}(X)$$

which satifies the following properties:

1. $\rho_1^{\mathbb{N}}(X)$, $\rho_1^{\mathbb{R}_+}(X)$ *have the structure of a groupoid.*
2. $\rho_2^{\mathbb{N}}(X)$, $\rho_2^{\mathbb{R}_+}(X)$ *have the structure of a categorical group.*
3. $\rho_3^{\mathbb{N}}(X)$, $\rho_3^{\mathbb{R}_+}(X)$ *have the structure of a braided categorical group.*
4. $\rho_q^{\mathbb{N}}(X)$, $\rho_q^{\mathbb{R}_+}(X)$ *have the structure of a symmetric categorical group for* $q \geq 4$.

The notion of exactness considered in Theorem above is the given in

E.M. Vitale, *A Picard-Brauer exact sequence of categorical groups*, J. Pure Applied Algebra, 175 (2002), 383-408.

To obtain a proof we can take the exact sequence of categorical groups associated to the fibration $X^{\mathbb{R}_+} \longrightarrow X^{\mathbb{N}}$, see:

A. R. Garzón, J. G. Miranda, A. Del Río, *Tensor structures on homotopy groupoids of topological spaces*, International Mathematical Journal 2, 2002, pp. 407-431.

8. Exterior $\mathbb{R}_+$-$[n, n+1]$-types and the $\mathbb{R}_+$-fundamental exterior categorical group

Consider the functor

$$p\colon \mathbf{Top}^* \to \mathbf{E}^{\mathbb{R}_+} \quad p(X) = \mathbb{R}_+ \bar{\times} X$$

and its rigtht adjoint

$$(\cdot)^{\mathbb{R}_+}\colon \mathbf{E}^{\mathbb{R}_+} \to \mathbf{Top}^*, \quad Y \to Y^{\mathbb{R}_+}$$

Lemma 8.1 *Suposse that $f\colon X \to X'$ is a map in $\mathbf{Top}^*$ and $g\colon Y \to Y'$ is a map in $\mathbf{E}^{\mathbb{R}_+}$. Then*

(i) if $\pi_q(f)$ is an isomorphism, then $\pi_q^{\mathbb{R}_+}(p(f))$ is an isomorphism,

(ii) if $\pi_q^{\mathbb{R}_+}(g)$ is an isomorphism, then $\pi_q(g^{\mathbb{R}_+})$ is an isomorphism,

(iii) the unit $X \to (X \bar{\times} \mathbb{R}_+)^{\mathbb{R}_+}$ and the counit $\mathbb{R}_+ \bar{\times} Y^{\mathbb{R}_+} \to Y$ are weak equivalences.

The functor p induces a covariant functor

$$p(S \wedge \Delta^+)\colon \mathcal{C} \longrightarrow \mathbf{E}^{\mathbb{R}_+}$$

and the corresponding singular an realization functors

$$\mathsf{Sing}^{\mathbb{R}_+}\colon \mathbf{E}^{\mathbb{R}_+} \longrightarrow \mathbf{Set}_{pp}^{\mathcal{C}^{op}}$$

$$|\cdot|^{\mathbb{R}_+}\colon \mathbf{Set}_{pp}^{\mathcal{C}op} \longrightarrow \mathbf{E}^{\mathbb{R}_+}$$

On the other hand, we also have the adjunction

$$G\colon \mathbf{Set}_{pp}^{\mathcal{C}^{op}} \longrightarrow \mathbf{CG}$$

$$U\colon \mathbf{CG} \longrightarrow \mathbf{Set}_{pp}^{\mathcal{C}op}$$

Taking the composites $G\mathsf{Sing}^{\mathbb{R}_+} \cong \rho_2^{\mathbb{R}_+}$ and $B^{\mathbb{R}_+} = |\cdot|^{\mathbb{R}_+}U$, one has that

Theorem 8.1 *The functors $\rho_2^{\mathbb{R}_+}$ and $B^{\mathbb{R}_+}$ induce an equivalence of categories*

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{R}_+}^{[1,2]}]^{-1} \longrightarrow \mathbf{CG}[\Sigma]^{-1}$$

where Σ is the class weak equivalences (equivalences) in $\mathbf{CG}$.

Similarly one has

Theorem 8.2 *The functors $\rho_3^{\mathbb{R}_+}$ and $\rho_n^{\mathbb{R}_+}$ of the following diagrams:*

$$
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[d, shift left, "p"] \arrow[drr, shift left, "\rho_3"] & & \mathbf{Set}_{pp}^{\mathcal{BC}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
\mathbf{E}^{\mathbb{R}_+} \arrow[u, shift left, "(\cdot)^{\mathbb{R}_+}"] \arrow[rr, shift left, "\rho_3^{\mathbb{R}_+}"] & & \mathbf{BCG} \arrow[ll, shift left, "B"] \arrow[u, shift left, "U"] \arrow[ull, shift left, "B"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[d, shift left, "p"] \arrow[drr, shift left, "\rho_n"] & & \mathbf{Set}_{pp}^{\mathcal{SC}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
\mathbf{E}^{\mathbb{R}_+} \arrow[u, shift left, "(\cdot)^{\mathbb{R}_+}"] \arrow[rr, shift left, "\rho_n^{\mathbb{R}_+}"] & & \mathbf{SCG} \arrow[ll, shift left, "B"] \arrow[u, shift left, "U"] \arrow[ull, shift left, "B"]
\end{tikzcd}
$$

induce category equivalences of $\mathbb{R}_+$-$[2,3]$-types and $\mathbb{R}_+$-$[n,n+1]$-types ($n \geq 3$) of rayed exterior spaces and the categories of categorical groups, braided categorical groups and symmetric categorical group ut to weak equivalences, respectively.

9. Exterior $\mathbb{N}$-[1,2]-types and the $\mathbb{N}$-fundamental exterior categorical group

Consider the functor $c\colon \mathbf{Top}^* \to \mathbf{E}^{\mathbb{R}_+}$ given by

$$c(X) = (\mathbb{R}_+ \sqcup (\sqcup_0^\infty X))/n \sim *_n$$

where $n \geq 0$ is a natural number and $*_n$ denotes the base point of the corresponding copy of X . Its rigtht adjoint is given by

$$(\cdot)^{\mathbb{N}}\colon \mathbf{E}^{\mathbb{R}_+} \to \mathbf{Top}^*, \quad Y \to Y^{\mathbb{N}}$$

Lemma 9.1 *Suposse that $f\colon X \to X'$ is a map in $\mathbf{Top}^*$ and $g\colon Y \to Y'$ is a map in $\mathbf{E}^{\mathbb{R}_+}$ .Then*

(i) if $\pi_q(f)$ is an isomorphism, then $\pi_q^{\mathbb{N}}(c(f))$ is an isomorphism,

(ii) if $\pi_q^{\mathbb{N}}(g)$ is an isomorphism, then $\pi_q(g^{\mathbb{N}})$ is an isomorphism.

Note that in this case, in general the unit $X \to (c(X))^{\mathbb{N}}$ and the counit $c(Y^{\mathbb{N}}) \to Y$ are not weak equivalences.

Theorem 9.1 *The functors $\rho_2^{\mathbb{N}_+}$ $\rho_3^{\mathbb{N}_+}$ and $\rho_n^{\mathbb{N}_+}$ of the following diagrams:*

$$
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[d, shift left, "c"] \arrow[dr, shift left, "\rho_2"] & \mathbf{Set}_{pp}^{\mathcal{C}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
\mathbf{E}^{\mathbb{R}_+} \arrow[u, shift left, "(\cdot)^{\mathbb{N}}"] \arrow[r, shift left, "\rho_2^{\mathbb{N}}"] & \mathbf{CG} \arrow[l, shift left, "B"] \arrow[u, shift left, "U"] \arrow[ul, shift left, "B"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[d, shift left, "c"] \arrow[dr, shift left, "\rho_3"] & \mathbf{Set}_{pp}^{\mathcal{BC}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
\mathbf{E}^{\mathbb{R}_+} \arrow[u, shift left, "(\cdot)^{\mathbb{N}}"] \arrow[r, shift left, "\rho_3^{\mathbb{N}}"] & \mathbf{BCG} \arrow[l, shift left, "B"] \arrow[u, shift left, "U"] \arrow[ul, shift left, "B"]
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{Top}^* \arrow[r, shift left, "\mathrm{Sing}"] \arrow[d, shift left, "c"] \arrow[dr, shift left, "\rho_n"] & \mathbf{Set}_{pp}^{\mathcal{SC}^{op}} \arrow[l, shift left, "|\cdot|"] \arrow[d, shift left, "G"] \\
\mathbf{E}^{\mathbb{R}_+} \arrow[u, shift left, "(\cdot)^{\mathbb{N}}"] \arrow[r, shift left, "\rho_n^{\mathbb{N}}"] & \mathbf{SCG} \arrow[l, shift left, "B"] \arrow[u, shift left, "U"] \arrow[ul, shift left, "B"]
\end{tikzcd}
$$

induce functors from the categories of $\mathbb{N}$-$[1,2]$-types, $\mathbb{N}$-$[2,3]$-types and $\mathbb{N}$-$[n,n+1]$-types ($n \geq 3$) of rayed exterior spaces to the categories of categorical groups, braided categorical groups and symmetric categorical group ut to weak equivalences, respectively.

Take an exterior rayed space X (for example, $X = \mathbb{R}_+ \bar{\times} S^1$) such that

$\mathrm{limtow}\, \pi_1 \varepsilon(\mathrm{X}) \neq 1$

We can prove that the space $B\rho_2^{\mathbb{N}}(X)$ satisfies that

$\mathrm{limtow}\, \pi_1 \varepsilon(B\rho_2^{\mathbb{N}}(X)) = 1$

This implies that X and $B\rho_2^{\mathbb{N}}(X)$ have different $\mathbb{N}$-1-type and then different $\mathbb{N}$-$[1, 2]$-type.

Open question: Is it possible to modify the notion of categorical group to obtain an new algebraic model for $\mathbb{N}$-$[1,2]$-types?

Perhaps, a partial answer can be obtained by taking a monoid $\mathbb{M}$ of endomorphisms of the exterior space $\mathbb{R}_+ \sqcup (\sqcup_0^\infty S^1))/n \sim *_n$, and a new extension of the category $\bar{4}$ obtained by adding an arrow for each element of the monoid. This gives a new type of presheaf that will induce a categorical group enriched with an action of the monoid $\mathbb{M}$.

We think that the new enriched categorical group and the new corresponding functors will give an equivalence of a large class of exterior $\mathbb{N}$-$[1,2]$-types and the corresponding $\mathbb{M}$-categorical groups. This class of exterior $\mathbb{N}$-$[1,2]$-types contains the subcategory of proper $\mathbb{N}$-$[1,2]$-types. Consequently, we will obtain a category of algebraic models for proper $\mathbb{N}$-$[1,2]$-types.

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