# Categorical groups in $K$-theory and number theory

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Workshop on Categorical Groups

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# 0-additive invariants...

Let $\mathcal{W}$ be a Waldhausen category, with

- a zero object $*$,
- cofibre sequences $A \rightarrowtail B \twoheadrightarrow B/A$,
- weak equivalences $A \xrightarrow{\sim} A'$,

satisfying certain axioms.

Example:

- If $\mathcal{A}$ is an abelian category, then the category of bounded chain complexes $\mathrm{Ch}^b\mathcal{A}$ is a Waldhausen category.

A 0-additive invariant on $\mathcal{W}$ is given by

- an abelian group $G$
- a function $\mathrm{Ob}\mathcal{W} \xrightarrow{\phi} G$ satisfying

$$\begin{array}{rcccl} A \rightarrowtail B \twoheadrightarrow B/A & \in & \mathcal{W} & \Rightarrow & \phi(B) = \phi(A) + \phi(B/A) \\ A \xrightarrow{\sim} A' & \in & \mathcal{W} & \Rightarrow & \phi(A') = \phi(A) \end{array}$$

# 0-additive invariants... and $K_0$

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**Easy Theorem**

There is a universal 0-additive invariant $\phi : \mathrm{Ob}\mathcal{W} \to G$ given by $G = K_0\mathcal{W}$ and $\phi(A) = [A]$.

$$\begin{tikzcd} \mathrm{Ob}\mathcal{W} \arrow[r] \arrow[dr, "\forall"'] & K_0\mathcal{W} \arrow[d, "\exists!"] \\ & G \end{tikzcd}$$

$K_0\mathcal{W}$ is the abelian group with

- generators $[A]$ for each object $A$ of $\mathcal{W}$
- relations $[B] = [B/A] + [A]$ and $[A'] = [A]$
  for each cofibre sequence and weak equivalence in $\mathcal{W}$.

Any $x \in K_0\mathcal{W}$ can be expressed as $[A_1] - [A_2]$, $A_i \in \mathrm{Ob}\mathcal{W}$.

# The 0-type of the $K$-theory spectrum

The functor $K_0$ fits into a commutative square

$$\begin{tikzcd}
\text{Waldhausen cats.} \arrow[r, "K_0"] \arrow[d, "\mathbb{K}"'] & \mathcal{M}_0 \arrow[d, "\cong"] \arrow[dl, dashed, "EM"'] \\
\text{Connective spectra} \arrow[r, two heads, "P_0"'] & \text{Ho}\,\text{spec}_0^0
\end{tikzcd}$$

Here $EM$ is the Eilenberg–MacLane spectrum on the category $\mathcal{M}_0$ of abelian groups, $\mathbb{K}$ is the Waldhasen $K$-theory spectrum, and $P_0$ is the 0-th Postnikov section.

## Easy Theorem

The 0-th Postnikov section of the Eilenberg–MacLane spectrum of $K_0\mathcal{W}$ coincides with that of $\mathbb{K}\mathcal{W}$.

# The 1-type of the $K$-theory spectrum

After the preceding discussion, it is clear the next step is to find an algebraic model $\mathcal{DW}$ which classifies the 1-type of the Waldhausen $K$-theory spectrum. We present

- a category of algebraic models $\mathcal{M}_1$
- a functor $\mathcal{D}$ from Waldhausen categories to $\mathcal{M}_1$
- a classifying functor $B$ from $\mathcal{M}_1$ to connective spectra
- a definition of 1-additive invariants
- a universal 1-additive invariant

$$\begin{tikzcd}
\mathcal{W}^{\mathrm{weq}} \arrow[r] \arrow[dr, "\forall 1\text{-add}"'] & \mathcal{DW} \arrow[d, "\exists! \in \mathcal{M}_1"] \\
 & G
\end{tikzcd}$$

such that the following square commutes:

$$\begin{tikzcd}
\text{Waldhausen cats} \arrow[r, "\mathcal{D}"] \arrow[d, "\mathbb{K}"'] & \mathcal{M}_1 \arrow[r, two heads] \arrow[dl, dashed, "B"] & \operatorname{Ho} \mathcal{M}_1 \arrow[d, "\cong"] \\
\text{Connective spectra} \arrow[rr, two heads, "P_1"'] & & \operatorname{Ho} \mathrm{spec}_0^1
\end{tikzcd}$$

# An ‘easy’ model $\mathcal{DW}$ for the 1-type of $\mathbb{K}\mathcal{W}$

Our model $\mathcal{DW}$ for $K_0$ and $K_1$ consists not of a pair of abelian groups but of a stable quadratic module

$$\begin{tikzcd}
 & (\mathcal{D}_0\mathcal{W})^{\mathrm{ab}} \otimes (\mathcal{D}_0\mathcal{W})^{\mathrm{ab}} \arrow[d, "\langle\cdot,\cdot\rangle"'] & & \\
K_1\mathcal{W} \arrow[r, hook] & \mathcal{D}_1\mathcal{W} \arrow[r, "\partial"] & \mathcal{D}_0\mathcal{W} \arrow[ul, dashed, "H"'] \arrow[r, two heads] & K_0\mathcal{W}.
\end{tikzcd}$$

in which the bottom row is exact.
Important features:

- small — generators given just by the objects, weak equivalences and cofibre sequences of the category $\mathcal{W}$.
- minimal nilpotency degree — $\mathcal{D}_0\mathcal{W}$ and $\mathcal{D}_1\mathcal{W}$ are class 2 nilpotent groups.
- encodes the 1-type $P_1\mathbb{K}\mathcal{W}$ in a functorial way; there is a bijection between homotopy classes of morphisms $\mathcal{DW} \to \mathcal{DW}'$ and of morphisms $P_1\mathbb{K}\mathcal{W} \to P_1\mathbb{K}\mathcal{W}'$.

A stable quadratic module can be seen as a special case of:

- a commutative monoid in the category of crossed modules
- a commutative strict categorical group,
- a (commutative, strict) Picard category,
- a symmetric monoidal category, in which all objects have inverses with respect to the monoidal structure.

  We therefore have a notion of the classifying spectrum of a stable quadratic module.

A stable quadratic module consists of a diagram of groups

$$H^{\mathrm{ab}} \otimes H^{\mathrm{ab}} \xrightarrow{\langle\cdot,\cdot\rangle} G \xrightarrow{\partial} H$$

satisfying

$$\begin{aligned} \partial\langle h_1, h_2\rangle &= [h_2, h_1], \\ \langle \partial(g_1), \partial(g_2)\rangle &= [g_2, g_1], \\ \langle h_1, h_2\rangle + \langle h_2, h_1\rangle &= 0. \end{aligned}$$

Here $(-)^{\mathrm{ab}}$ is abelianization and $[a, b] = -a - b + a + b$.

# Stable quadratic modules *v* monoids in crossed complexes *v* symmetric monoidal categories. . .

- There is a group action of $H$ on $G$ via

  $$g^h = g + \langle h, \partial g\rangle$$

  and $\partial : G \to H$ is a crossed module of nil$_2$-groups;

  $\langle\ ,\ \rangle$ gives a monoid stucture on this crossed module.
- One can define a strict symmetric monoidal category with objects $h \in H$ and morphisms $(h, g) : h \to h + \partial g$;

  the symmetry isomorphism is given by the bracket:

  $$(h_1 + h_2, \langle h_2, h_2\rangle) : h_1 + h_2 \longrightarrow h_2 + h_1$$

Note: the kernel and cokernel of the boundary map $\partial$ are abelian,

$$\pi_1(\partial) := \ker(\partial), \qquad \pi_0(\partial) := \operatorname{coker}(\partial)$$

# A 1-additive invariant to a Picard category $G$...

is a functor from the subcategory of weak equivalences

$$\det\colon \mathcal{W}^{\mathrm{weq}} \to G$$

together with additivity data in $G$,

$$\det(\Delta)\colon\ \det(B/A)\otimes\det(A) \to \det(B),$$

for each cofibre sequence $\Delta = (A \rightarrowtail B \twoheadrightarrow B/A)$ satisfying naturality, associativity and commutativity conditions:

**Naturality:** For a weak equivalence of cofibre sequences $\Delta \xrightarrow{\sim} \Delta'$ the following diagram must commute in $G$,

$$\begin{tikzcd}
\det(B/A)\otimes\det(A) \arrow[r, "\det(\Delta)"] \arrow[d] & \det(B) \arrow[d] \\
\det(B'/A')\otimes\det(A') \arrow[r, "\det(\Delta')"] & \det(B')
\end{tikzcd}$$

**Associativity:** For a diagram of cofibre sequences as follows

$$\Delta_f = (A \overset{f}{\rightarrowtail} B \twoheadrightarrow B/A), \quad \Delta_g = (B \overset{g}{\rightarrowtail} C \twoheadrightarrow C/B),$$
$$\Delta_{gf} = (A \overset{gf}{\rightarrowtail} C \twoheadrightarrow C/A), \quad \widetilde{\Delta} = (B/A \rightarrowtail C/A \twoheadrightarrow C/B),$$

$$\begin{tikzcd}
 & & C/B \\
 & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\
A \arrow[r, tail] & B \arrow[r, tail] \arrow[u, two heads] & C \arrow[u, two heads]
\end{tikzcd}$$

the following diagram must commute in $G$,

$$\begin{tikzcd}
\det(C/B) \otimes \det(B/A) \otimes \det(A) \arrow[r, "\det(\widetilde{\Delta})\otimes 1"] \arrow[d, "1\otimes\det(\Delta_f)"'] & \det(C/A) \otimes \det(A) \arrow[d, "\det(\Delta_{gf})"] \\
\det(C/B) \otimes \det(B) \arrow[r, "\det(\Delta_g)"'] & \det(C)
\end{tikzcd}$$

**Commutativity:** Let $A$, $B$ be two objects in $\mathcal{W}$ with the following cofibre sequences equences associated to inclusions and projections of a coproduct $A \sqcup B$,

$$\Delta_1 = (A \rightarrowtail A \sqcup B \twoheadrightarrow B), \qquad \Delta_2 = (B \rightarrowtail A \sqcup B \twoheadrightarrow A).$$

Then the following diagram must commute in $G$,

$$\begin{tikzcd}
 & \det(A \sqcup B) & \\
\det(B) \otimes \det(A) \arrow[ur, "det(\Delta_1)"] \arrow[rr, tail, "\text{symmetry}"'] & & \det(A) \otimes \det(B) \arrow[ul, "\det(\Delta_2)"']
\end{tikzcd}$$

# History and context

- Deligne[1] introduces 1-additive determinant functors for exact categories and non-strict Picard categories,
- Burns–Flach[2] use this formalism in the Tamawara number conjecture with coefficients in non-commutative rings.
- Fukaya–Kato[3] give an adhoc construction of universal determinant functors in their extension of Burns–Flach.
- Muro–Tonks[4] give a construction of the universal determinant functor which is more explicit and general.
- Breuning[5] gives a construction 'by hand' of universal determinant functors on triangulated categories.
- Witte uses the Muro–Tonks construction in his thesis Noncommutative Iwasawa Main Conjectures for Varieties over Finite Fields (May 2008)

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[1] Le déterminant de la cohomologie (1987)

[2] Tamagawa numbers for motives with (non-commutative) coefficients (2001)

[3] A formulation of conjectures on $p$-adic zeta functions in non-commutative Iwasawa theory (2006)

[4] The 1-type of a Waldhausen $K$-theory spectrum (2007)

[5] Determinant functors on triangulated categories (preprint 2006)

# Explicit presentation of the universal object

$\mathcal{D}\mathcal{W}$ is the stable quadratic module with generators

$$\begin{aligned}
[A] &\in \mathcal{D}_0\mathcal{W} && \text{for any object in } \mathcal{W},\\
[A \overset{\sim}{\to} A'] &\in \mathcal{D}_1\mathcal{W} && \text{for any weak equivalence in } \mathcal{W},\\
[A \rightarrowtail B \twoheadrightarrow B/A] &\in \mathcal{D}_1\mathcal{W} && \text{for any cofibre sequence in } \mathcal{W},
\end{aligned}$$

and seven types of relations:

$$[* \rightarrowtail * \twoheadrightarrow *] = 0 \tag{R0}$$

$$\partial[A \overset{\sim}{\to} A'] = -[A'] + [A], \tag{R1}$$

$$\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A], \tag{R2}$$

(Hence coker($\partial$) gives us $K_0$).
For any composite of weak equivalences $A \overset{\sim}{\to} B \overset{\sim}{\to} C$,

$$[A \overset{\sim}{\to} C] = [B \overset{\sim}{\to} C] + [A \overset{\sim}{\to} B], \tag{R3}$$

# (R4) and (R5): For all diagrams of the form. . .

$\boxed{x_{2,1}}$

$$\begin{tikzcd}
A' \arrow[r, tail] & B' \arrow[r, two heads] & B'/A' \\
A \arrow[u, "\sim"] \arrow[r, tail] & B \arrow[u, "\sim"] \arrow[r, two heads] & B/A \arrow[u, "\sim"]
\end{tikzcd}$$

$\boxed{x_{3,0}}$

$$\begin{tikzcd}
 & & C/B \\
 & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\
A \arrow[r, tail] & B \arrow[u, two heads] \arrow[r, tail] & C \arrow[u, two heads]
\end{tikzcd}$$

$$[B\xrightarrow{\sim}B']+[A\rightarrowtail B\twoheadrightarrow B/A]=[A'\rightarrowtail B'\twoheadrightarrow B'/A']+[A\xrightarrow{\sim}A']+[B/A\xrightarrow{\sim}B'/A']^{[A]}$$

(R4)

$$[B\rightarrowtail C\twoheadrightarrow C/B]+[A\rightarrowtail B\twoheadrightarrow B/A]=[A\rightarrowtail C\twoheadrightarrow C/A]+[B/A\rightarrowtail C/A\twoheadrightarrow C/B]^{[A]}$$

(R5)

# Coproduct relation (R6)

For any coproduct diagram $\begin{tikzcd} A \arrow[r, shift left, "i_1"] & A \sqcup B \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \end{tikzcd}$ in $\mathcal{W}$,

$$\langle [A], [B] \rangle \;=\; -[A \overset{i_1}{\rightarrowtail} A \sqcup B \overset{p_2}{\twoheadrightarrow} B] + [B \overset{i_2}{\rightarrowtail} A \sqcup B \overset{p_1}{\twoheadrightarrow} A]. \qquad \text{(R6)}$$

The first Postnikov invariant of this stable quadratic module is

$$\begin{array}{rcl} K_0\mathcal{W} \otimes \mathbb{Z}/2 & \xrightarrow{\cdot\eta} & K_1\mathcal{W} \\ [A] \cdot \eta & = & \langle [A], [A] \rangle \end{array}$$

Relations (R4) and (R6) imply

$$[A] \cdot \eta \;=\; [A \sqcup A \xrightarrow{\cong} A \sqcup A]$$

# The abelian group $K_1\mathcal{W}$

As a first application of this presentation we have

### Theorem [MT06]

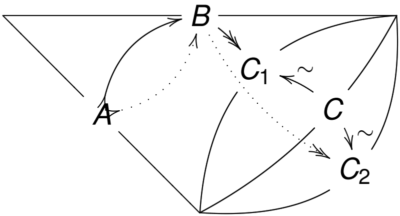
Any $x \in K_1\mathcal{W}$ can be expressed as

$$[\Delta_1] - [\Delta_2] + [w_1] - [w_2]$$

for cofibre sequences $\Delta_i$ and weak equivalences $w_i$

$$\begin{tikzcd}
 & & C_1 & \\
A \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \arrow[ur, tail] \arrow[dr, tail] & & C. \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
 & & C_2 &
\end{tikzcd}$$

# Elements of $K_1\mathcal{W}$



# Where does this presentation come from?

Waldhausen's definition of $\mathbb{K}\mathbf{C}$ is a classifying spectrum associated to a simplicial category

$$\begin{tikzcd}
\{*\} = wS_0\mathcal{W} \arrow[r, bend left=50, "s_0"] & wS_1\mathcal{W} \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[r, bend left=50, "s_0"] \arrow[r, bend left=30, "s_1"] & wS_2\mathcal{W} \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2, "{d_0,d_1,d_2}"] \arrow[r, bend left=50, "{s_0,s_1,s_2}"] & wS_3\mathcal{W} \arrow[l, "{d_0,d_1,d_2,d_3}"] \arrow[r, bend left=50] & \cdots \arrow[l]
\end{tikzcd}$$

Each $wS_m\mathcal{W}$ has objects the sequences of cofibrations

$$a \;=\; (A_1 \rightarrowtail A_2 \rightarrowtail A_3 \rightarrowtail \cdots \rightarrowtail A_{m-1} \rightarrowtail A_m)$$

and associated cofibre sequences $A_j/A_i \rightarrowtail A_k/A_i \twoheadrightarrow A_k/A_j$, and morphisms levelwise weak equivalences between them.

Taking the nerve gives a bisimplicial set $X$ with $(m, n)$-simplices

$$x_{m,n} \;=\; \left(a \xrightarrow{\sim} a' \xrightarrow{\sim} a'' \xrightarrow{\sim} \cdots \xrightarrow{\sim} a^{(n)}\right).$$

The $K$-theory of $\mathcal{W}$ is then just the homotopy theory of $X$.

# The recipe for defining $\mathcal{DW}$

- Consider the topological space $|X|$ filtered by

$$|X|_k = \bigcup_{m+n\leq k} \Delta^m \times \Delta^n \times X_{m,n} \Big/ \sim$$

- the total crossed complex

$$\Pi(X) = \left( \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1, \right)$$

given by $C_k = \pi_k(|X|_k, |X|_{k-1})$ and connecting maps $\partial$

$\mathcal{DW}$ is the s.q.m. associated to the crossed module $\phi\Pi X$

$$\pi_2(X) \rightarrowtail \mathrm{coker}(\partial_3) \xrightarrow{\partial_2} C_1 \twoheadrightarrow \pi_1(X)$$

with monoid stucture

$$\phi\Pi X \otimes \phi\Pi X \to \phi\Pi(X \times X) \xrightarrow{\sqcup} \phi\Pi X$$

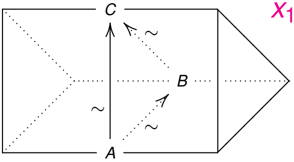
# Presenting $\mathcal{DW}$; pictures of (R3) and (R4)

The cell structure of $X$ gives generators and relations for $\mathcal{DW}$.

The generators are the simplices $x_{m,n}$ with $m+n=1$ or 2:

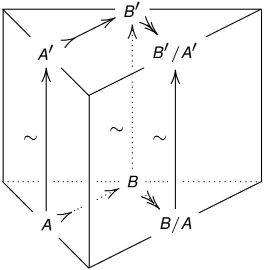
- elements $[A]$ of $X_{1,0}$
- elements $[A \overset{\sim}{\to} A']$ of $X_{1,1}$
- elements $[A \rightarrowtail B \twoheadrightarrow B/A]$ of $X_{2,0}$

The simplicial boundary maps give the relations (R1) and (R2)
For $m+n=3$ we have 3 types of relation $\partial_3(x_{m,n})=0$.



$x_{1,2}$

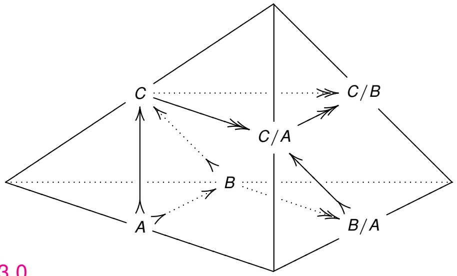
$x_{2,1}$



$$[B\xrightarrow{\sim}C]+[A\xrightarrow{\sim}B] \;=\; [A\xrightarrow{\sim}C],$$

$$[B\xrightarrow{\sim}B']+[A\rightarrowtail B\twoheadrightarrow B/A]=[A'\rightarrowtail B'\twoheadrightarrow B'/A']+[A\xrightarrow{\sim}A']+[B/A\xrightarrow{\sim}B'/A']^{[A]}$$

# The ‘octahedral axiom’ (R5)



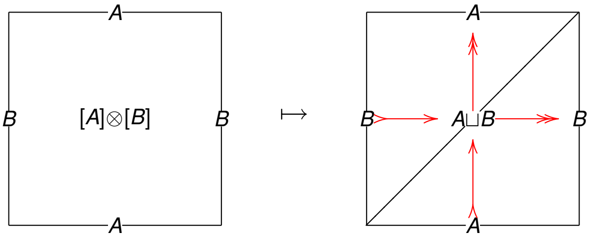
$X_{3,0}$

$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A] = [A \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B]^{[A]}$$

# The monoid structure

The bracket $\langle [A], [B] \rangle$ is given by the image of $[A] \otimes [B]$ under the Eilenberg–Zilber equivalence

$$\phi\Pi X \otimes \phi\Pi X \to \phi\Pi(X \times X) \xrightarrow{\sqcup} \phi\Pi X$$



$$\langle [A], [B] \rangle \;=\; -[A \overset{i_1}{\rightarrowtail} A \sqcup B \overset{p_2}{\twoheadrightarrow} B] + [B \overset{i_2}{\rightarrowtail} A \sqcup B \overset{p_1}{\twoheadrightarrow} A]. \qquad \text{(R6)}$$