DERIVED CATEGORIES OF COHERENT SHEAVES ON CALABI-YAU 3-FOLDS

CRISTINA MARTÍNEZ RAMÍREZ

MPIM (BONN)

November 7^{th}, 2007

Let X be a smooth complex projective variety and $D^b(X)$ the bounded derived category of coherent sheaves.

- How much information about X is contained in $D^b(X)$?

Certain invariants of X can be shown to depend only on $D^b(X)$.

- The dimension of X
- If it is Calabi-Yau ($\omega_X \cong \mathcal{O}_X$) or not.
- $K^*_{top}(X)$

<u>Bondal and Orlov</u>. If the canonical sheaf ω_X or its inverse is ample, X can be enterely reconstructed from $D^b(X)$.

D. Orlov Let A_1 and A_2 be abelian varieties. Then the derived categories $D^b(A_1)$ and $D^b(A_2)$ are equivalent as triangulated categories iff there exists an isometric isomorphisms

$$f : A_1 \times \hat{A}_1 \xrightarrow{\sim} A_2 \times \hat{A}_2 .$$

Given an abelian variety A, one can consider the abelian variety $\hat{A} := \mathrm{Pic}^0(A)$,

There exists an equivalence $D^b(A) \cong D^b(\hat{A})$ induced by the Poincaré bundle on $A \times \hat{A}$.

Def. Hochschild cohomology of X

$HH^*(X) = \mathrm{Ext}^*_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ depends only on $D^b(X)$.

$$HH^r(X) \cong \bigoplus_{p+q=r} H^p(X, \Lambda^q TX)$$

· For Calabi-Yau manifolds

$$\dim HH^r(X) = \sum_{p+q=r} h^{p,n-q}(X)$$

In mirror symmetry, these are the Betti numbers of the mirror manifold.

Let X, Y be two smooth complex projective varieties; The Fourier-Mukai transform (FMT) by an object $\mathcal{P} \in D^b(X \times Y)$ is the exact functor

$$\Phi_{\mathcal{P}} : D^b(X) \to D^b(X)$$

$$\Phi_{\mathcal{P}}(\mathcal{G}) = R\pi_{2*}\left(\pi_1^* \mathcal{G} \overset{L}{\otimes} \mathcal{P}\right)$$

where $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ are the projections.

· If $\Phi_{\mathcal{P}}$ is an equivalence, then $\mathcal{P}$ must satisfy a partial Calabi-Yau condition:

$$\mathcal{P} \otimes \pi_1^* \omega_X \otimes \pi_2^* \omega_Y^{-1} \cong \mathcal{P}$$

Orlov. Let X, Y be smooth projective varieties. Suppose that $F: D^b(X) \xrightarrow{\sim} D^b(Y)$ is an exact functor and an equivalence of triangulated categories. Then there is a unique (up to an isomorphism) object $\mathcal{E} \in D^b(X \times Y)$ such that the functor F is isomorphic to the functor $\Phi_{\mathcal{E}}$

DEF. Two CY manifolds X, X' are derived equivalent if there exists a Fourier-Mukai equivalence

$$\Phi: D^b(X) \to D^b(X')$$

Let X, X' two CY 3-folds s.t. $D^b(X) \cong D^b(X')$

▶ Which information can we recover from the equivalence?

This equivalence should be of FMT, that is, there is an object $\mathcal{P} \in D^b(X \times X')$

$$\Phi_{\mathcal{P}} : D^b(X) \to D^b(X')$$

$$\Phi_{\mathcal{P}}(L) = R\pi_{2*}(\pi_1^* L \otimes \mathcal{P}),\ \mathcal{P} \text{ satisfies a CY-condition}$$

■ Example Equivalence of K3-surfaces

▪ Def A K3 surface is a 2-dim complex projective smooth variety with trivial canonical bundle and first Betti number $b_1 = 0$.

■ Mukai's construction

Suppose that (X, ℓ) is a $2d$ polarized K3 surface with $\ell^2 = 2d = 2rs$ and $(r,s) = 1$

Consider the fine moduli space $M(r, l, s)$ parametrizing ℓ-stable sheaves E on X s.t. $rk(E) = r$, $c_1(E) = \ell$, and $\chi(E) = r + s$.

$M(r, l, s)$ is a K3 surface as well!

$\hat{X} := M(r, l, s)$ is a K3 dual surface to X.

(1) Mukai's duality is an involution $\hat{\hat{X}} = X$

(2) There exists an equivalence

$D^b(X) \cong D^b(\hat{X})$ induced by the universal family on $X \times \hat{X}$.

(3) $\hat{X}$ has a polarization l', s.t. $(l')^2 = 2d$

(4) There exists a Hodge isometry

$$f: \tilde{H}(X, \mathbb{Z}) \longrightarrow \tilde{H}(Y, \mathbb{Z})$$

$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$ (Lattice theory + Hodge structures)

(Derived Torelli Theorem)

Q. Can we make a similar construction for moduli of Calabi-Yau's?

A Calabi-Yau threefold is a smooth compact connected threefold with vanishing first Betti-number & trivial canonical class.

Given C_1, CY threefold $\Rightarrow$ Construct a dual C_2

(fibration over C_1)

(A. Strominge, S.-T. Yau, E. Zaslow)

SYZ mirror symmetry conjecture

Mirror dual CY manifolds should be fibered over the same base in such a way that generic fibers are dual tori and each fiber of any of these two fibrations is a Lagrangian submanifold.

Then mirror symmetry is related with duality.

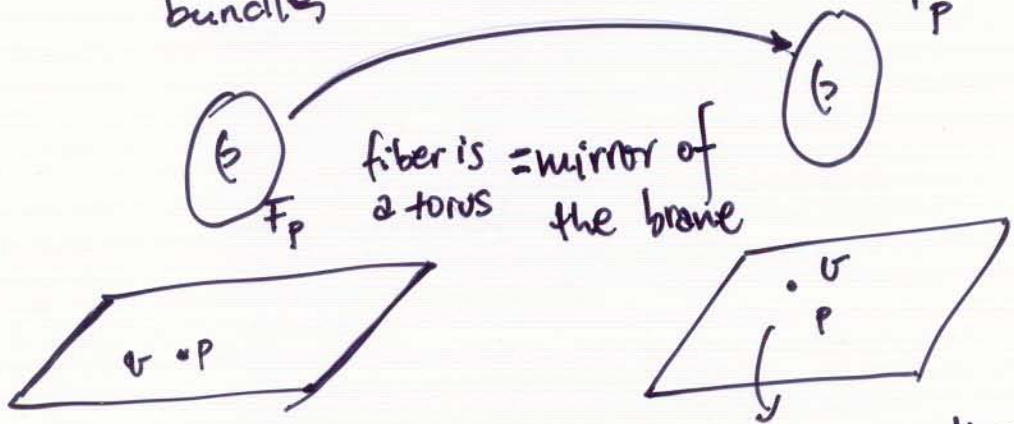
Geometric - Langland program: a verification of a particular case of mirror symmetry.

Let C be a Riemann surface,

$$\mathrm{Hom}(\pi_1(C), G_{\mathbb{C}})/_{\text{conj}} = \text{moduli space of flat } G_{\mathbb{C}}\text{-connections on } C$$

$$:= \mathcal{M}_{flat}(C, G_{\mathbb{C}})$$

$$\underset{\substack{\text{Higgs}\\ \text{bundles}}}{\mathcal{M}_{stable}(G, C)} \cong \mathcal{M}_{Hit}(G, C)_I$$



$$F_p \cong ({}^L F_p)^\vee$$

sitting on the dual point, its flat connection corresponds to v

There is a way to produce CY-threefolds that are derived equivalent.

DEF Two Calabi-Yau manifolds X, X' are derived equivalent if there exists a Fourier-Mukai equivalence

$$\Phi : D^b(X) \to D^b(X')$$

- Elliptic curves For smooth projective curves a derived equivalence always corresponds to an isomorphism.
- Bernardara If there is an equivalence between the derived categories of two smooth projective curves, then there is an isomorphism between the Jacobians of the curves, that preserves the principal polarization.

• Bridgeland. If X is a projective threefold with terminal singularities and $Y_1 \xrightarrow{f_1} X$, $Y_2 \xrightarrow{f_2} X$ are crepant resolutions, then there is an equivalence of derived categories of coherent sheaves $D(Y_1) \to D(Y_2)$.

In particular two birational Calabi-Yau threefolds have equivalent derived categories, and therefore they have the same Hodge numbers!

But not all CY that are derived equivalent are birrational.

If two projective, smooth Calabi-Yau varieties X, X' have equivalent derived categories of coherent sheaves, then there is an isomorphism between the rational cohomology groups

$$H^*(X, \mathbb{Q}) \underset{f}{\cong} H^*(X', \mathbb{Q})$$

In terms of f, one can introduce an isometry between the tangent space to the moduli space of algebraic structures on $\hat{X}$ and the space of deformations of the complexified Kähler structure on X

[BBRP]

• Mirror symmetry should consist in the identification of the moduli space of complex structures on an n-dimensional Calabi-Yau manifold X with the moduli space of "complexified Kähler structures" on the mirror manifold $\hat{X}$.

$\boxed{\text{HMSC}}$ (Homological mirror symmetry conjecture)

$$D^b(X) \cong DF(X')$$

HMSC Mirror symmetry should be understood as an equivalence of categories, one being the derived category $D^b(X)$ of coherent sheaves and the other one, being the Fukaya category.

$$\boxed{\begin{array}{l} X, X' \text{ CYs are related by mirror symmetry} \\ \quad \text{iff } D^b(X) \cong DF(X') \text{ and } DF(X) \cong D^b(X') \end{array}}$$

→ $D^b(X)$:= Derived category of coherent sheaves on X

One has $Coh(X) \xrightarrow[\text{pass}]{} D^b(X)$ by formally inverting all quasi-isomorphic complexes

→ $DF(X)$:= Fukaya category

Objects: triples (Y, E, ∇) → connection

Y: Lagrangian submanifold

E: E bundle

Morphisms (Floer homology)

$DF(X) \subseteq$ Cat of A-branes.

full subcategory

• CONSEQUENCES OF HMSC

1. Calabi-Yau manifolds that are mirror have equivalent derived categories of coherent sheaves.

2. There is an isomorphisms at the level of cohomology groups of CYs mirror.

In general, $D(X) \cong D(X') \Rightarrow H(X, \mathbb{Q}) \cong H(X', \mathbb{Q})$

3. From the point of view of enumerative geometry, mirror symmetry explains formulae for the number of rational curves on some 3-dimensional CY manifolds. These formulae involve Hodge theory on mirror dual Calabi-Yau manifolds. The problem of counting rational curves in CY manifolds has been embedded in the general context of theory of GW invariants.

THE ELLIPTIC CURVE: A CONCRETE EXAMPLE OF VERIFICATION OF HMSC

E_τ Elliptic curve with Teichmüller parameter τ

It's an algebraic group

e the unity

The ring of e-preserving endomorphisms of E_τ contains $\mathbb{Z}$ as a subring

▷ For a generic elliptic curve $\mathrm{End}(E_\tau) \cong \mathbb{Z}$

▷ For $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ with CM (complex multiplication)

$(\tau \in \mathbb{Q}(\sqrt{-D})$, a quadratic number field$)$

We can identify τ in the upper half plane $\mathbb{H}$,

$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PLS(2,\mathbb{Z})$$

The Kähler class $[\omega] \in H^2(E, \mathbb{C})$

$\int_E \omega = 2\pi i t$, is parametrized by $t \in \mathbb{H}$. Mirror ~~symmetry~~ simmetry

$$E_\tau \longleftrightarrow E_t$$