Tensor product of coherent functors

Vincent Franjou

Laboratoire Jean-Leray, Université de Nantes & CRM

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Joint work with Teimuraz Pirashvili

The project started when T. Pirashvili saw that J. A. Green's 1987 paper *On three functors of M. Auslander's* covered some of the material used in M. Chałupnik's 2005 paper on functor cohomology.
So this is where I'll start.

`I shall not use derived categories.`

On three functors of M. Auslander's

$\mathbb{K}$ is a (finite) field
A and B are (f.d.) $\mathbb{K}$-algebras
A-*Mod* (A-*mod*) is the category of (f.d.) A-*mod*ules
M is an $A - B$-bimodule

Examples

- M is in A-*mod* and $B = \mathrm{End}_A(M)$, e.g.
- $A = \mathbb{K}\mathfrak{S}_d$, $M = V^{\otimes d}$ for a $\mathbb{K}$-vector space V of dimension $\geq d$, $B = \mathrm{End}_{\mathfrak{S}_d}(M)$ is the Schur algebra.

On three functors of M. Auslander's

The bimodule M ties representations:

$$A\text{-}mod^{op} \to B\text{-}Mod$$
$$X \mapsto \mathrm{Hom}_A(X, M) =: \mathrm{h}_X(M)$$

$$A\text{-}mod \to B\text{-}Mod$$
$$X \mapsto X \otimes_A M =: \mathrm{t}_X(M)$$

Or putting both in one exact functor:

$$\mathcal{L}(A\text{-}mod, \mathbb{K}\text{-}Mod) \to B\text{-}Mod$$
$$\mathrm{f} \mapsto \mathrm{f}(M)$$

On three functors of M. Auslander's

The evaluation functor

$$\begin{aligned} j^* : \mathcal{L}(A\text{-}mod, \mathbb{K}\text{-}mod) &\to B\text{-}mod \\ \mathrm{f} &\mapsto \mathrm{f}(M) \end{aligned}$$

admits a right adjoint j_*:

$$(j_* Y)(X) = \mathrm{Nat}(\mathrm{h}_X, j_* Y) = \mathrm{Hom}_B(\mathrm{Hom}_A(X, M), Y).$$

The adjoint j_* is a right inverse of j^* in our example.

On three functors of M. Auslander's

There is also a left adjoint $j_!$, so we get a recollement situation:

$$\begin{tikzcd} kernel \arrow[r] & \mathcal{L}(A\text{-}mod, \mathbb{K}\text{-}mod) \arrow[rr, "j^*"] & & B\text{-}mod \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "j_*"] \end{tikzcd}$$

The three functors of Auslander's are the two adjoints and the *extension intermédiaire* of this recollement, that is the image of the norm: $j_! \to j_*$.

The middle term in the above is quite large. However, the adjoints take values in a smaller class.

Plan of the talk

- *On three functors of M. Auslander's*
- Coherent functors
- Tensor products of coherent functors
- Modules over the Schur algebra as polynomial functors
- Cohomological applications

Coherent functors

Definition

A linear functor f is *coherent* if it is finitely presented, that is if there are X_0 and X_1 in *A-mod* and an exact sequence:

$$\mathrm{h}_{X_1} \to \mathrm{h}_{X_0} \to \mathrm{f} \to 0.$$

For a given group G, we let $\mathcal{C}(G)$ be the category of coherent functors in $\mathcal{L}(A\text{-}mod, \mathbb{K}\text{-}mod)$. It is an abelian category whose projectives are the h_Xs.

Proposition

The adjoints $j_!$ and j_* take values in $\mathcal{C}(G)$.

Extension to coherent functors

Recall that the functors

$$\begin{array}{rr} \mathbb{K}G\text{-}mod^{op} \to B\text{-}mod & \mathbb{K}G\text{-}mod \to B\text{-}mod \\ X \mapsto \mathrm{Hom}_A(X, M) & X \mapsto X \otimes_A M \end{array}$$

extend to an exact functor: $\mathcal{C}(G) \to B$-*mod*.

Proposition

1. *For an additive functor* $T : \mathbb{K}G\text{-}mod \to \mathcal{E}$, *there exists a (unique) left exact functor* $\bar{T} : \mathcal{C}(G) \to \mathcal{E}$ *such that:* $\bar{T}(\mathrm{t}_X) = T(X)$.
2. *For an additive functor* $H : \mathbb{K}G\text{-}mod^{op} \to \mathcal{E}$, *there exists a (unique) right exact functor* $\bar{H} : \mathcal{C}(G) \to \mathcal{E}$ *such that:* $\bar{H}(\mathrm{h}_X) = H(X)$.
3. *If* T *is right exact,* $\bar{T}$ *is exact; if* H *is left exact,* $\bar{H}$ *is exact; if furthermore* $T(P) = H(P^{\#})$ *for all projective* P, *then* $\bar{T} = \bar{H}$.

Tensor products of coherent functors

One similarly extends the external tensor product:
$\mathbb{K}G\text{-}mod \times \mathbb{K}H\text{-}mod \to \mathbb{K}(G \times H)\text{-}mod$.

Proposition

There is a right exact balanced symmetric functor:

$$\underset{\ell}{\boxtimes} : \mathcal{C}(G) \times \mathcal{C}(H) \to \mathcal{C}(G \times H)$$

such that:

$$\mathrm{h}_X \underset{\ell}{\boxtimes} \mathrm{h}_Y = \mathrm{h}_{X \otimes Y} ,$$

and a natural transformation $\mathrm{f} \underset{\ell}{\boxtimes} \mathrm{g} \to \mathrm{f}(\mathrm{g}(\mathrm{Res}_H^{G\times H}))$ *which is an isomorphism if* g *is projective.*

Dually, there is $\overset{r}{\boxtimes}$ and $\mathrm{f}(\mathrm{g}(\mathrm{Res}_H^{G\times H})) \to \mathrm{f} \overset{r}{\boxtimes} \mathrm{g}$ etc.

Polynomial functors [Friedlander& Suslin 1997] [Pirashvili 2003]

For the rest of the talk, let us consider Schur's example. That is, we fix a positive integer d and evaluate a coherent functor in $\mathcal{C}(\mathfrak{S}_d)$ on the d-th tensor $\otimes^d(V) := V^{\otimes d}$ for dim $V \geq d$. To vary from Schur's thesis, $\mathbb{K}$ is a finite field of characteristic p.

We start with the d-th divided power functor of a $\mathbb{K}$-vector space V:

$$\Gamma^d(V) := (V^{\otimes d})^{\mathfrak{S}_d} = \mathrm{h}_{\mathbb{K}}(V^{\otimes d})$$

and consider the category $\Gamma^d(\mathbb{K}\text{-}mod)$ with same objects as $\mathbb{K}$-*mod* and with morphisms

$$\mathrm{Hom}_{\Gamma^d(\mathbb{K}\text{-}mod)}(V, W) := \Gamma^d(\mathrm{Hom}_{\mathbb{K}}(V, W)).$$

Polynomial functors

Definition

An homogeneous polynomial functor of degree d is a $\mathbb{K}$-linear functor $\Gamma^d(\mathbb{K}\text{-}mod) \to \mathbb{K}\text{-}mod$. We let $\mathcal{P}_d = \mathcal{L}(\Gamma^d(\mathbb{K}\text{-}mod), \mathbb{K}\text{-}mod)$ be the category of (natural transformations between) homogeneous polynomial functors of degree d.

That is: the structural map is an homogeneous degree d polynomial.

The functor $P_V := \Gamma^d(\mathrm{Hom}(V, -))$ is a projective generator for dim $V \geq d$. Since

$$\mathrm{End}_{\mathcal{P}_d}(P_V) = \Gamma^d(\mathrm{End}(V)) = \mathrm{End}_{\mathfrak{S}_d}(V^{\otimes d}) = B$$

is the Schur algebra, the category $\mathcal{P}_d$ is indeed equivalent to the category B-*mod* of f. d. degree d polynomial representations of GL_n for $n \geq d$.

Polynomial functors

Examples

- the d-th tensor power functor $\otimes^d$;
- for any f. d. representation M of $\mathfrak{S}_d$, $\mathrm{Hom}_{\mathfrak{S}_d}(M, \otimes^d)$; e.g. Γ^d;
- $d = p^r$: the r-th Frobenius twist $I^{(r)}$, which send V to $V^{(r)}$ - same as V additively but with scalar action given by base change along the r-th power of the Frobenius.

Polynomial functors

Because

$$\mathrm{Hom}_{\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}) = \Gamma^d(\mathrm{Hom}_{\mathbb{K}}(V, W))$$

the d-th tensor power defines a full embedding of $\Gamma^d(\mathbb{K}\text{-}mod)$ in $\mathbb{K}\mathfrak{S}_d\text{-}mod$, whose precomposition defines our recollement with:

$$\begin{tikzcd} kernel \arrow[r] & \mathcal{C}(\mathfrak{S}_d) \arrow[r, "j^*"] & \mathcal{P}_d \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

$$j^*\mathrm{f}: \qquad \Gamma^d(\mathbb{K}\text{-}mod) \xrightarrow{\otimes^d} \mathbb{K}\mathfrak{S}_d\text{-}mod \xrightarrow{\mathrm{f}} \mathbb{K}\text{-}mod$$

Polynomial functors

$$\begin{tikzcd} \mathcal{C}(\mathfrak{S}_d) \arrow[r, "j^*"] & \mathcal{P}_d \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Examples

- The functors $j_!$ and j_* take the same value on projectives (tensor products of divided powers), injectives (tensor products of symmetric powers), or tensor products of exterior powers;
- the s. exact sequence in $\mathcal{P}_p$:

$$0 \to I^{(1)} \to S^p \to \Gamma^p \to I^{(1)} \to 0$$

implies: $j_*(I^{(1)}) = \hat{\mathrm{H}}^{-1}(\mathfrak{S}_p, -)$ and $j_!(I^{(1)}) = \hat{\mathrm{H}}^0(\mathfrak{S}_p, -)$

Polynomial functors

Several features are best seen through functors:

- Duality: $\mathbb{K}$-linear duality induces a duality for coherent functors

$$\mathrm{Df}(X) := \mathrm{f}(X^{\#})^{\#}$$

exchanging h_X and t_X, and similarly for polynomial functors;
the functor j^* respects duality, and $j_!\mathrm{D} = \mathrm{D}j_*$.
- Tensor product taken at the target: $\mathcal{P}_m \times \mathcal{P}_n \to \mathcal{P}_{m+n}$.
- Composition or plethysm: $\mathcal{P}_m \times \mathcal{P}_n \to \mathcal{P}_{mn}$.

Cohomological features:

- Ext in $\mathcal{P}_d$ computes rational cohomology of GL_n for $n \geq d$;
- finite cohomological dimension of $\mathcal{P}_d$ [Donkin 1989];
- Ext injectivity by pre-composition.

Functor cohomology computations

Most computations stem from the following:

Theorem

$E_r := \mathrm{Ext}_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is a divided power algebra on a generator e_0 in degree 2 truncated at height r. Thus, it is concentrated in even degrees up to $2(p^r - 1)$ and one-dimensional in those.

This has been widely extended in [FFSS 1999], which provided the basic computations for Chałupnik's 2005 paper.

Cohomology computations (after Chałupnik)

Chałupnik then proves formulae expressing $\mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(F^{(r)}, G^{(r)})$ in terms of f, g and the graded $\mathfrak{S}_d^{op} \times \mathfrak{S}_d$ permutation representation

$$T_r := \mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(\otimes^{d(r)}, \otimes^{d(r)}) = E_r^{\otimes^d} \otimes \mathbb{K}\mathfrak{S}_d.$$

In many cases, for instance for a projective F, it is given by:

$$\mathrm{Ext}^*_{\mathcal{P}_{dp^r}}(F^{(r)}, G^{(r)}) \cong j_* G((j_* \mathrm{D}F)(T_r)).$$

This is better written as a left exact functor $j_* G \overset{r}{\boxtimes} j_*(\mathrm{D}F)(T_r)$: resolving a general F by projectives gives then rise to a spectral sequence.

Touzé's cohomology computations

Theorem (F-Pirashvili)

There is a natural graded functor $e(F,G)$ *in* $\mathcal{C}(\mathfrak{S}_d^{op} \times \mathfrak{S}_d)$ *such that* $e(F,G)(T_r)$ *is the second page of a spectral sequence converging to* $\mathrm{Ext}_{\mathcal{P}}(F^{(r)}, G^{(r)})$*, for all polynomial functors* F *and* G *of degree* d*.*

Antoine Touzé uses A. Troesch resolutions in $\mathcal{P}$ to derive a more explicit version of the spectral sequence. He has shown in many cases that the spectral sequence collapses.

Note, when this is the case, the amazing formula:

$$\mathrm{Ext}^*_{\mathcal{P}}(F^{(r)}, G^{(r)}) = \mathrm{Hom}_{\mathcal{P}}(F(E_r^*), G).$$

which seem to indicate that in these computations, derived Homs are unnecessary.