Analytic Theory of Numbers

Def An <u>arithmetic function</u> is a function $f: \mathbb{N} \to \mathbb{C}$.

f is <u>multiplicative</u> if whenever $(m,n)=1$, $f(mn)=f(m)\cdot f(n)$.

f is <u>completely multiplicative</u> if $f(mn)=f(m)f(n)$ $\forall m,n \in \mathbb{N}$.

f is <u>additive</u> if whenever $(m,n)=1$, $f(mn)=f(m)+f(n)$

f is <u>completely additive</u> if $f(mn)=f(m)+f(n)$ $\forall m,n \in \mathbb{N}$.

Examples:

- $f(n)=1$ $\forall n$ is completely multiplicative.
- $f(n)=0$ $\forall n$ is completely additive & multiplicative. (however, it will be usually excluded as mult.)
- Unit function: $e(n)=\begin{cases}1 & n=1\\ 0 & n\neq 1\end{cases}$ is completely multiplicative.
- Identity function: $f(n)=n$ is completely multiplicative.
- $\log n$ is completely additive.
- Powers: $f(n):=n^{\alpha}$ $(\alpha\in\mathbb{C})$ is completely multiplicative.
- $d(n)=\tau(n)=\#\{d\in\mathbb{N}: d\mid n\}$.

<u>Lemma 1</u>: Let $(m,n)=1$. Any divisor $d\mid mn$ can be uniquely written as $d=d_1d_2$ where $d_1\mid m$, $d_2\mid n$. (uniqueness fails if $(m,n)>1$).

Pf Let $d_1=(d,m)$, $d_2=(d,n)$.

A divisor d of mn has a prime factorization $d=p_1\cdots p_k\, q_1\cdots q_\ell$, $p_i\mid m$, $q_i\mid n$.

which is unique (up to order) thanks to $(m,n)=1$.

Then $d_1=p_1\cdots p_k$, $d_2=q_1\cdots q_\ell$. And so done. //

Thus, there's a bijection between divisors $d\mid mn$ and pairs (d_1,d_2) with $d_1\mid m, d_2\mid n$.

The number of d is $d(mn)$. The number of pairs is $d(m)d(n)$.

Thanks to the bijection, $d(mn)=d(m)d(n)$. $\to$ $d(n)$ is multiplicative.

(not completely multiplicative: $d(2\cdot2)=d(4)=3$, $d(2)\cdot d(2)=4$).

(note: if $n = p_1^{e_1}\cdots p_k^{e_k}$, $d(n) = (e_1+1)\cdots(e_k+1)$, and so it's easy to see that it's multiplicative).

If f is multiplicative, and $n = p_1^{e_1}\cdots p_k^{e_k}$, then $f(n) = f(p_1^{e_1})\cdots f(p_k^{e_k})$.

Theorem 1.1: f is multiplicative $\iff f(n) = \prod_{p^e \| n} f(p^e) \quad \forall n \in \mathbb{N}$

f is completely mult $\iff f(n) = \prod_{p^e\|n} f(p)^e$

~~Pf~~ //

Def: Add to the definition of multiplicative function: f is not identically zero. (Technical reasons)

Rk: similarly for f additive (resp. comp. additive) $\iff f(n) = \sum_{p^e\|n} f(p^e)$ (resp $\sum_{p^e\|n} e f(p)$).

· Euler ϕ-function: $\phi(n) := \#\{1 \le m \le n : (m,n)=1\}$.

As $\phi(p_1^{e_1}\cdots p_j^{e_j}) = p_1^{e_1-1}(p_1-1)\cdots p_j^{e_j-1}(p_j-1)$, it's multiplicative. (not completely mult).

Another formula: $\phi(n) = n\cdot\prod_{p|n}\left(1-\frac{1}{p}\right)$

An interesting identity: $n = \sum_{d|n}\phi(d)$

~~Pf~~ If $1 \le m \le n$, then $(m,n)=f$ for $f|n$.

Therefore, $n = \sum_{f|n} \#\{1\le m\le n : (m,n)=f\}$

If $(m,n)=f$, then $\left(\frac{m}{f},\frac{n}{f}\right)=1$ and $f|m$, $f|n$.

So $\#\{1\le m\le n : (m,n)=f\} = \#\{1\le m\le n : f|m \text{ and } \left(\frac{m}{f},\frac{n}{f}\right)=1\}$.

$= \#\{1 \le m' \le \frac{n}{f} : (m', \frac{n}{f})=1\} = \phi\left(\frac{n}{f}\right)$ //

Two additive functions:

For $n = p_1^{e_1}\cdots p_j^{e_j}$ define $\omega(n) := j$, $\Omega(n) = e_1+\cdots+e_j$

Thm 1.2:

1) If f, g are mult. (resp. completely mult). so is $f \cdot g$ and $1/g$ (if $g(n) \neq 0$ $\forall n \in \mathbb{N}$).

2) If f, g are additive, so is $\alpha f + \beta g$ $\forall \alpha, \beta \in \mathbb{C}$.

3) If g is additive, and $C > 0$, then C^g is multiplicative.

4) If f is multiplicative and $f(n) > 0$ $\forall n \in \mathbb{N}$, then $\log f$ is additive.

5) All the above hold with mult. replaced with completely mult (= additive $\to$ comp. additive).

The Möbius function $\mu(n)$:

$\mu(n)$: if n is the product of $k \geq 0$ distinct primes, then $\mu(n) = (-1)^k$.

If n is divisible by p^2 for some prime p, then $\mu(n) = 0$.

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1

Lemma: $\mu(n)$ is multiplicative.

Pf/ obvious.

Thm 1.3: For $n \in \mathbb{N}$, $\sum_{d|n} \mu(d) = e(n) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$

Pf/ true when $n = 1$.

Suppose now $n = p_1^{e_1} \cdots p_j^{e_j}$ $(j \geq 1)$.

For $d|n$, $\mu(d) \neq 0 \iff d = p_1^{f_1} \cdots p_j^{f_j}$ $f_i \in \{0,1\}$.

(and then $\mu(d) = (-1)^{f_1 + \cdots + f_j}$).

Thus $$\sum_{d|n} \mu(d) = \sum_{\substack{f_1, \ldots, f_j \\ f_i \in \{0,1\}}} (-1)^{f_1 + \cdots + f_j} = \prod_{i=1}^{j} \overbrace{\left(\sum_{f_i=0}^{1} (-1)^{f_i} \right)}^{=0} = 0. \quad (j \geq 1)$$

//

- Using μ to handle coprimality conditions.

Fact: $(a,b)=1 \iff \sum_{d|(a,b)} \mu(d) = 1$.

So suppose one wants to understand, for a given k

$$\sum_{\substack{n\in S\\(n,k)=1}} f(n) = \sum_{n\in S} f(n) \underbrace{\sum_{d|(n,k)}}_{\substack{d|n\\ \&\\ d|k}} \mu(d) = \sum_{d|k}\mu(d)\sum_{\substack{n\in S\\ d|n}} f(n)$$

easier to deal with.

Example:

$$\phi(n) = \sum_{\substack{1\le m\le n\\(m,n)=1}} 1 = \sum_{m=1}^{n}\sum_{d|(m,n)}\mu(d) = \sum_{d|n}\mu(d)\sum_{\substack{1\le m\le n\\ d|m}} 1 = \sum_{d|n}\mu(d)\frac{n}{d}$$

- Ramanujan's sum: $c_q(n) = \sum_{\substack{1\le a\le q\\(a,q)=1}} e^{2\pi i a n/q}$ (in particular, $c_q(1)$ = sum of primitive q^{th} roots of 1)

$$c_q(1) = \sum_{1\le a\le q} e^{2\pi i a/q}\sum_{d|(a,q)}\mu(d) = \sum_{d|q}\mu(d)\sum_{\substack{1\le a\le q\\ d|a}} e^{2\pi i a/q}$$

Substituting $b=\frac{a}{d}$, the inner sum is: $\sum_{b=1}^{q/d} e^{2\pi i \frac{b}{(q/d)}}$ = sum of the zeros of $z^{q/d}-1$

which is 0 unless $\frac{q}{d}=1$.

So we get $\boxed{c_q(1)=\mu(q)}$.

Liouville function

$\lambda(n) := (-1)^{\Omega(n)}$ is $\mu(n)$ if n is squarefree. It is also completely multiplicative.

Theorem 1.4: For a real $x \geq 1$, $\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1$.

Proof: WLOG, $x \in \mathbb{N}$, write $x = N$.

Let $S(N) = \sum_{n \leq N} e(n) = 1$. By theorem 1.3, ↙ close to what we look for

$$1 = S(N) = \sum_{n \leq N} e(n) = \sum_{n \leq N} \sum_{d \mid n} \mu(d) \overset{\text{interchange sum}}{=} \sum_{d \leq N} \mu(d) \sum_{\substack{n \leq N \\ d \mid n}} 1 = \sum_{d \leq N} \mu(d) \left\lfloor \frac{N}{d} \right\rfloor$$

$$\Big| \underbrace{S(N)}_{=1} - N \sum_{d \leq N} \frac{\mu(d)}{d} \Big| = \Big| \sum_{d \leq N} \mu(d) \left(\left\lfloor \frac{N}{d} \right\rfloor - \frac{N}{d} \right) \Big| \leq N - 1$$

Dividing by N and using the triangle inequality, the result follows. //

In fact, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$, but we can't prove it now. It's equivalent to the prime number theorem.

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \iff \text{PNT} \iff \lim_{x \to \infty} \left\{ \frac{1}{x} \sum_{n \leq x} \mu(n) \right\} = 0.$$

• von Mangoldt function

$$\Lambda(n) := \begin{cases} \log(p) & \text{if } n = p^e \ (p \text{ is prime}, e \geq 1) \\ 0 & \text{o.w.} \end{cases}$$

e.g. $\Lambda(4) = \log 2$, $\Lambda(5) = \log 5$, $\Lambda(6) = 0$

We will see later that $\sum_{n \leq x} \Lambda(n) \sim x \iff$ PNT.

Theorem 1.6 ← use numbering system from Hildebrand's notes:

$$\sum_{d \mid n} \Lambda(d) = \log n$$

Pf: $\text{LHS} = \sum_{p^e \mid n} \log p = \sum_{p^e \| n} e \log p = \log n$. //

Dirichlet Convolution

Given two arithmetic function f, g, define:

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right) \qquad \text{(Dirichlet convolution)}$$

e.g. $(f*g)(1) = f(1)g(1)$

$(f*g)(p) = f(1)g(p) + f(p)g(1) \quad (p \text{ prime}).$

<u>Some examples:</u>

1) $\sum_{d|n} \phi(d) = n.$ The LHS is $\phi * 1$. So $\phi * 1 = id$

2) $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} \longrightarrow \phi = \mu * id$

3) $\sum_{d|n} \mu(d) = e(n) \longrightarrow \mu * 1 = e$

4) $\sum_{d|n} \Lambda(d) = \log n \longrightarrow \Lambda * 1 = \log$

<u>Others:</u>

5) $d = 1 * 1 \quad (d(n) = \#\text{ divisors of } n).$

6) $\sigma = 1 * id$

<u>Theorem 1.8</u>

i) e is a unit: $\forall$ arithmetic function f, $f * e = e * f = f$.

ii) $f * g = g * f$

iii) $f * (g * h) = (f * g) * h$

iv) <u>inverses</u>: if $f(1) \neq 0$, $\exists! f^{-1}$ so that $f * f^{-1} = e$

Pf (i), (ii) obvious. For (iii), both sides equal $\sum_{cde=n} f(c) g(d) h(e)$, so done.

(iv) Note that if $f(1) = 0$, there cannot exist an inverse, as $(f*g)(1) = f(1)g(1)$.

~~Suppose $f \cdot g = e$~~

d

(iv) We will construct g, by induction.

$f(1)g(1)=1$, then set $g(1)=\frac{1}{f(1)}$.

Suppose $g(1),\dots,g(k)$ have been uniquely constructed. Then,

$$(f*g)(k+1)=0 \Leftrightarrow \sum_{d|k+1} f(k+1)\,g\left(\frac{k+1}{d}\right) = f(1)g(k+1) + \sum_{\substack{d|k+1\\ d>1}} f(k+1)\,g\left(\frac{k+1}{d}\right)$$

So we need to define $g(k+1) \equiv \dfrac{-\sum(\cdot)}{f(1)}$ (unique)

Examples:

1) $\mu * 1 = e$, so $\mu^{-1}=1$.

2) $d=1*1 \Rightarrow \mu * d = \mu * 1 * 1 = e * 1 = 1$, so:

$$\sum_{c|n} d(c)\mu\left(\frac{n}{c}\right)=1.$$

3) $\Lambda * 1 = \log \Rightarrow \Lambda * 1 * \mu = \mu * \log \Rightarrow \Lambda = \mu * \log$.

Theorem 1.9 (Möbius inversion formula).

If $f=g*1$, then $g=f*\mu$.

Pf $f=g*1 \Rightarrow f*\mu = g*1*\mu = g*e = g$

Note: if f is multiplicative and $f(n)\neq 0$, then $f(n\cdot 1)=f(n)\cdot f(1) \Rightarrow f(1)=1\neq 0$

Hence any multiplicative function has a Dirichlet inverse.

Theorem 1.10:

i) f,g multiplicative $\Rightarrow f*g$ is multiplicative.

ii) f mult $\Rightarrow f^{-1}$ is multiplicative.

iii) if $h=f*g$ and h and f are multiplicative, so is g.

iv) if f and g are mult, and h is completely multiplicative, then $h(f*g)=hf*hg$.

Pf Let $(m,n)=1$. Then $f*g = \sum_{d|mn} f(d)\, g\left(\frac{mn}{d}\right)$.

i) Write $d=d_1 d_2$, $(d_1|m,\ d_2|n)$. Then $f*g = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1 d_2)\, g\left(\frac{mn}{d_1 d_2}\right)$

Now $(d_1, d_2)=1$ and $\left(\frac{m}{d_1}, \frac{n}{d_2}\right)=1 \Rightarrow$

$\Rightarrow \sum_{\substack{d_1|m \\ d_2|n}} f(d_1) f(d_2)\, g\left(\frac{m}{d_1}\right) g\left(\frac{n}{d_2}\right) = \sum_{d_1|m} f(d_1)\, g\left(\frac{m}{d_1}\right) \sum_{d_2|n} f(d_2) g\left(\frac{n}{d_2}\right) = (f*g)(m)(f*g)(n)$ //

ii) induction on mn : $f^{-1}(mn) = f^{-1}(m)\, f^{-1}(n)$. $(*)$

$mn=1 \Rightarrow m=n=1$. $1=(f*f^{-1})(1) = f(1)\cdot f^{-1}(1) \Rightarrow f^{-1}(1)=1$.

Assume $(*)$ for $mn<k$. Take $mn=k$, $(m,n)=1$.

From the proof of Thm 1.18, $f^{-1}(mn) = -\sum_{\substack{d|mn \\ d>1}} f(d)\, f^{-1}\left(\frac{mn}{d}\right) = -\sum_{\substack{d_1 d_2>1 \\ d_1|m \\ d_2|n}} f(d_1 d_2) f^{-1}\left(\frac{m}{d_1}\right) f^{-1}\left(\frac{n}{d_2}\right)$ //

iii) immediate from (i) + (ii) //

iv) $(hf * hg)(n) = \sum_{d|n} h(d) f(d)\, h\left(\frac{n}{d}\right) f\left(\frac{n}{d}\right) = h(n)\cdot \sum_{d|n} f(d) f\left(\frac{n}{d}\right)$ //

<u>Note</u>:

Part (i) on the theorem is false if "mult" is replaced by "completely mult".

(e.g: $d = 1*1$).

<u>Example</u> (a) Find a "simple" formula for $f = 1*\mu^2$.

f is multiplicative, so look at prime powers:

$$f(p^e) = \sum_{d|p^e} \mu^2(d) = \sum_{i=0}^{e} \mu^2(p^i) = \mu^2(1) + \mu^2(p) = 2.$$

So $f(n) = 2^{\omega(n)}$.

↓

Example

b) Identify $g = 1 * \frac{\mu^2}{\phi}$ ← it's multiplicative!)

$$g(p) = \frac{\mu^2(1)}{\phi(1)} + \frac{\mu^2(p)}{\phi(p)} = 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

$$g(p^e) = \sum_{i=0}^{e} \frac{\mu^2(p^i)}{\phi(p^i)} = \sum_{i=0}^{k} (\ ) = \frac{p}{p-1}$$

So if $n = p_1^{e_1} \cdots p_k^{e_k}$, $g(n) = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} = \frac{n}{\phi(n)}$.

c) Solve for g: $\lambda = \mu * g$.

As $\mu^{-1} = 1$, $\lambda * 1 = g$. So $g(p) = \lambda(1) + \lambda(p) = 0$

$g(p^2) = 1$, $g(p^3) = 0$ ..

So $g(p^k) = \begin{cases} 0 & k \text{ odd} \\ 1 & k \text{ even} \end{cases}$

Hence $g(n) = \begin{cases} 0 & \text{if } n \text{ is not a square} \\ 1 & \text{if } n \text{ is a square} \end{cases} = \chi_{\text{squares}}$.

So $\boxed{\lambda(n)} = \sum_{d \mid n} g(d)\mu\left(\frac{n}{d}\right) = \boxed{\sum_{e^2 \mid n} \mu\left(\frac{n}{e^2}\right)}$

d) $\sigma_k(n) = \sum_{d \mid n} d^k = 1 * id^k \Rightarrow \sigma_k$ is multiplicative. $(k \in \mathbb{R})$.

Theorem 1.11: If f is completely multiplicative, then $f^{-1}(n) = \mu(n) f(n)$.

Pf/ Let $g(n) = \mu(n) f(n)$. Then $(g * f)(n) = \sum_{d \mid n} g(n) f\left(\frac{n}{d}\right) = \sum_{d \mid n} \mu(d) \underbrace{f(d) f\left(\frac{n}{d}\right)}_{f(n)} = f(n) \underbrace{\sum_{d \mid n} \mu(d)}_{e(n)} = e(n)$. //

Theorem 1.12: If f is multiplicative,

$$\sum_{d \mid n} f(d) = \prod_{p^e \| n} \left(1 + f(p) + \cdots + f(p^e)\right)$$

Theorem 1.13: Let $G(x) = \sum_{n \leq x} F\left(\frac{x}{n}\right)$ $\quad (x \in \mathbb{R}_{\geq 1})$.

Then $F(x) = \sum_{k \leq x} \mu(k)\, G\left(\frac{x}{k}\right)$ $\quad (n, k \in \mathbb{N})$.

Pf

$$\sum_{k \leq x} \mu(k)\, G\left(\frac{x}{k}\right) = \sum_{k \leq x} \mu(k) \sum_{n \leq \frac{x}{k}} F\left(\frac{x}{kn}\right) = \sum_{kn \leq x} \mu(k)\, F\left(\frac{x}{kn}\right).$$

Let $kn = d$, then becomes $\displaystyle\sum_{d \leq x} F\left(\frac{x}{d}\right) \underbrace{\sum_{kn=d} \mu(k)}_{\text{" } 0 \text{ unless } d=1} = F(x)$. //

Example: $F(x) = 1$, $G(x) = \lfloor x \rfloor$.

$\therefore$ $1 = \sum_{k \leq x} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor$ (already knew that).

II. Sums of arithmetic functions

Notation: (Big-O, little-o):

- $f(x) = O(g(x))$ means that $\exists C$ s.t. $|f(x)| \leq C g(x)$ for all x in an "appropriate range" (will have to be specified.)

  (so $f(x)$ can be complex-valued, but $g(x)$ will be nonnegative).

- $f(x) = o(g(x))$ $(x \to \infty)$ means that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

- $f(x) \sim g(x)$ $(x \to \infty)$ ← asymptotic to, if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

Examples:

- $3x^2 + 19x + 731 = O(x^2)$ $(x \geq 1)$ (take $C = 753$).
- $\sin(x) = O(1)$ $(x \in \mathbb{R})$ and $\sin(x) = O(x)$ $(x \geq 0)$ ← good when $x \to 0$.
- $x^2 + \sin x = O(x^2)$ $(x \geq 1)$.
- $x^{10} = O(e^x)$ $(x \geq 0)$. In fact, $x^{10} = o(e^x)$.
- $3x^2 + 19x + 731 \sim 3x^2$.
- $\frac{1}{x+7} = O\left(\frac{1}{x}\right)$ $(x > 0)$.
- $\frac{1}{x} = O\left(\frac{1}{x^2}\right)$ $(0 < x \leq 1)$, $\frac{1}{x^2} = O\left(\frac{1}{x}\right)$ $(x \geq 1)$.

More notation:

$f(x) \ll g(x)$ means the same as $f(x) = O(g(x))$.

$f(x) \asymp g(x)$ means $[f(x) \ll g(x)$ & $g(x) \ll f(x)]$.

Examples:

- $3x^2 + 119 \asymp x^2 \quad (x \geq 1)$.
- $\frac{1}{x+7} = \frac{1}{x} + O\left(\frac{1}{x^2}\right) \quad (x \geq 1)$: $\left|\frac{1}{x+7} - \frac{1}{x}\right| = \left|\frac{-7}{x(x+7)}\right| \leq \frac{7}{x^2}$.
- $\log(1+x) = O(|x|) \quad (|x| \leq 1/2)$.
- $5x^3 + x^4 = o(x) \quad (x \to 0)$

True or false?

$|\sin x| \asymp 1 \quad (x \in \mathbb{R})$ (false)

$2 + \sin x \asymp 1 \quad (x \in \mathbb{R})$ (true)

- Dependence on a parameter

Ex: $\log x = O_\varepsilon(x^\varepsilon) \quad (x \geq 1)$ for every $\varepsilon > 0$.
this means that the constant depends on ε.

Ex: $ax^2 + 15x = O_a(x^2) \quad (x \geq 1)$.

(but $ax^2 + 15x = O(ax^2) \quad (x \geq 1, a \geq 1)$.)

Fact: $f(x) = o(g(x)) \quad (x \to \infty) \Rightarrow \exists x_0$ $f(x) = O(g(x)) \quad (x \geq x_0)$

- Taylor series expansion of $f(x)$:

$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ spq. it converges for $|x-c| < r$.

Then, for any $R < r$, $k \geq 1$,

$$f(x) = a_0 + a_1(x-c) + \cdots + a_k(x-c)^k + O_{R,k}((x-c)^{k+1}), \quad (|x-c| < R).$$

(e.g. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O_c(x^4)$ for $|x| < c < 1$.)

<u>Lemma</u>: If $f(y) = O(g(y))$ for $y \geq a$, then

$$F(x) = \int_a^x f(y)\,dy = O\left(\int_a^x g(y)\,dy\right).$$

Pf $\frac{|f(y)|}{g(y)} \leq C$ for some C, for some $y \geq a$. Then $|F(x)| \leq \int_a^x |f(y)|\,dy \leq$

$\leq \int_a^x C g(y)\,dy = C \int_a^x g(y)\,dy.$ //

This is not true with little-o!! ← exercise.

- If $f(x) = o(g(x))$ and $\boxed{g(x) \to \infty}$ $(x \to \infty)$, then $e^{f(x)} = o(e^{g(x)})$.
  (consider $f(x) = 0$, $g(x) = 1$!!).
- If $f(x) = g(x) + O(1)$, then $e^{f(x)} \asymp e^{g(x)}$ can be reversed.
- If $f(x) = g(x) + o(1)$, then $e^{f(x)} \sim e^{g(x)}$

Place the following functions in increasing order as $x \to \infty$
(i.e. put $f(x)$ before $g(x)$ if $f(x) = o(g(x))$)

$f = x^{1/100}$, $g = (\log x)^{10}$, $h = e^{\sqrt{\log x}}$, $k = e^{e^{\sqrt{\log\log x}}}$

($\to g = o(f)$.)

Take logs: get $\frac{1}{100} Y$, $10 \log Y$, $\sqrt{Y}$, $\exp(\sqrt{\log Y})$

Then have $g < h < f$.

"Take another log at it": get $c + Z$, $10 \log Z$, $\frac{1}{2} Z$, $\sqrt{Z}$

So $g < k < h < f$.

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Theorem 2.1: For each $k \in \mathbb{N}$,

$$li(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x}\left(\sum_{i=0}^{k-1} \frac{i!}{(\log x)^i} + O_k\left(\frac{1}{(\log x)^k}\right)\right) \qquad \text{(Asymptotic expansion)}$$

Pf/ exercise. (Note: $\sum i!\, z^i$ diverges!!)

Summations

Theorem 2.17: $\sum_p \frac{1}{p}$ diverges (Euler).

Proof:
Let $N \in \mathbb{N}$, and let $f(n) = \begin{cases} \frac{1}{n} & \text{if all prime factors of } n \text{ are } \leq N. \\ 0 & \text{otherwise.} \end{cases}$

Note that f is multiplicative.

$f(a) = \frac{1}{a}$ for $n \leq N$. So $\sum_{n \leq N} \frac{1}{n} \leq \sum_{n=1}^{\infty} f(n) = \prod_p \left(1 + f(p) + f(p^2) + \cdots\right)$

But $\prod_p \left(1 + f(p) + \cdots\right) = \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \leq N}\left(\frac{1}{1 - 1/p}\right) = \prod_{p \leq N}\left(1 + \frac{1}{p-1}\right)$.

Taking logs, $\log\left(\sum_{n \leq N} \frac{1}{n}\right) \leq \sum_{p \leq N}\left(-\log\left(1 - \frac{1}{p}\right)\right) \underset{\text{taylor}}{=} \sum_{p \leq N}\left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \cdots\right)$

$$= \sum_{p \leq N} \frac{1}{p} + \sum_{p \leq N} \sum_{k \geq 2} \frac{1}{kp^k}$$

Note that $\frac{1}{2p^2} + \frac{1}{3p^3} + \cdots \leq \frac{1}{2p^2} + \frac{1}{2p^3} + \cdots = \frac{1}{2p(p-1)}$.

So $\log\left(\sum_{n \leq N} \frac{1}{n}\right) \leq \sum_{p \leq N} \frac{1}{p} + \underbrace{\sum_{p} \frac{1}{2p(p-1)}}_{\text{if } p \text{ replaced with } n\text{, it's a telescopic sum}} \leq \sum_{p \leq N} \frac{1}{p} + \frac{1}{2}$.

- Sums of smooth functions

Theorem 2.18: If f is decreasing on (m,∞) $(m\in\mathbb{Z})$.

Then,
$$\int_{m+1}^{n+1} f(t)\,dt \leq \sum_{j=m+1}^{n} f(j) \leq \int_m^n f(t)\,dt$$

Pf/ For $j \geq m+1$, clearly $\int_j^{j+1} f(t)\,dt \leq f(j) \leq \int_{j-1}^{j} f(t)\,dt$ //

Example: $f(n)=\frac{1}{n}$. Then $\int_1^{N+1}\frac{dt}{t} \leq \sum_{n=1}^{N}\frac{1}{n} \leq \int_0^N \frac{dt}{t}$

So $\log(N+1) \leq \sum_{n=1}^{\infty}\frac{1}{n} = 1+\sum_{n\geq 2}\frac{1}{n} \leq 1+\int_1^N \frac{dt}{t} = 1+\log N$

$\quad\log(N+1) \geq \log N$.

So we get $(\log N) \leq \sum_{n=1}^{N}\frac{1}{n} \leq (\log N)+1$

Can write that $\sum_{n=1}^{N}\frac{1}{n} = \log N + O(1)$.

So $\sum_{p\leq N}\frac{1}{p} \geq \log\log(N+1) - \frac{1}{2}$ $\quad(N\geq 2)$.

Later we will see that this is actually the ~~...~~ rate of growth of this sum.

Example: $f(n)=\frac{1}{n^2}$. Estimate for $\sum_{n>x}\frac{1}{n^2}$ $(x\geq 1)$.

$\sum_{n>x}\frac{1}{n^2} = \sum_{n=\lfloor x\rfloor+1}^{\infty}\frac{1}{n^2}$. Let $m := \lfloor x \rfloor$. Then

get $\int_{\lfloor x\rfloor+1}^{\infty}\frac{dt}{t^2} \leq \sum_{n\geq x}\frac{1}{n^2} \leq \int_{\lfloor x\rfloor}^{\infty}\frac{dt}{t^2} \Rightarrow \frac{1}{\lfloor x\rfloor+1} \leq \sum\frac{1}{n^2} \leq \frac{1}{\lfloor x\rfloor}$

So $\sum_{n>x}\frac{1}{n^2} = \frac{1}{\lfloor x\rfloor + O(1)} = \frac{1}{x+O(1)} = \frac{1}{x}+O\left(\frac{1}{x^2}\right)$

$$\frac{1}{x+O(1)} = \frac{1}{x}\cdot\underbrace{\frac{1}{1+O\left(\frac{1}{x}\right)}}_{\text{Taylor: } 1+O\left(\frac{1}{x}\right)}$$

check

i.e. $\exists x_0$ s.t. $x \leq x_0 \Rightarrow \sum_{n>x}\frac{1}{n^2} = O(1) = \frac{1}{x}+O\left(\frac{1}{x^2}\right)$ //

<u>Some notation</u>: $\{x\} := x - \lfloor x \rfloor$ is called the <u>fractional part</u> of x. (eg. $\{-2.9\} = 0.1$)

<u>Theorem 2.2</u> (Euler-Maclaurin summation formula) (simple version):
If f has continuous first derivative ($\mathcal{C}^1([y,x])$), then:

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \underbrace{\int_y^x \{t\}\cdot f'(t)\,dt}_{\text{if } f \text{ is very "flat", this is small.}} - \{x\}f(x) + \{y\}f(y).$$

endpoints

Pf/ Suppose $n \in \mathbb{Z}$.

$$\int_{n-1}^n f(t)\,dt = f(n) + \int_{n-1}^n (f(t) - f(n))\,dt = f(n) - \int_{n-1}^n \left(\int_t^n f'(u)\,du\right) dt =$$

$$= f(n) - \int_{n-1}^n f'(u)\cdot(u-(n-1))\,du = f(n) - \int_{n-1}^n f'(u)\cdot\{u\}\cdot du$$

So $f(n) = \int_{n-1}^n f(t)\,dt + \int_{n-1}^n f'(t)\{t\}\,dt$. Summing from $\lfloor y \rfloor + 1$ to $\lfloor x \rfloor$,

get $$\sum_{y<n\le x} f(n) = \int_{\lfloor y\rfloor+1}^{\lfloor x \rfloor} f(t)\,dt + \int_{\lfloor y\rfloor+1}^{\lfloor x \rfloor} f'(t)\cdot\{t\}\cdot dt \qquad (1).$$

By integration by parts, $\int_{\lfloor y \rfloor}^y f'(u)\{u\}\,du = f(y)\cdot\{y\} - f(\lfloor y \rfloor)\overset{0}{\cancel{\{\lfloor y \rfloor\}}} - \int_{\lfloor y \rfloor}^y f(t)\,dt$

Similarly, $\int_{\lfloor x \rfloor}^x f'(u)\{u\}\,du = f(x)\{x\} - \int_{\lfloor x \rfloor}^x f(t)\,dt$.

Combining it with (1), get the theorem. //

Theorem 2.4. (Harmonic Series): For real $x \geq 1$,

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right) \qquad \gamma = 0.5772.. \text{ (Euler-Mascheroni constant)}$$

Pf Apply thm 2.2 with $y=1$:

$$\sum_{n \leq x} \frac{1}{n} = 1 + \sum_{1 < n \leq x} \frac{1}{n} = 1 + \int_1^x \frac{dt}{t} \cancel{+} - \int_1^x \frac{\{t\}}{t^2} dt - \{x\} \cdot \frac{1}{x} + 0.$$

$$= 1 + \log x - \int_1^x \frac{\{t\}}{t^2} dt + O\left(\frac{1}{x}\right)$$

Let $I(x) = \int_1^x \frac{\{t\}}{t^2} dt$. Note that $\lim_{x \to \infty} I(x) =: I$ exists $\left(\left|\frac{\{t\}}{t^2}\right| < \frac{1}{t^2} \right)$

Therefore, $I(x) = I - \int_x^\infty \frac{\{t\}}{t^2} dt = I + \int_x^\infty \frac{O(1)}{t^2} dt = I + O\left(\int_x^\infty \frac{dt}{t^2}\right) = I + O\left(\frac{1}{x}\right)$

Hence, $\sum_{n \leq x} \frac{1}{n} = \log x + 1 - I + O\left(\frac{1}{x}\right)$. Rk: $\gamma := 1 - I = 0.5772\ldots$ (Euler Mascheroni ct.)

Note: $\gamma = 1 - \sum_{n=1}^{\infty} \left(\log\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} \right)$

Note: The implied constant can be taken to be 2 (from the proof).

(eg. $\sum_{n \leq 10^{100}} \frac{1}{n} = \log 10^{100} + \gamma + \left(\text{extremely small, at most } \frac{2}{10^{100}}\right)$).

Theorem 2.5: $\sum_{n=1}^{N} \log n = N(\log N - 1) + \frac{1}{2} \log N + C + O\left(\frac{1}{N}\right)$ for some ct. c. $(N \in \mathbb{N})$

Pf Use thm 2.2 with $x = N$, $y = 1$:

$$\sum_{n=1}^{N} \log n = \int_1^N \log t \, dt + \int_1^N \frac{\{t\}}{t} dt + 0 + 0 = N \log N - N + 1.$$

We will show that $\int_1^N \frac{\{t\}}{t} dt = \frac{1}{2} \log N + c' + O\left(\frac{1}{N}\right)$

1) $\int_1^N \frac{\{t\}}{t}\,dt = \frac{1}{2}\log N + \int_1^N \frac{\{t\}-\frac{1}{2}}{t}\,dt$

(2) Integrate by parts: $\int_1^N \frac{\{t\}-\frac{1}{2}}{t}\,dt = \frac{R(t)}{t}\Big]_1^N + \int_1^N \frac{R(t)}{t^2}\,dt$, $R(t) = \int_1^t (\{u\}-\frac{1}{2})\,du$

Since $\int_n^{n+1}(\{u\}-\frac{1}{2})\,du = 0$, then $|R(t)| = O(1)$.

So $S = \int_1^\infty \frac{R(t)}{t^2}\,dt$ exists, and the tail $\int_N^\infty \frac{R(t)}{t^2}\,dt = O(\frac{1}{N})$

~~Also~~, $\int_1^N \frac{\{t\}}{t}\,dt = \frac{1}{2}\log N + S + O(\frac{1}{N})$ //

Fact: $c = \frac{1}{2}\log(2\pi)$ (not proven ..)

o **Euler-Maclaurin using Riemann-Stieltjes integrals**: (see handout).

$$\sum_{y\le n\le x} f(n) = \int_y^x f(t)\,d\lfloor t\rfloor = \int_y^x f(t)\,dt - \int_y^x f(t)\,d\{t\} \underset{\text{int. by parts}}{=} \int_y^x f(t)\,dt + \int_y^x f'(t)\{t\}\,dt + f(x)\{x\} - f(y)\{y\}$$ //

Corollary (Stirling's formula):

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,\left(1+O\left(\frac{1}{n}\right)\right)$$

Pf/ $\log(n!) = \sum_{m=1}^n \log m = n(\log n - 1) + \frac{1}{2}\log n + c + O(\frac{1}{n})$.

Exponentiate, get $n! = \left(\frac{n}{e}\right)^n \sqrt{n}\, e^{c+O(\frac{1}{n})}$.

If we know that $c = \frac{1}{2}\log(2\pi)$, then $n! = \left(\frac{n}{e}\right)^n \sqrt{n}\sqrt{2\pi}\, e^{O(1/n)}$.

Now, $e^{O(\frac{1}{n})} = 1 + O(\frac{1}{n})$.

(since $e^y = 1 + O_c(|y|)$ if $|y|$ is bounded by c $\forall c>0$) //

Corollary 2.7: For real $x \geq 2$, $\sum_{n\le x}\log n = x(\log x - 1) + O(\log x)$

Pf/ Apply thm 2.6 with $N = \lfloor x\rfloor$.

$$\sum_{n\le x}\log x = \lfloor x\rfloor(\log\lfloor x\rfloor - 1) + O(\log\lfloor x\rfloor) = (x+O(1))(\log(x+O(1)) - 1) + O(\log x)$$ //

(Note: we used here that $\log(x+O(1)) = \log\left(x\left(1+O\left(\frac{1}{x}\right)\right)\right) = \log x + \log\left(1+O\left(\frac{1}{x}\right)\right) = \log x + O\left(\frac{1}{x}\right)$).

Theorem 2.19 (Euler-Maclaurin summation formula, full version).

Let $k \geq 0$, suppose f has $k+1$ continuous derivatives on (y,x). $(k, y \in \mathbb{Z})$

Then
$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \sum_{r=0}^{k} \frac{(-1)^{r+1} B_{r+1}}{(r+1)!}\left(f^{(r)}(x) - f^{(r)}(y)\right) + \frac{(-1)^{k+1}}{(k+1)!}\int_y^x B_{r+1}(\{t\}) f^{(k+1)}(t)\,dt$$

where $B_j(t)$ are the Bernoulli polynomials, and $B_j = B_j(0)$ are the Bernoulli numbers.

Defined by: $B_1(x) = x - \frac{1}{2}$, $B_n'(x) = nB_{n-1}(x)$ s.t. $\int_0^1 B_n(x)\,dx = 0$ $(n \geq 2)$.

(ex: $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$).

Example, $k=2$, $f(n) = \frac{1}{n}$, $y=1$:

$$\sum_{n\leq x} \frac{1}{n} = 1 + \int_1^x \frac{dt}{t} + \left(\frac{-1}{2}\right)\left(\frac{1}{x} - \frac{1}{1}\right) + \frac{1}{6\cdot 2}\left(\frac{-1}{x} - \frac{-1}{1^2}\right) + 0 - \frac{1}{3!}\int_y^x B_3(\{t\})\cdot\frac{2}{t^3}\,dt$$

$$= \log x + \left(1 - \frac{1}{2} + \frac{1}{2}\right) + \frac{1}{2x} - \frac{1}{12x^2} + C + \frac{1}{6}\underbrace{\int_x^\infty \frac{2B_3(\{t\})}{t^3}\,dt}_{O\left(\frac{1}{x^2}\right)}$$

we get $\log x + \overset{\gamma}{\overset{\|}{C'}} + \frac{1}{2x} + O\left(\frac{1}{x^2}\right)$ if x is an integer!

Dirichlet's hyperbola method.

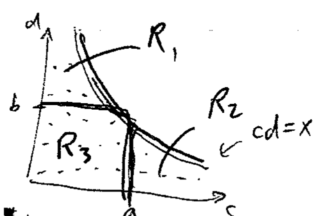
Theorem 2.20: Let f, g be arithmetic functions. Define $F(x) = \sum_{n\leq x} f(n)$, $G(x) = \sum_{n\leq x} g(n)$.

If $a>0$, $b>0$, $ab = x$, then:

$$\sum_{n\leq x}(f*g)(n) = \sum_{n\leq a} f(n)G\left(\frac{x}{n}\right) + \sum_{n\leq b} g(n)F\left(\frac{x}{n}\right) - F(a)G(b)$$

Pf

$$\sum_{n\leq x} f*g(n) = \sum_{n\leq x}\sum_{cd=n} f(c)g(d) = \sum_{cd\leq x} f(c)g(d) =$$



$$= \sum_{(c,d)\in R_1\cup R_3} + \sum_{(c,d)\in R_2\cup R_3} - \sum_{R_3} = \sum_{c\leq a} f(c)G\left(\frac{x}{c}\right) + \sum_{d\leq b} g(d)F\left(\frac{x}{d}\right) - F(a)G(b).$$

Example: estimating $\sum_{n \leq x} d(n)$. As $d = 1 * 1$:

Naive: $\sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d \mid n} 1 = \sum_{d \leq x} 1 \cdot \sum_{n \leq \frac{x}{d}} 1 = \sum_{d \leq x} \lfloor \frac{x}{d} \rfloor = \sum_{d \leq x} \left(\frac{x}{d} + O(1)\right) = x \overbrace{\sum_{d \leq x} \frac{1}{d}}^{x(\log x + \gamma + O(\frac{1}{x})) + O(1)} + O(x)$

$= x \log x + O(x)$.

Hyperbola Method: $a = b = \sqrt{x}$, $d = 1 * 1$:

$F(x) = G(x) = \lfloor x \rfloor$

By thm 2.20, $\sum_{n \leq x} d(n) = \sum_{n \leq \sqrt{x}} \lfloor \frac{x}{n} \rfloor + \sum_{n \leq \sqrt{x}} \lfloor \frac{x}{n} \rfloor - (\lfloor \sqrt{x} \rfloor)^2 =$

$= 2 \sum_{n \leq \sqrt{x}} \left(\frac{x}{n} + O(1)\right) - \left(\sqrt{x} + O(1)\right)^2 = 2x \sum_{n \leq \sqrt{x}} \frac{1}{n} - x + O(\sqrt{x}) =$

$= 2x \left(\log \sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\right) - x + O(\sqrt{x}) = x \log x + x(2\gamma - 1) + O(\sqrt{x})$

a lot better

The Convolution Method.

Goal: estimate $\sum_{n \leq x} f(n)$.

idea: Write $f = f_0 * g$, where f_0 is "close to f", and for which we have good estimates (for $\sum_{n \leq x} f_0(x)$).

g is "close to $e(n)$".

Focus on prime values to make $f_0(p)$ close to $f(p)$.

Then $\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{d \mid n} f_0(d) g\left(\frac{n}{d}\right) \left(= \sum_{d \leq x} f_0(d) \sum_{m \leq \frac{x}{d}} g(m)\right) = \sum_{m \leq x} g(m) \sum_{d \leq \frac{x}{m}} f(d)$

Good Choices:

$f(n)$	$\phi(n)$	$\sigma(n)$	$\mu^2(n)$	$2^{\omega(n)}$
$f_0(n)$	n	n	1	$d(n)$

$\Leftarrow$ chosen s.t. the values at p are very similar.

<u>Theorem 2.9</u>: $\displaystyle\sum_{n \le x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$

Pf/ Write $\phi = id * g \to g = \mu \quad (\phi = id * \mu)$

Also, $\displaystyle\sum_{n \le y} n = \frac{\lfloor y \rfloor (\lfloor y \rfloor + 1)}{2} = \frac{(y + O(1))(y + O(1))}{2} = \frac{y^2}{2} + O(y)$

Thus, $\displaystyle\sum_{n \le x} \phi(n) = \sum_{m \le x} \mu(m) \cdot \sum_{d \le \frac{x}{m}} d = \sum_{m \le x} \mu(m) \cdot \left(\left(\frac{x}{m}\right)^2 \cdot \frac{1}{2} + O\left(\frac{x}{m}\right)\right) =$

$$= \frac{x^2}{2} \sum_{m \le x} \mu(m)/m^2 + \sum_{m \le x} O\left(\frac{x}{m}\right)$$

The total of the error terms is $\displaystyle O\Big(x \sum_{m \le x} \frac{1}{m}\Big) = O\big(x \cdot (\log x + O(1))\big) = O(x \log x)$

<u>Note</u>: to do this, we have to know that $O\left(\frac{x}{m}\right)$ has the same constant, independent of m!

The sum $\displaystyle\sum \frac{\mu(m)}{m^2}$ converges absolutely, so we can write

$$\sum_{m \le x} \frac{\mu(m)}{m^2} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} - \underbrace{\sum_{m > x} \frac{\mu(m)}{m^2}}_{T(x)}$$

$$|T(x)| \le \sum_{m > x} \frac{1}{m^2} = \int_x^\infty \frac{dt}{t^2} - \int_x^\infty \{t\} \frac{2}{t^3} + \frac{\{x\}}{x^2} = \frac{1}{x} + O\left(\int_x^\infty \frac{1}{t^3} dt\right) + O\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{x} + O\left(\frac{1}{x^2}\right).$$

So $T(x) = O\left(\frac{1}{x}\right)$, and $\displaystyle\sum \phi(n) = \frac{x^2}{2}\left(c + O\left(\frac{1}{x}\right)\right) + O(x \log x)$

where $\displaystyle c = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2}$

$$= \frac{x^2 c}{2} + O(x \log x)$$

By exercise 5, can write $\displaystyle c = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^2} + 0 + 0 + 0 \cdots\right) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^2}\right)$

Then $\displaystyle c^{-1} = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \underset{\substack{\uparrow \\ \text{Euler}}}{=} \frac{\pi^2}{6}$, so $c = \frac{6}{\pi^2}$

Corollary A: Let $F(Q)$ be the number of Farey fractions of order Q,
i.e. fractions $\frac{a}{q}$ with $(a,q)=1$ $0 \leq a \leq q \leq Q$

$$F(Q) \overset{\frac{0}{1}}{=} 1 + \sum_{1 \leq q \leq Q} \phi(q) = 1 + \frac{3}{\pi^2}Q^2 + O(Q\log Q) = \frac{3}{\pi^2}Q^2 + O(Q\log Q).$$

Corollary B: A lattice point in $\mathbb{Z}^2$ considered as a subset of $\mathbb{R}^2$.
A lattice point is visible from $(0,0)$ if the line segment connecting it to the origin passes through no other lattice point.

It is easy to check that a point (a,b) is visible $\Leftrightarrow$ $\overset{\text{gcd}}{(a,b)} = 1$.

Q: How many lattice points visible from 0 are there in $[-N,N]^2$?

A:
$$\# = 8 \cdot \#\{(m,n) : 1 \leq n < m \leq N,\ (n,m)=1\} + \cancel{O(1)}$$
$$= 8 \cdot \sum_{m \leq N} \phi(m) + O(1) = \frac{24}{\pi^2}N^2 + O(N\log N).$$

The ~~total~~ number of lattice points in such a square is $(N \in \mathbb{N})$
is $(2N+1)^2 = 4N^2 + O(N)$.

So as $N \to \infty$, if we pick a lattice point at random in $\mathbb{Z}^2 \subseteq \mathbb{R}^2$,
the prob. that it's visible will be $\simeq \dfrac{\frac{24}{\pi^2}N^2}{4N^2} = \dfrac{6}{\pi^2}$.

"**Corollary C**": What is the probability that a random integer is square-free?

Heuristic: n squarefree $\Leftrightarrow$ $p^2 \nmid n$ $\forall$ prime p.

The prob. that n is squarefree should be $\approx \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}$

Theorem 2.10: (Estimate for $\sum_{n \leq x} \mu^2(n)$) $= \frac{6}{\pi^2}x + O(\sqrt{x})$.

~~If~~ write $\mu^2 = 1 * g$, g is multiplicative, and actually $g = \mu * \mu^2$.

$$g(p^m) = \sum_{j=0}^{m} \mu(p^j)\mu^2(p^{m-j}) = \begin{cases} 0 & m=1 \\ -1 & m=2 \\ 0 & m > 2 \end{cases}$$

~~So $g(n) \neq 0 \iff$ the only primes ... is a perfect square.~~

So $g(n)=0$ unless $n=m^2$ with m squarefree.

So $$\sum_{n\le x}\mu^2(n) = \sum_{n\le x}\sum_{d|n} g(d) = \sum_{n\le x}\sum_{m^2|n}\mu(m) = \sum_{m\le\sqrt{x}}\mu(m)\cdot\left\lfloor \frac{x}{m^2}\right\rfloor =$$

$$= \sum_{m\le\sqrt{x}}\mu(m)\left(\frac{x}{m^2}+O(1)\right) = x\sum_{m\le\sqrt{x}}\frac{\mu(m)}{m^2} + \underbrace{O\Big(\sum_{m\le\sqrt{x}}|\mu(m)|\Big)}_{\ll O(\sqrt{x})}$$

$$= x\cdot\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} - \underbrace{x\sum_{m>\sqrt{x}}\frac{\mu(m)}{m^2}}_{O\left(\frac{1}{\sqrt{x}}\right)} + O(\sqrt{x}) = x\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} + O(\sqrt{x}) =$$

$$= x\cdot\prod_p\left(1-\frac{1}{p^2}\right) + O(\sqrt{x}) = x\cdot\frac{6}{\pi^2} + O(\sqrt{x})$$

Now, dividing both sides by x, and letting $x\to\infty$, get prob. $= \frac{6}{\pi^2}$.

- **Averages of $2^{\omega(n)}$** (exercise).

Use the fact $2^{\omega} = d * g$ for some g, and find an asymptotic for $\sum_{n\le x} 2^{\omega(n)}$ with error term $O(\sqrt{x}\log x)$.

- **Mean values:**

If it exists, $M(f) = \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} f(n)$ is called the mean value of f.

(eg $M(\mu^2)=\frac{6}{\pi^2}$, $M(d)$ = not exist, $M(\Lambda)=1 \iff$ PNT (prime # theorem) $\iff M(\mu)=0$

Theorem 2.11 (Wintner): Suppose $f=1*g$ with $\sum_{n=1}^{\infty}\frac{|g(n)|}{n}<\infty$.

Then $M(f)=\sum_{n=1}^{\infty}\frac{g(n)}{n}$.

Pf $\frac{1}{x}\sum_{n\leq x} f(n) = \frac{1}{x}\sum_{dm\leq x} g(d) = \frac{1}{x}\sum_{d\leq x} g(d)\cdot\lfloor \frac{x}{d}\rfloor = \sum_{d\leq x}\frac{g(d)}{d} + O\left(\frac{1}{x}\sum_{d\leq x}|g(d)|\right) =$

$$= \sum_{d\geq 1}^{\infty}\frac{g(d)}{d} + o(1) + O\Bigg(\frac{1}{\sqrt{x}}\cdot\overbrace{\sum_{d\leq\sqrt{x}}\frac{|g(d)|}{d}}^{\text{converges!}} + \underbrace{\sum_{\sqrt{x}<d\leq x}\frac{|g(d)|}{d}}_{\text{tail of convergent series.}}\Bigg)\quad (x\to\infty) =$$

$$= \sum_{d=1}^{\infty}\frac{g(d)}{d} + \underbrace{o(1) + O\left(\frac{1}{\sqrt{x}}\right) + o(1)}_{o(1)}$$

- Partial summation (or summation by parts).

Q: How to sum $\sum_{n\leq x} a_n f(n)$, f smooth, a_n = arithmetic function?

Theorem 2.13 (Abel): Let $0<y\leq x$, and let $f(t)$ have continuous derivative on (y,x), and $a:\mathbb{N}\to\mathbb{C}$ be an arithmetic function.

Let $A(t)=\sum_{n\leq t} a(n)$. Then,

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

Pf in terms of Riemann-Stieltjes integration,

$$\sum_{y<n\leq x} a_n f(n) = \int_y^x f(t)\,dA(t)$$ and apply integration by parts.

Direct pf:

Uses $S=\sum_{y<n\leq x} f(n)\left(A(n)-A(n-1)\right) = \sum_{y<n\leq x} A(n)\left(f(n)-f(n+1)\right) + A(x)f(\lfloor x\rfloor) - A(y)f(\lfloor y\rfloor+1)$

and use $f(n)-f(n+1) = -\int_n^{n+1} f'(t)\,dt$ ...

Special case: $y\to 1^-$: $\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,dt$

(just note $A(y)=0$).

Application:

1) Estimate $\sum_{n \leq x} \frac{\phi(n)}{n}$. One approach is using that $\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$.
$(= A(x))$

Then $f(t) = \frac{1}{t}$, $f'(t) = \frac{-1}{t^2}$.

$$\text{So } \sum_{n \leq x} \frac{\phi(n)}{n} = A(x) \cdot \frac{1}{x} + \int_1^x A(t)/t^2 \, dt = \frac{3}{\pi^2} x + O(\log x) + \int_1^x \left(\frac{3}{\pi^2} + O\left(\frac{\log t}{t}\right) \right) dt =$$

$$= \frac{3}{\pi^2} x + O(\log x) + \frac{3}{\pi^2} x - O(1) + O\Bigg(\underbrace{\int_1^x \frac{\log t}{t} dt}_{\frac{1}{2}\log^2 x} \Bigg) = \frac{6}{\pi^2} x + O(\log^2 x)$$

2) $\sum_{2 \leq n \leq x} \frac{d(n)}{\log n}$. Now $\frac{d(n)}{\log n}$ is not multiplicative, so we have less options. But we still can use the previous result, as $\frac{1}{\log n}$ is smooth.

Let $D(x) = \sum_{n \leq x} d(n)$. Then: $D(x) = x \log x + (2\gamma - 1) x + O(\sqrt{x})$

$$\sum_{n \leq x} \frac{d(n)}{\log n} = \int_{2^-}^x \frac{dD(t)}{\log t} = \frac{D(t)}{\log t}\Big|_{2^-}^x - \int_{2^-}^x D(t) \left(\frac{1}{\log t}\right)' dt = x + (2\gamma - 1)\frac{x}{\log x} + O\left(\frac{\sqrt{x}}{\log x}\right) (+O(1)) + \int_2^x \frac{t\log t + (2\gamma-1)t + O(\sqrt{t})}{t \log^2 t} dt$$

$$= x + (2\gamma - 1)\frac{x}{\log x} + O\left(\frac{\sqrt{x}}{\log x}\right) (+O(1)) + \int_2^x \left(\frac{1}{\log t} + \frac{2\gamma - 1}{\log^2 t} + O\left(\frac{1}{\sqrt{t}\log^2 t}\right) \right) dt$$

We can use then $\int_2^x O\left(\frac{1}{\log t}\right) dt = O(li(x)) = O\left(\frac{x}{\log x}\right)$, thus obtaining

$$\sum_{2 \leq n \leq x} \frac{d(n)}{\log n} = x + O\left(\frac{x}{\log x}\right).$$

Logarithmic mean:

$L(f) := \lim_{x \to \infty} \left(\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} \right)$, if it exists.

Note: if $f(n) = ct.$, then $\frac{1}{\log x} \sum_{n \leq x} \frac{c}{n} = \frac{c}{\log x}\left(\log x + \gamma + O\left(\frac{1}{x}\right)\right) \xrightarrow[x \to \infty]{} c$

So for $f = ct$, $L(f) = M(f) = f$.

Theorem: if $M(f)$ exists, so does $L(f)$ and $M(f) = L(f)$.

The converse is false: $\exists f$ so that $L(f)$ exists but $M(f)$ doesn't.

Pf First, the counterexample:

Let $f(n) = \begin{cases} n & \text{if } n = 2^k \text{ some } k \\ 0 & \text{ow.} \end{cases}$ (multiplicative!)

$$\frac{1}{\log x} \sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{\log x} \sum_{k \leq \frac{\log x}{\log 2}} 1 = \frac{1}{\log x} \cdot \left\lfloor \frac{\log x}{\log 2} \right\rfloor = \frac{1}{\log 2} + O\left(\frac{1}{\log x}\right) \xrightarrow{x \to \infty} \frac{1}{\log 2}$$

However:

$\frac{1}{x} \sum_{n \leq x} f(n)$ has jumps of size 1 at every power of 2 $\Rightarrow$ diverges.

Now suppose $M(f)$ exists, let $M = M(f)$, and:

$$S(x) = \sum_{n \leq x} f(n), \quad T(x) = \sum_{n \leq x} \frac{f(n)}{n}$$

By partial summation,

$$T(x) = \int_{1^-}^{x} \frac{dS(t)}{t} = \overbrace{\frac{S(x)}{x}}^{M + o(1) \leftarrow \text{little-oh}} + \int_1^x \frac{S(t)}{t^2}\, dt = \overbrace{M + o(1)}^{O(1)} + \int_1^x \frac{Mt + o(t)}{t^2}\, dt$$

Write $S(t) = Mt + E(t)$ where $E(t) = o(t)$.

Then $$\int_1^x \frac{S(t)}{t^2}\, dt = \int_1^x \frac{M}{t}\, dt + \int_1^x \frac{E(t)}{t^2}\, dt = M \log x + \int_1^x \frac{E(t)}{t^2}\, dt$$

We hope that $\int_1^x \frac{E(t)}{t^2}\, dt = o(\log x)$.

Know: $\forall \varepsilon, \exists t_0$ s.t. $t \geq t_0 \Rightarrow |E(t)| \leq \varepsilon \cdot t$.

If $x \geq t_0$, then $$\left| \int_1^x \frac{E(t)}{t^2}\, dt \right| \leq \left| \int_1^{t_0} \frac{E(t)/t}{t}\, dt \right| + \left| \int_{t_0}^x \frac{\varepsilon}{t}\, dt \right| = g(t_0) + \varepsilon \log\left(\frac{x}{t_0}\right)$$

$$= \varepsilon \log x + O_{t_0}(1).$$

Thus for large x, $\left| \int_1^x \frac{E(t)}{t^2}\, dt \right| \leq 2\varepsilon \log x$ (for any $\varepsilon > 0$). $\Rightarrow$ ✓.

- Sums over prime numbers

$\sum_{p\leq x} f(p)$? Let $a_n := \begin{cases} 1 & n \text{ is prime} \\ 0 & \text{ow} \end{cases}$

Then $\sum_{p\leq x} f(p) = \sum_{n\leq x} a_n f(n)$.

If f has continuous derivative on $(2,x)$, then can apply partial summation:

Let $\pi(x) := \sum_{n\leq x} a_n$ = # primes up to x.

So $\pi(x) f(x) - \int_2^x \pi(t) f'(t)\,dt \implies$ need to understand $\pi(x)$.

This will allow us to understand "all" sums over primes. But can do something without PNT?

§3. Elementary distribution of primes

1) Chebyshev's estimates (1848-52).

$\pi(x) = \sum_{p\leq x} 1$; $\theta(x) := \sum_{p\leq x} \log p$, $\psi(x) = \sum_{n\leq x} \Lambda(n)$ $\left(\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \\ 0 \end{cases}\right)$

Trivial estimates (Thm 3.0)

i) $\pi(x) = O(x)$

ii) $\theta(x), \psi(x) = O(x\log x)$.

iii) $\psi(x) = \theta(x) + O(\sqrt{x}\log x)$

iv) $\theta(x) = \pi(x)\log x\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) + O\left(\frac{x}{\log x}\right)$

Pf iii) $\psi(x) = \theta(x) + \sum_{p^2\leq x} \log p + \sum_{p^3\leq x}\log p + \cdots + \sum_{p^k\leq x} \log p$, $k = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$ ← the smallest prime

$= \theta(x) + O(\log x\sqrt{x})$

iv) $\theta(x) = \theta\left(\frac{x}{\log^2 x}\right) + \sum_{\frac{x}{\log^2 x} \leq p\leq x} \log p = O\left(\frac{x}{\log x}\right) + \left(\log x + O(\log\log x)\right)\left(\underbrace{\pi(x) - \pi\left(\frac{x}{\log^2 x}\right)}_{O\left(\frac{x}{\log^2 x}\right)}\right)$

$\log p = \log x + O(\log\log x)$

Consequence: The function $\frac{\theta(x)}{x}$, $\frac{\psi(x)}{x}$, $\frac{\pi(x)\log x}{x}$ all have the same lim inf and lim sup when $x \to \infty$.

So,

Corollary 3.3: The relations

$\pi(x) \sim \frac{x}{\log x}$, $\theta(x) \sim x$, $\psi(x) \sim x$ are all equivalent.

Theorem (Chebyshev) 3.1:

$\pi(x) \asymp \frac{x}{\log x}$, $\theta(x) \asymp x$, $\psi(x) \asymp x$. (for $x \geq 2$)

Pf only need to prove one of them, by previous comments. We will prove $\pi(x) \succeq \frac{x}{\log x}$.

First, for $n \in \mathbb{N}$, consider $\binom{2n}{n}$.

Claim: $2^n \leq \binom{2n}{n} \leq 4^n$

Pf Since $(1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} > \binom{2n}{n}$, then $\binom{2n}{n} < 4^n$.

Also, $\binom{2n}{n} = \frac{(2n)(2n-1)\cdots(n+1)}{n(n-1)\cdots 1} \geq 2^n$.

Thus, $n\log 2 \leq \log\binom{2n}{n} \leq 2n\log 2$.

For $m \in \mathbb{N}$, $m! = \prod_{p \leq m} p^{\alpha(p)}$, $\alpha(p) = \lfloor \frac{m}{p} \rfloor + \lfloor \frac{m}{p^2} \rfloor + \cdots$

So $\log m! = \sum_{p \leq m} \alpha(p) \log p$.

So $\log\binom{2n}{n} = \sum_{p \leq 2n} \left(\sum_{1 \leq k \leq \frac{\log 2n}{\log p}} \lfloor \frac{2n}{p^k} \rfloor - 2\lfloor \frac{n}{p^k} \rfloor \right) \log p$

Note that for real $x > 0$, $\lfloor 2x \rfloor - 2\lfloor x \rfloor \in \{0,1\}$

So $\log\binom{2n}{n} \leq \sum_{p \leq 2n} \frac{\log 2n}{\log p} \log p = \pi(2n) \log 2n$

Therefore, for $n \in \mathbb{N}$, $\pi(2n) \geq \frac{\log\binom{2n}{n}}{\log 2n} \geq \frac{n\log 2}{\log(2n)}$

for $x \geq 4$ $(x-2 \geq \sqrt{x})$

For $x \geq 2$, let $n = \lfloor \frac{x}{2} \rfloor$. Then $\pi(x) \geq \pi(2n) \geq \frac{n\log 2}{\log 2n} \geq \frac{(\frac{x}{2}-1)\log 2}{\log(x-2)} \geq \frac{\log 2}{8} \frac{x}{\log x}$

(cont proof)

Upper bound:

Notice that $\binom{2n}{n} = \frac{2n(2n-1)\cdots(n+1)}{n!}$ (is integer) divisible by all primes in $(n, 2n]$.

So by the inequality $n\log 2 \leq \log\binom{2n}{n} \leq 2n\log 2$,

$$2n\log 2 \geq \log\binom{2n}{n} \geq \sum_{n<p\leq 2n} \log p = \theta(2n) - \theta(n)$$

Apply this by $n = 1, 2, 2^2, \dots, 2^r$ and sum:

$$2^{r+2}\log(2) \geq 2\log(2)(1+2+\dots+2^r) \geq \sum_{k=0}^{r}\left(\theta(2^{k+1}) - \theta(2^k)\right) = \theta(2^{r+1}) - \overset{0}{\theta(1)} = \theta(2^{r+1})$$

So $2^{r+2}\log(2) \geq \theta(2^{r+1})$ $r \geq 0$, $r \in \mathbb{Z}$.

Let now x be arbitrary; define r by $2^r < x \leq 2^{r+1}$.

Then $\theta(x) \leq \theta(2^{r+1}) \leq 2^{r+2}\log 2 \leq (4\log 2)\cdot x$

By partial summation (or could use thm. 3.0).

$$\pi(x) = \sum_{p\leq x} \frac{1}{\log p}\log p = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t}\,dt \leq (4\log 2)\left(\frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}\right)$$

$$\int_2^x \frac{dt}{\log^2 t} \leq \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2\sqrt{x}} \leq \frac{\sqrt{x}}{(\log 2)^2} + \frac{4x}{\log^2 x} = O\left(\frac{x}{\log^2 x}\right) \quad (x \geq 2).$$

Thus, $\pi(x) = O\left(\frac{x}{\log x}\right)$ $(x \geq 2)$. //

Chebyshev proved later, using the same methods, that for large x,

$$0.921 < \frac{\pi(x)\log x}{x} < 1.106.$$

Theorem 3.2: For $x \geq 2$,

- $\theta(x) = \psi(x) + O(\sqrt{x})$
- $\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right)$

Pf/ The first follows from $\psi(x) = \theta(x) + \sum_{p^2 \leq x} \log p + \sum_{\substack{k \geq 3 \\ p^k \leq x}} \log p = \theta(x) + \theta(\sqrt{x}) + \sum_{3 \leq k \leq \frac{\log x}{\log 2}} \theta(x^{\frac{1}{k}})$

$= \theta(x) + O(\sqrt{x}) + O(\log x) \cdot O(x^{1/3}) = \theta(x) + O(\sqrt{x})$ //

For the second,

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt \overset{\text{proved that before}}{=} \frac{\theta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right) = \frac{\psi(x)}{\log x} + O\left(\cancel{\frac{\sqrt{x}}{\log x}}\right) + O\left(\frac{x}{\log^2 x}\right)$$ //

- Mertens (1874):

$L(\Lambda) = 1$ (logarithmic mean). (recall that PNT $\Leftrightarrow M(\Lambda) = 1$).

Consequence: if $M(\Lambda)$ exists, then PNT.

First, note:

Prop: $\log n = \Lambda * 1 (n) = \sum_{d | n} \Lambda(d)$. (Chebyshev identity).

Pf/ If $n = p^\alpha$, $\log n = \alpha \log p$. $\Lambda(d) = \begin{cases} \log p & \text{for } d = p, p^2, \dots, p^\alpha \\ 0 & \text{else} \end{cases}$ $\Rightarrow$ ✓.

- If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then $\log n = \sum \alpha_i \log p_i$.

For the RHS, $\sum_{d|n} \Lambda(d) = ?$ Note $\Lambda(d) \neq 0$ for $d = p_1, p_1^2, \dots, p_1^{\alpha_1}$; $p_2, p_2^2, \dots, p_2^{\alpha_2}$; $\vdots$; $p_r, \dots, p_r^{\alpha_r}$. So ok. //

Now, the theorem:

Thm (Mertens Estimate): (3.4) we'd like to have a little-oh here but not now...

i) $\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$.

ii) $\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$.

Pf

(1) $\sum_{n\le x}\log n = \sum_{n\le x}\sum_{ij=n}\Lambda(i)1(j)$. The LHS $= \log(\lfloor x\rfloor !) = x\log x + O(x)$ ← weak Stirling formula.

The RHS $= \sum_{\substack{i\cdot j\le x\\ i,j>0}}\Lambda(i)1(j) = \sum_{i\le x}\left(\sum_{j\le\lfloor x/i\rfloor}1(j)\Lambda(i)\right) = \sum_{i\le x}\Lambda(i)\cdot\lfloor \frac{x}{i}\rfloor =$

$$= \sum_{i\le x}\frac{x}{i}\Lambda(i) + O(1)\cdot\sum_{i\le x}\Lambda(i) = x\sum_{i\le x}\frac{\Lambda(i)}{i} + O(x).$$

(2) It suffices to show that:

$$\sum_{n\le x}\frac{\Lambda(n)}{n} - \sum_{p\le x}\frac{\log p}{p} = O(1).$$

But first term $= \sum_{p^\alpha\le x}\frac{\log p}{p^\alpha}$. So the difference is $\sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}}\frac{\log p}{p^\alpha}\ge 0$.

Also, $\sum_{p^\alpha\le x}\frac{\log p}{p^\alpha} - \sum_{p\le x}\frac{\log p}{p} = \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}}\frac{\log p}{p^\alpha} \le \sum_{\substack{p\\ \alpha\ge 2}}\frac{\log p}{p^\alpha} = \sum_p \log p\cdot\sum_{\alpha=2}^{\infty}\frac{1}{p^\alpha} =$

$$= \sum_p \log p\,\frac{1/p^2}{1-1/p} \le \sum_p \frac{2\log p}{p^2} \le 2\sum_n \frac{\log n}{n^2} \overset{\text{integral test}}{=} O(1).$$

Corollary 3.5: $\int_1^x \frac{\psi(t)}{t^2} = \log x + O(1)$.

Pf Start with $\sum_{n\le x}\frac{\Lambda(n)}{n} = \log x + O(1)$. Write LHS $= \int_1^x \frac{d\psi(t)}{t} \overset{\text{by parts}}{=} \frac{\psi(x)}{x} - \frac{\psi(1)}{1} + \int_1^x \frac{\psi(t)}{t^2}dt$

$$= O(1) - 0 + \int_1^x \frac{\psi(t)}{t^2}dt.$$

Corollary 3.6: $\varliminf_{x\to\infty}\frac{\psi(x)}{x} \le 1 \le \varlimsup_{x\to\infty}\frac{\psi(x)}{x}$ (∴ if limit exists, it is 1)

Pf (lim inf): suppose not. $\varliminf \frac{\psi(x)}{x} > 1 \Rightarrow \exists\,\varepsilon>0,\ x_0$ s.t. $\frac{\psi(x)}{x} > 1+\varepsilon\ \forall x\ge x_0$.

By cor 3.5, $\log x + O(1) = \left(\int_1^{x_0} + \int_{x_0}^x\right)\frac{\psi(t)}{t^2}dt \ge \int_{x_0}^x \frac{\psi(t)}{t^2}dt \ge \int_{x_0}^x \frac{1+\varepsilon}{t}dt = (1+\varepsilon)(\log x - \log x_0)$

Divide through by $\log x$ and let $x\to\infty$. Then $1 \ge 1+\varepsilon \Rightarrow$!!

(Same for lim sup)

Theorem 3.4 II (Mertens):

(iii) $\displaystyle\sum_{p\le x} \frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right).$ $\quad (x > e).$

(iv) $\displaystyle\prod_{p\le x}\left(1-\frac{1}{p}\right) = \frac{c}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right).$ $\quad (c = e^{-\gamma}$, in fact$)$

Pf (iii) Let $\displaystyle S(x)=\sum_{p\le x}\frac{\log p}{p} \left(\overset{\text{pt (ii)}}{=} \log x + O(1)\right)$

Then
$$\sum_{p\le x}\frac{1}{p} = \int_1^x \frac{1}{\log t}\, dS(t) = \int_{3/2}^x \frac{1}{\log t}\, dS(t) \overset{\text{IB-parts}}{=} \frac{S(t)}{\log t}\Big|_{3/2}^x + \int_{3/2}^x \frac{S(t)}{t\log^2 t}\,dt =$$

at 1, log 1 = 0!

$$= \frac{\overset{\log x + O(1)}{S(x)}}{\log x} + \int_{3/2}^x \frac{S(t)}{t\log^2 t}\,dt = 1 + O\left(\frac{1}{\log x}\right) + \int_{3/2}^x \frac{1}{t\log t}\,dt + \int_{3/2}^x\frac{O(1)}{t\log^2 t}\,dt =$$

$$= 1 + O\left(\frac{1}{\log x}\right) + \log\log x - \log\log\frac{3}{2} + \underbrace{\int_{3/2}^\infty \frac{O(1)}{t\log^2 t}\,dt}_{\text{converges}} - \underbrace{\int_x^\infty \frac{O(1)}{t\log^2 t}\,dt}_{\le K\int_x^\infty \frac{dt}{t\log^2 t} = K \frac{-1}{\log t}\Big|_x^\infty = \frac{K}{\log x}}$$

$$= \log\log x + B + O\left(\frac{1}{\log x}\right)$$

(iv) Take logs:

$$\log\prod_{p\le x}\left(1-\frac{1}{p}\right) = \sum_{p\le x}\log\left(1-\frac{1}{p}\right) = -\sum_{p\le x}\left(\frac{1}{p} + \frac{1}{2p^2}+\frac{1}{3p^3}+\cdots\right) =$$

$$= -\sum_{p\le x}\left(\frac{1}{p} + \text{rest}\right), \text{ where } |\text{rest}| \le \sum_{p} \frac{1}{2p^2}+\frac{1}{2p^3}+\frac{1}{2p^4} \underbrace{\cdots = O(1)}_{\text{geometric series}} - \sum_{p>x}(\ ) = ct + O\left(\frac{1}{x}\right)$$

Exponentiating, the RHS becomes
$$= e^{-\sum_{p\le x}\frac{1}{p}}\cdot e^{O(x)} = \frac{1}{\log x}\cdot c\cdot \underbrace{e^{O\left(\frac{1}{\log x}\right)}}_{=\,1+O\left(\frac{1}{\log x}\right)} \cdot e^{O\left(\frac{1}{x}\right)}$$

• Applications of Mertens' bounds.

1) let $\mathcal{P}$ be the set of primes $\leq y$; $P := \prod_{p \leq y} p$, and

$$F(x,y) = \left| \{ n \leq x : p \mid n \Rightarrow p > y \} \right|.$$

By Theorem 3.4 (iv), and homework exercise from #1,

$$F(x,y) = \sum_{\substack{n \leq x \\ (n,P)=1}} 1 = \sum_{n \leq x} \sum_{d \mid (n,P)} \mu(d) = \sum_{\substack{d \mid P \\ (d \leq x)}} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{d \mid P} \frac{\mu(d)}{d} + \sum_{d \mid P} O(1) \overset{HW}{=}$$

$$= x \cdot \prod_{p \leq y} \left(1 - \frac{1}{p}\right) + O(d(P)).$$

$p \mid P \Leftrightarrow p \leq y$

By Mertens' estimate, $\prod_{p \leq y} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log y} \left(1 + O\left(\frac{1}{\log y}\right)\right)$.

Also, $d(P) = 2^{\pi(y)}$. Thus $F(x,y) = x \frac{e^{-\gamma}}{\log y} + O\left(\frac{x}{\log^2 y} + 2^{\pi(y)}\right)$

Known as <u>Legendre's sieve</u>.

<u>Corollary</u>: if $y \to \infty$ as $x \to \infty$ and $y = O(\log x)$, then:

$$F(x,y) \sim \frac{x e^{-\gamma}}{\log y} \quad (x \to \infty) \qquad \left(\text{ie } F(x, y(x)) \sim \frac{x e^{-\gamma}}{\log y(x)}\right).$$

Pf Since $y \to \infty$, $O\left(\frac{x}{\log^2 y}\right) = o\left(\frac{x}{\log y}\right)$.

Also, by Chebyshev's estimate, $\pi(y) \leq C \frac{y}{\log y}$. And $y \leq K \log x$ for some constant K. Thus $\pi(y) \leq C \frac{K \log x}{\log(K \log x)} \leq C' \frac{\log x}{\log \log x}$ for some C'.

Then $2^{\pi(y)} \leq 2^{C' \frac{\log x}{\log \log x}}$

Now $2^{C' \frac{\log x}{\log \log x}} = x^{\frac{C' \log 2}{\log \log x}} \leq x^{1/2}$ if x is large enough.

Hence the main term is $\gg \frac{x}{\log y} \gg \frac{x}{\log \log x}$. //

Another application:

- Let $\pi_k(x) = |\{n \leq x : \omega(n) = k\}|$ (numbers with exact k prime factors).

Note: $\pi_1(x) = \pi(x) + \pi(\sqrt{x}) + \cdots = \pi(x) + O(\sqrt{x})$.

Theorem 3.10 (Hardy - Ramanujan (1917)): There exist constants C_1, C_2 so that $\forall k \geq 1, x \geq 2$,

$$\pi_k(x) \leq C_1 \cdot \frac{x(\log\log x + C_2)^{k-1}}{(k-1)! \log x}$$

Pf Note that for $k=1$, $\pi_1(x) \leq K \frac{x}{\log x}$ for some cst. K (by Chebyshev).

By Mertens' estimates, $\exists$ constants D and E s.t.:

$$\sum_{p \leq x} \frac{1}{p} < \log\log x + D \quad (x \geq 2)$$

$$\sum_{p \leq x} \frac{\log p}{p} < E \log x \quad (x \geq 2)$$

Let $J = \sum_{s=2}^{\infty} (s+1) \sum_p p^{-s}$. The double sum converges, since $\sum_p p^{-s} \leq \sum_{n=2}^{\infty} n^{-s} \leq$

$$\leq \frac{1}{2^s} + \int_2^{\infty} \frac{dt}{t^s} = \frac{1}{2^s} + \frac{2^{1-s}}{s-1} \leq \frac{3}{2^s}$$

Claim: $C_1 = K$, $C_2 = D + E + J$ are admissible.

Suppose $k \geq 1$, $\omega(n) = k+1$, $n \leq x$, $n = p_1^{a_1} \cdots p_{k+1}^{a_{k+1}}$, where $p_1 < \cdots < p_{k+1}$.

For $1 \leq i \leq k$, $p_j^{a_j+1} \leq x$, and $\omega(n \cdot p_j^{-a_j}) = k$.

So $k \cdot \pi_{k+1}(x) \leq \sum_{\substack{p, a \\ p^{a+1} \leq x}} \pi_k\left(\frac{x}{p^a}\right)$

If $\pi_k(x) \leq K \dfrac{x(\log\log x + D + E + J)^{k-1}}{(k-1)! \log x}$ $(x \geq 2)$.

Then $\pi_{k+1}(x) \leq \frac{1}{k} \sum_{\substack{p^{a+1} \leq x \\ p, a}} K \dfrac{\left(\frac{x}{p^a}\right)(\log\log x/p^a + D + E + J)^{k-1}}{(k-1)! \log(x/p^a)}$ (ok since $\frac{x}{p^a} \geq p \geq 2$).

$$\cdots \leq \frac{K\cdot x}{k!}\,\frac{(\log\log x + D+E+J)^{k-1}}{1}\sum_{\substack{p,a\\ p^a\leq x}} \frac{1}{p^a \log\frac{x}{p^a}}$$

<u>When $a=1$</u>, $p\leq\sqrt{x}$. So $\displaystyle\sum_{p\leq\sqrt{x}}\frac{1}{p\log\frac{x}{p}} \leq \frac{2}{\log x}\sum_{p\leq\sqrt{x}}\frac{1}{p} \leq \frac{2}{\log x}\sum_{p\leq x}\frac{1}{p} \leq$

$$\leq \frac{2}{\log x}(\log\log x + D).$$

Alternately, $\displaystyle\frac{1}{\log\frac{x}{p}} = \frac{1}{\log x - \log p} = \frac{1}{\log x}\,\frac{1}{1-\frac{\log p}{\log x}} = \frac{1}{\log x}\left(1+\frac{\log p}{\log x}+\left(\frac{\log p}{\log x}\right)^2+\cdots\right)$

$$\leq \frac{1}{\log x}\left(1+2\frac{\log p}{\log x}\right).$$

By Mertens, $\displaystyle\sum_{p\leq\sqrt{x}}\frac{1}{p\log\frac{x}{p}} \leq \frac{1}{\log x}\sum_{p\leq\sqrt{x}}\left(\frac{1}{p}+\frac{2\log p}{p\log x}\right) \leq \frac{1}{\log x}\Bigg(\log\log x + D + \frac{2}{\log x}\underbrace{\sum_{p\leq\sqrt{x}}\frac{\log p}{p}}_{\leq E\log\sqrt{x}}\Bigg)$

So get $\displaystyle\leq \frac{1}{\log x}\left(\log\log x + D + E\right)$.

<u>When $a\geq 2$</u> and $p^{a+1}\leq x$,

$\displaystyle\log\frac{x}{p^a} \geq \log x^{\frac{1}{a+1}} = \frac{\log x}{a+1}$. Thus $\displaystyle\sum_{a\geq 2}\sum_{p^{a+1}\leq x}\frac{1}{p^a\log\frac{x}{p^a}} \leq \sum_{a\geq 2}\frac{a+1}{\log x}\sum_p\frac{1}{p^a} = \frac{J}{\log x}$.

Therefore, $\displaystyle\sum_{\substack{p,a\\ p^a\leq x}}\frac{1}{p^a\log\frac{x}{p^a}} \leq \frac{\log\log x + D+E+J}{\log x}$.

In 1900, Landau proved, for <u>fixed</u> $k\geq 1$, $\displaystyle\#\{n\leq x : \omega(n)=k\} \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}$ $(x\to\infty)$

With the M-R formula, one can fix x and vary k.

The maximum occurs at $k = \log\log x + O(1)$. $\Longrightarrow$ Showed that most $n\leq x$ have about $\log\log x$ prime factors (can be made precise).

<u>Precisely</u>: $\forall\varepsilon>0$, $\#\{n\leq x : |\omega(K) - \log\log x| \geq \varepsilon\log\log x\} = o(x)$ $(x\to\infty)$.

Exercise: $\liminf_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma}$.

In particular, $\phi(n) >> \frac{n}{\log\log n}$.

Corollary: $\sum_{n\leq x} \frac{1}{\phi(n)} << \sum_{n\leq x} \frac{\log\log n}{n} \leq \log\log x \sum_{n\leq x} \frac{1}{n} << (\log x)\cdot(\log\log x)$.

Theorem 3.7: The PNT implies:

(i) the n^{th} prime p_n satisfies $p_n \sim n\log n$ $(n\to\infty)$.

(ii) given $\varepsilon>0$, $\exists X_0$ so that $X\geq X_0$, there is a prime in $(X, X(1+\varepsilon)]$.

(iii) The set of rationals $\frac{p}{q}$ with p, q prime is dense in $\mathbb{R}^+=(0,\infty)$.

(iv) $\limsup_{n\to\infty} \frac{\omega(n)}{\frac{\log n}{\log\log n}} = 1$

(v) $\sum \frac{\Lambda(n)}{n} = \log x + c_1 + o(1)$

(vi) $\sum_{p\leq x} \frac{\log p}{p} = \log x + c_2 + o(1)$

These are actually equivalent to PNT.

Pf

(i) PNT $\Rightarrow \pi(x) \sim \frac{x}{\log x}$ $\Rightarrow \pi(x) = \frac{x}{\log x}(1+o(1))$. So $\log\pi(x) = \log\left(\frac{x}{\log x}\right) + \log(1+o(1)) \sim \log x$.

Hence $\pi(x)\log\pi(x) \sim x$. Let $x=p_n$. Then $p_n \sim \pi(p_n)\log(\pi(p_n)) = n\log n$. //

(ii) Fix $\varepsilon>0$. By PNT, $\frac{\pi(X(1+\varepsilon))}{\pi(X)} \sim \frac{X(1+\varepsilon)}{\log X(1+\varepsilon)}\cdot\frac{\log X}{X} = \frac{(1+\varepsilon)\log X}{\log X + \log(1+\varepsilon)} \xrightarrow{X\to\infty} 1+\varepsilon > 1$.

Therefore, for X large, $\pi(X(1+\varepsilon)) > \pi(X) \Rightarrow \checkmark$. //

(iii) Need $\forall\alpha\in\mathbb{R}^+$, $\forall\varepsilon>0$, $\exists\frac{p}{q}$ with $|\alpha-\frac{p}{q}|<\varepsilon$.

Equivalently, $(\alpha-\varepsilon)q < p < (\alpha+\varepsilon)q$

Apply (ii) with $X=(\alpha-\varepsilon)q$, $\varepsilon'=\frac{\varepsilon}{\alpha}$. Then for q large, $\exists$ a prime in $(X, X(1+\varepsilon')] = ((\alpha-\varepsilon)q, (\alpha-\varepsilon)q(\frac{\varepsilon}{\alpha}+1)]$ and $(\alpha-\varepsilon)q(1+\frac{\varepsilon}{\alpha}) < \alpha q$. //

(cont proof)

(iv) Given n with $\omega(n)=k$, the smallest such n is $p_1 \cdots p_k =: n_k$

Then $\displaystyle \limsup_{n\to\infty} \frac{\omega(n)}{\frac{\log n}{\log\log n}} = \limsup_{n\to\infty} \frac{k}{\frac{\log n_k}{\log\log n_k}}$

By PNT in the form $\theta(x) \sim x$ $\left(\log n_k = \sum_{p \le p_k} \log p = \theta(p_k) \right)$, then $\log n_k \sim \overset{\text{pt (i)}}{\overset{\downarrow}{p_k}} \sim k \log k$. $(k\to\infty)$

Also, $\log\log n_k = \log\left(k\log k + \log(1+o(1))\right) =$

$= \log\left((k\log k)\cdot(1+o(1))\right) = \log k + \log\log k + o(1) \sim \log k$.

Hence $\dfrac{\log n_k}{\log\log n_k} \sim k$. So done. //

(v) $\Rightarrow$ (vi)

$$\sum_{n\le x} \frac{\Lambda(n)}{n} - \sum_{p\le x} \frac{\log p}{p} = \sum_{\substack{p,\ a\ge 2 \\ p^a \le x}} \frac{\log p}{p^a} = C + o(1) \quad \text{as } x\to\infty$$

the sum without condition $p^a \le x$ is convergent. //

(v) Using Stirling's formula and $\log = 1 * \Lambda$, let $A \in \mathbb{Z}$, $A \ge 2$.

$$\sum_{n\le Ax} \log n = \sum_{m \le Ax} \left\lfloor \frac{Ax}{m} \right\rfloor \Lambda(m)$$

Stirling $\rightarrow$ ||

$Ax\log(Ax) - Ax + O(\log Ax)$

By PNT, $\forall \varepsilon > 0$ $\exists x_0$ s.t. $x \ge x_0(\varepsilon)$, $|\psi(x) - x| \le \varepsilon x$ (as PNT says $\psi(x)\sim x$)

If $x \ge x_0$, $\displaystyle \sum_{m \le x} \left\lfloor \frac{Ax}{m} \right\rfloor \Lambda(m) = \sum_{m\le x} \left(\frac{Ax}{m} + O(1)\right)\Lambda(m) = Ax \sum_{m\le x} \frac{\Lambda(m)}{m} + \underbrace{O\Big(\sum_{n\le x}\Lambda(n)\Big)}_{O(x) \text{ by Chebyshev's}}$

!!

Also,

$$\sum_{x < m \le Ax} \left\lfloor \frac{Ax}{m} \right\rfloor \Lambda(m) = \sum_{k=1}^{A-1} k \cdot \sum_{\frac{Ax}{k+1} < m \le \frac{Ax}{k}} \Lambda(m) = \sum_{k=1}^{A-1} k\left(\psi\left(\frac{Ax}{k}\right) - \psi\left(\frac{Ax}{k+1}\right)\right) =$$

$$= \sum_{k=1}^{A-1} \frac{Ax}{k+1} + O(\varepsilon A^2 x) = Ax\left(\log A - 1 + \gamma + O\left(\frac{1}{A}\right) + O(\varepsilon A)\right)$$

End of the proof:

Divide all by Ax, obtaining:

independ of ε, A.

$$\sum_{m\leq x}\frac{\Lambda(m)}{m}=\log x-\gamma+O\left(\frac{\log Ax}{Ax}+\frac{1}{A}+\varepsilon A\right).$$

Now let A be large, $\varepsilon:=\frac{1}{A^2}$, let $x\to\infty$ and $A\to\infty \Rightarrow$ ✓.

Theorem 3.8: $M(x):=\sum_{n\leq x}\mu(n)=o(x) \iff$ P.N.T.

Proof: Recall $\Lambda=\mu*\log$, so $\Lambda(n)=\sum_{d|n}\mu(d)\log\frac{n}{d}=\cancel{(\log n)\sum_{d|n}\mu(d)}^{\,0}-\sum_{d|n}\mu(d)\log d=$

$$=-\sum_{d|n}\mu(d)\log d$$

Thus, $\Lambda=(-\mu\log)*1$. By Möbius inversion, $\mu\log=-\Lambda*\mu$ (*).

Now, if PNT, then $\psi(x)\sim x$.

Add $e(n)=(1*\mu)(n)$ to both sides of (*), and sum on $n\leq x$:

$$(**)\quad 1+\sum_{n\leq x}\mu(n)\log n=\sum_{n\leq x}\sum_{d|n}\left(1-\Lambda\left(\tfrac{n}{d}\right)\right)\mu(d)$$

$|M(t)|\leq t$

By partial summation, LHS of (**) is $1+M(x)\log x-\int_1^x\frac{M(t)}{t}dt=$

$$=M(x)\log x+O(x)$$

$\forall\varepsilon>0,\ \exists x_0\geq 3$ s.t. $x\geq x_0\Rightarrow|\psi(x)-x|\leq\varepsilon x$. Also, $\psi(x)\leq k(x)$ $\forall x\geq 1$, $k\in\mathbb{R}$.

The RHS of (**) is, in absolute value,

$$\left|\sum_{n\leq x}\sum_{d|n}\left(1-\Lambda\left(\tfrac{n}{d}\right)\right)\mu(d)\right|\leq\left|\sum_{d|m}\mu(d)\sum_{m\leq\frac{x}{d}}(1-\Lambda(m))\right|\leq\sum_{d|n}\left(\left\lfloor\tfrac{x}{d}\right\rfloor-\psi\left(\tfrac{x}{d}\right)\right)\leq$$

$$\leq\sum_{d\leq\frac{x}{x_0}}\left(\frac{\varepsilon x}{d}+O(1)\right)+\sum_{\frac{x}{x_0}<d\leq x}\frac{kx}{d}\leq\varepsilon x\left(\log\left(\tfrac{x}{x_0}\right)+O(1)\right)+O\left(\tfrac{x}{x_0}\right)+kx\left(\log x_0+O\left(\tfrac{x_0}{x}\right)\right)$$

$\leq\varepsilon x\log x+O_{\varepsilon,x_0}(x).\leq 2\varepsilon x\log x$ if x is large enough (ε, x_0 fixed).

So $|M(x)| \leq 3\varepsilon x$ for large x. Since ε is arbitrary, $|M(x)| = o(x)$.

Conversely, assume now $M(x) = o(x)$. We will prove that $\psi(x) \sim x$.

Let $f(n) = d(n) - \log n - 2\gamma$.

Note that $d = 1 * 1 \Rightarrow 1 = \mu * d$, so:

$$\lfloor x \rfloor - \psi(x) - 2\gamma = \sum_{n \leq x} \left(1 - \Lambda(n) - 2\gamma e(n)\right) = \sum_{n \leq x} \sum_{q \mid n} \mu(q) \left(d\left(\tfrac{n}{q}\right) - \log \tfrac{n}{q} - 2\gamma \right) =$$

$$= \sum_{q \leq x} \mu(q) \sum_{r \leq x/q} f(r).$$

So (*) $$\psi(x) - x = -\sum_{q \leq x} \mu(q) \sum_{r \leq x/q} f(r) + O(1) = -\sum_{qr \leq x} \mu(q) f(r) + O(1).$$

Now, if $F(x) = \sum_{n \leq x} f(n)$, then by Thms 2.7 and 2.8,

So $$F(x) = x \log x - (2\gamma - 1)x + O(\sqrt{x}) - \left(x \log x - x + O(\log x)\right) - 2\gamma \lfloor x \rfloor$$

$$= O(\sqrt{x}).$$

The sum on q, r is estimated with the hyperbola method:

Let B be such that $|F(x)| \leq B\sqrt{x}$ $\forall x \geq 1$.

Choose a, b st $ab = x$. By Thm. 2.20,

$$\sum_{qr \leq x} \mu(q) f(r) = \sum_{n \leq a} f(n) M\left(\tfrac{x}{n}\right) + \sum_{n \leq b} \mu(n) F\left(\tfrac{x}{n}\right) - F(a)M(b).$$

$1 + (2\sqrt{b} - 2) \leq 2\sqrt{b}$ ← Euler-Maclaurin or integral test

The second sum is, in abs. value, $\leq \sum_{n \leq b} \left|F\left(\tfrac{x}{n}\right)\right| \leq B\sqrt{x} \sum_{n \leq b} \frac{1}{\sqrt{n}} \leq 2\sqrt{bx}\, B = \frac{2Bx}{\sqrt{a}}$

$\forall \varepsilon > 0$, $\exists x_0(\varepsilon)$ st $x \geq x_0 \Rightarrow |M(x)| \leq \varepsilon x$. If $b = \frac{x}{a} \geq x_0$, and so $x \geq a x_0$,

then $$\left| \sum_{n \leq a} f(n) M\left(\tfrac{x}{n}\right) \right| \leq \sum_{n \leq a} |f(n)| \cdot \left(\frac{\varepsilon x}{n}\right) = \varepsilon x \cdot \sum_{n \leq a} \frac{|f(n)|}{n} = \varepsilon x \underbrace{K(a)}_{\text{define it here.}}$$

Lastly, $|F(a)M(b)| \leq (B\sqrt{a}) \cdot (\varepsilon b) = \frac{\varepsilon B x}{\sqrt{a}}$.

Combining these estimates together we find:

$$\Big| \sum_{qr\leq x} \mu(q) f(r) \Big| \leq x\left(\varepsilon K(a) + \frac{23+\varepsilon B}{\sqrt{a}}\right) \text{ if } x \geq a x_0.$$

Take $\varepsilon = \frac{1}{1+\sqrt{a}K(a)}$, and let $a \to \infty$. Then $\psi(x) - x = o(x) \Rightarrow$ PNT.

Theorem 3.11: PNT $\Leftrightarrow \sum_{n\leq x} \frac{\mu(n)}{n} = o(1)$

Pf/ We will actually show $M(x) = o(x) \Leftrightarrow \sum_{n\leq x} \frac{\mu(n)}{n} = o(1)$

Recall the formula $\sum_{n\leq x} \mu(n) \lfloor \frac{x}{n} \rfloor = 1$ $(*)$ obtained by summing $e = 1 * \mu$ over $n \leq x$.

Let $A \in \mathbb{N}$, $A \geq 2$. Replace x with Ax in $(*)$.

$$1 = \sum_{n\leq Ax} \mu(n) \left\lfloor \frac{Ax}{n} \right\rfloor = \sum_{x < n \leq Ax} \mu(n) \left\lfloor \frac{Ax}{n} \right\rfloor + \sum_{n\leq x} \mu(n) \frac{Ax}{n} + O\Big(\sum_{n\leq x} 1\Big)$$

$$= \sum_{k=1}^{A-1} k\left(M\left(\tfrac{Ax}{k}\right) - M\left(\tfrac{Ax}{k+1}\right)\right) + Ax \sum_{n\leq x} \frac{\mu(n)}{n} + O(x).$$

Assume $M(x) = o(x)$. Then $\forall \varepsilon > 0$, $\exists x_0$ s.t. $x \geq x_0 \Rightarrow |M(x)| \leq \varepsilon x$.

If $x \geq x_0$ we have:

$$\sum_{n\leq x} \frac{\mu(n)}{n} = O\left(\tfrac{1}{A}\right) - \frac{1}{Ax} \sum_{k=1}^{A-1} k\left(M\left(\tfrac{Ax}{k}\right) - M\left(\tfrac{Ax}{k-1}\right)\right) = O\left(\tfrac{1}{A}\right) + O\left(\frac{1}{Ax}\sum_{k=1}^{A-1} k\left(\frac{\varepsilon Ax}{k} + \frac{\varepsilon Ax}{k+1}\right)\right)$$

$$= O\left(\tfrac{1}{A}\right) + O(\varepsilon \cdot A) = \overset{\text{indep of } \varepsilon, A.}{O}\left(\tfrac{1}{A} + \varepsilon \cdot A\right). \text{ Take } \varepsilon = \tfrac{1}{A^2} \text{ and } A \to \infty.$$

Conversely, assume $\sum_{n\leq x} \frac{\mu(n)}{n} = o(1)$. By partial summation, if $C(x) = \sum_{n\leq x} \frac{\mu(n)}{n}$,

then $M(x) = \int_{1^-}^{x} t\, dC(t) = xC(x) - \int_1^x C(t)\,dt = o(x) - \int_1^x C(t)\,dt$

By HW set #2, pblm 2, as $\int_1^x 1\,dt$ diverges, then $\int_1^x C(t)\,dt = o\left(\int_1^x 1\,dt\right) = o(x)$.

So $M(x) = o(x)$.

Note: if $M(x)=o(x)$, then

$$\sum_{n\le x}\frac{\mu(n)}{n} = \int_{1^-}^{x}\frac{dM(t)}{t} = \frac{M(x)}{x} + \int_1^x \frac{M(t)}{t^2}\,dt = \frac{o(x)}{x} + o\left(\int_1^x \frac{dt}{t}\right) = o(1) + o(\log x).$$

(too weak for the proof).

History

The statement $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$ was stated by Euler in 1748, but proved in 1897 by von Mangoldt (using PNT). Then shown that it was equivalent to it.

Riemann Zeta Function

$$\zeta(s) = \sum_{n\ge 1}^{\infty}\frac{1}{n^s} \qquad (s\in\mathbb{C},\ \operatorname{Re} s>1).$$

Note that $\frac{1}{n^s}$ is multiplicative.

Euler product: $\zeta(s) = \prod_p \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1}$ $\quad(\operatorname{Re} s>1)$.

Behavior for $s>1$, s real

Lemma 4.1: We have $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$ $\quad (s>1)$. (useful for $s\to 1$)
(for $s>2$, not useful already)

Pf: $$\sum_{n=1}^{N}\frac{1}{n^s} = 1 + \int_1^N \frac{dt}{t^s} - s\int_1^N \{t\}\,t^{-s-1}\,dt$$ by Euler-Maclaurin summation.

Both integrals converge when $N\to\infty$. So:

$$\cancel{\sum_{n=1}^{N}\frac{1}{n^s}}\quad \zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty}\{t\}\,t^{-s-1}\,dt \qquad (s>1).$$

Let $I(s) = \int_1^{\infty}\{t\}\,t^{-s-1}\,dt$. From the proof of thm 2.4 (summation of $\frac{1}{n}$),

$$I(1) = \int_1^{\infty}\{t\}\,t^{-2}\,dt = 1-\gamma$$

Also, $$|I(1)-I(s)| = \int_1^{\infty}\{t\}\left(\frac{1}{t^2}-\frac{1}{t^{s+1}}\right)dt \le \int_1^{\infty}\frac{1}{t^2}-\frac{1}{t^{s+1}}\,dt = 1-\frac{1}{s} = \frac{s-1}{s}$$

Thus, $$\zeta(s) = 1+\frac{1}{s-1} - s\big(I(1) + (I(s)-I(1))\big) = 1+\frac{1}{s-1} - \overbrace{s}^{1+(s-1)}(1-\gamma) + O(s-1) =$$
$$= \frac{1}{s-1} + \gamma - (s-1)(1-\gamma) + O(s-1) = \frac{1}{s-1} + \gamma + O(s-1).$$

Application: The constant in Theorem 3.4 (iv) is $e^{-\gamma}$, i.e.

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) \qquad (x\geq 2).$$

~~Proof~~ Recall that in the proof of 3.4 (iv), it's shown $\prod_{p\leq x}\left(1-\frac{1}{p}\right)=\frac{c}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right)$

for $c=e^{A-B}$ where
$$\begin{cases} A=\sum_p\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) \\ \sum_{p\leq x}\frac{1}{p}=\log\log x+B+O\left(\frac{1}{\log x}\right). \end{cases}$$

Define, for $\sigma>0$,

$$f(\sigma)=\log\zeta(1+\sigma)-\sum_p\frac{1}{p^{1+\sigma}}\overset{\text{converges}}{=}\sum_p\left(\log\left(\frac{1}{1-p^{-1-\sigma}}\right)-\frac{1}{p^{1+\sigma}}\right)$$

Since $\log\left(\frac{1}{1-p^{-1-\sigma}}\right)=-\log\left(1-p^{-1-\sigma}\right)=\frac{1}{p^{1+\sigma}}+\frac{1}{2p^{2+2\sigma}}+\dots=\frac{1}{p^{1+\sigma}}+E$

where $0\leq E\leq\frac{1}{2p^2}+\frac{1}{2p^3}+\dots\leq\frac{1}{p^2}$

Therefore, as $\sigma\to 0^+$ $f(\sigma)\to -A$

On the other hand, by lemma 4.1, if $0<\sigma\leq 1$

$$\log\zeta(1+\sigma)=\log\left(\frac{1}{\sigma}+O(1)\right)=-\log\sigma+O(\sigma)$$

As $1-e^{-\sigma}=\sigma+O(\sigma^2)$, $\log\zeta(1+\sigma)=\log\frac{1}{1-e^{-\sigma}+O(\sigma^2)}+O(\sigma)=$

$$=\log\frac{1}{(1-e^{-\sigma})(1+O(\sigma))}+O(\sigma)=\log\frac{1}{1-e^{-\sigma}}+O(\sigma)$$

Let $H(x)=\sum_{n\leq x}\frac{1}{n}$, $P(x)=\sum_{p\leq x}\frac{1}{p}$.

So $\log\zeta(1+\sigma)=-\log(1-e^{-\sigma})+O(\sigma)=\sum_{n=1}^{\infty}\frac{1}{ne^{n\sigma}}+O(\sigma)=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{ne^{n\sigma}}+O(\sigma)$

$$\overset{\text{partial summation}}{=}\lim_{N\to\infty}\left(\frac{H(N)}{e^{\sigma N}}+\sigma\int_0^N e^{-\sigma t}H(t)\,dt\right)+O(\sigma)=\sigma\int_0^\infty e^{-\sigma t}H(t)\,dt+O(\sigma).$$

Also, $\sum_p \frac{1}{p^{1+\sigma}} = \sigma \int_1^\infty u^{-1-\sigma} P(u)\,du \underset{u=e^t}{=} \sigma \int_0^\infty e^{-\sigma t} P(e^t)\,dt$.

Then $f(\sigma) = \sigma \int_0^\infty e^{-\sigma t}\left(H(t) - P(e^t)\right) dt + O(\sigma)$.

$$= \sigma O(1) + \int_1^\infty e^{-\sigma t}\left(\log t + \gamma + O(\tfrac{1}{t}) - \left(\log t + B + O(\tfrac{1}{t})\right)\right) dt + O(\sigma)$$

$$= \sigma \int_1^\infty e^{-\sigma t}\left(\gamma - B + O(\tfrac{1}{t})\right) dt + O(\sigma) = \sigma\left(\frac{\gamma - B}{\sigma}(e^{-\sigma})\right) + O\left(\sigma + \int_1^\infty \tfrac{1}{t} e^{-\sigma t} dt\right)$$

To estimate: $\int_1^\infty \frac{1}{t} e^{-\sigma t}\,dt \leq \int_1^{1/\sigma} \frac{dt}{t} + \frac{1}{\frac{1}{\sigma}} \int_{\frac{1}{\sigma}}^\infty e^{-\sigma t}\,dt = \log\frac{1}{\sigma} + \frac{1}{e}$

Thus the O becomes $O(\sigma \log \frac{1}{\sigma})$ for, say $0 < \sigma < \frac{1}{10}$.

So $f(\sigma) = \gamma - B + O(\sigma \log \frac{1}{\sigma}) \xrightarrow[\sigma \to 0^+]{} \gamma - B$.

So $-A = \gamma - B \Rightarrow B - A = \gamma$.

We also had $\prod_{p \leq x}(1 - \frac{1}{p}) = \frac{e^{A-B}}{\log x}\left(1 + O(\frac{1}{\log x})\right)$, the result follows. //

Attempt to prove PNT:

By Euler product, $\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s})$ $(\mathrm{Re}\, s > 1)$.

So $\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$. As $s \to 1^+$, $\frac{1}{\zeta(s)} \overset{\text{w.l.}}{\to} 0$ and therefore

$\lim_{s \to 1^+} \sum_{n \geq 1} \frac{\mu(n)}{n^s} = 0$.

Recall: PNT $\Leftrightarrow \sum_{n=1}^\infty \frac{\mu(n)}{n} = 0$

Does $\lim_{s \to 1^+} \sum_{n=1}^\infty \frac{\mu(n)}{n^s} = 0 \Rightarrow \sum_{n=1}^\infty \frac{\mu(n)}{n} = 0$??

Analogy: Consider $f(x) = \sum_{n=0}^\infty x^n = \frac{1}{1-x}$ (valid for $|x| < 1$).

$\lim_{x \to -1} f(x) = \frac{1}{2}$. But $\sum (-1)^n$ doesn't converge!

Dirichlet Series

Def A Dirichlet Series is a series $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$. Write $s = \sigma + it$

Theorem 4.1: Let f, g be arithmetic functions, with D.S. $F(s)$, $G(s)$.

Let $h = f * g$ with D.S. $H(s)$.

Then: If $F(s)$ and $G(s)$ are absolutely convergent at s, then so is $H(s)$

and $H(s) = F(s) \cdot G(s)$.

Pf
$$F(s)G(s) = \sum_{k=1}^{\infty} \frac{f(k)}{k^s} \sum_{\ell=1}^{\infty} \frac{g(\ell)}{\ell^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k\ell=n} f(k)g(\ell) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = H(s).$$

(Note: $H(s)$ is abs. convergent. $\sum_{n\geq 1}^{N} \frac{|h(n)|}{|n^s|} = \sum_{n=1}^{N} \frac{1}{n^\sigma} \left| \sum_{k\ell=n} f(k)g(\ell) \right| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} \sum_{k\ell=n} |f(k)||g(\ell)| =$

$$= \sum_{k\ell \leq N} \left|\frac{f(k)}{k^\sigma}\right| \left|\frac{g(\ell)}{\ell^\sigma}\right| \leq \sum_{k\leq N} \left|\frac{f(k)}{k^\sigma}\right| \sum_{\ell \leq N} \left|\frac{g(\ell)}{\ell^\sigma}\right|$$

Corollary 4.2: If f, g have DS given by $F(s)$, $G(s)$ and $g = f^{-1}$ (wrt $*$). Then: $G(s) = \frac{1}{F(s)}$ wherever both $F(s)$ and $G(s)$ both converge absolutely.

Pf Let $E(s) = \sum_{n=1}^{\infty} \frac{e(n)}{n^s} = 1$ (converges abs. unif. everywhere). Then thm 4.1 ⇒ $E(s) = F(s)G(s)$

Remark: the absolute conv. of $F(s) \not\Rightarrow$ abs. convergence of $G(s)$!

Eg: $f(1) = 1$, $f(2) = -1$, $f(n) = 0$ o.w.

Here $F(s) = 1 - 2^{-s}$. But $\frac{1}{1-2^{-s}} = \sum_{n\geq 0} \frac{1}{2^{ns}} = G(s)$ only converges for $\underset{\text{Re}(s)}{\sigma} > 0$.

Examples:

1) $f(n) = 1 \to F(s) = \zeta(s)$
2) $f(n) = \mu(n) \to F(s) = 1/\zeta(s)$

} Re $(s) > 1$

3) $f(n) = n \Rightarrow F(s) = \sum_{s\geq 1} \frac{1}{n^{s-1}} = \zeta(s-1)$ (Re $(s) > 2$).

More examples:

- $f(n)=\phi(n)$. As $\phi = id * \mu$,
$$F(s) = \zeta(s-1)\cdot\frac{1}{\zeta(s)} \qquad (\sigma>2)$$
- $f(n)=d(n)$. As $d=1*1$, $F(s)=\zeta(s)^2 \quad (\sigma>1)$.
- $f(n)=d_k(n) = \#\{$ decomposition of n as product of k factors $\} = 1 \overset{k}{*\cdots*} 1 \Rightarrow F(s)=\zeta(s)^k \quad (\sigma>1)$.
- Let $f(n) = \begin{cases} 1 & \text{if } n=m^2 \text{ for some } m \\ 0 & \text{else.} \end{cases}$

Then $\displaystyle\sum_{n\geq 1}\frac{f(n)}{n^s} = \sum_{m\geq 1}\frac{1}{m^{2s}} = \zeta(2s) \quad (\sigma>\tfrac{1}{2})$.

- Let $f(n)=\mu^2(n)$ (char. function on squarefree numbers): as $1 = \displaystyle\sum_{d^2|n}\mu^2(d^2)$,

Can think $1=\displaystyle\sum_{d^2|n}\mu^2(d^2) = \mu^2 * g$, where $g = \begin{cases} 1 & n=m^2 \text{ some } m \\ 0 & \text{else} \end{cases}$

Then $\zeta(s) = F(s)\cdot\zeta(2s) \Rightarrow F(s) = \dfrac{\zeta(s)}{\zeta(2s)} \quad (\sigma>1)$.

Logarithm: Let $f(n)=\log n$ (non-multiplicative!)
$$F(s) = \sum_{n\geq 1}\frac{f(n)}{n^s} = \sum_{n\geq 1}\frac{\log n}{n^s}.$$
Consider $\displaystyle\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \sum_{n\geq 1} e^{-s\log n} \text{ "}\Rightarrow\text{" } \zeta'(s) = -\sum_{n\geq 1}\frac{\log n}{n^s}\text{"}$

The series $\sum\frac{1}{n^s}$ is absolutely convergent in $\sigma>1$, but is actually uniformly convergent in $\sigma\geq 1+\varepsilon$ for any $\varepsilon>0$ $\left(\overset{\zeta(1+\varepsilon)}{\sum}\frac{1}{n^{1+\varepsilon}} \geq \sum\left|\frac{1}{n^s}\right|\right)$.

Thus, $\displaystyle\sum_{n\geq 1}\frac{1}{n^s}$ is analytic in $\sigma>1$, and termwise differentiation is justified.

So, in fact, it is true that $\displaystyle\zeta'(s) = -\sum\frac{\log n}{n^s}$.

Let now $f(n)=\Lambda(n)$. Then $1*\Lambda=\log \Rightarrow \Lambda=\mu*\log$.

Therefore, $\boxed{\displaystyle\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}=-\frac{\zeta'(s)}{\zeta(s)}}$.

General Euler Products

Let f be multiplicative. Then $F(s)=\displaystyle\sum_{n=1}^{\infty}\frac{f(n)}{n^s} \text{ "=" } \prod_p\left(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\dots\right)$. ← Euler product of $F(s)$

Recall: $\displaystyle\prod_n(1+a_n)$ converges absolutely means $\displaystyle\prod_n(1+|a_n|)$ converges $\Leftrightarrow \displaystyle\sum_{n\geq 1}|a_n|$ converges.

Q: is $F(s)$ equal to its Euler product?

Theorem 4.5:

i) If $F(s)$ converges absolutely, then so does its Euler product, and the two are equal.

ii) $\displaystyle\sum_{n\geq 1}\frac{f(n)}{n^s}$ converges absolutely $\Leftrightarrow \displaystyle\sum_{p,\,m\geq 1}\left|\frac{f(p^m)}{p^{ms}}\right|<\infty$.

↑ prime powers

Proof:

(i) Follows from HW set #1.

(ii) Assume $\displaystyle\sum_{p,m}\left|\frac{f(p^m)}{p^{ms}}\right|<\infty$. Let $N\geq 1$. Then by unique factorization,

$$\sum_{n=1}^{N}\left|\frac{f(n)}{n^s}\right|\leq\sum_{\substack{n\text{ s.t.}\\ p|n\Rightarrow p\leq N}}\left|\frac{f(n)}{n^s}\right|=\sum\left|\frac{f(n)\,g_N(n)}{n^s}\right| \quad\text{where } g(n)=\begin{cases}1 & \text{if } p|n\Rightarrow p\leq N\\ 0 & \text{else.}\end{cases}$$

↑ multiplicative ↑ multiplicative

By HW, $\displaystyle\sum\left|\frac{f(n)\,g(n)}{n^s}\right|=\prod_{p\leq N}\left(1+\left|\frac{f(p)}{p^s}\right|+\left|\frac{f(p^2)}{p^{2s}}\right|+\dots\right)$.

Since $\displaystyle\sum_{p,m}\left|\frac{f(p)}{p^{ms}}\right|<\infty$, the product over all primes converges.

Examples:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + p^{-s} + p^{-2s} + \cdots\right)$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)$$

As
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \overset{\sigma > 2}{=} \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p(p-1)}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \frac{p-1}{p^s - 1}\right) =$$

$$= \prod_p \frac{p^s - 1}{p^s - p} = \prod_p \left(\frac{1-p^{-s}}{1-p^{1-s}}\right) = \prod (1-p^{-s}) \cdot \prod (1 - p^{1-s})^{-1} = \frac{\zeta(s-1)}{\zeta(s)}.$$

Example: ($\omega(n)$ = # distinct prime factors of n)

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \frac{2}{p^s - 1}\right) = \prod_p \left(\frac{p^s+1}{p^s-1}\right) = \prod_p \left(\frac{1+p^{-s}}{1-p^{-s}}\right)$$

$$= \zeta(s) \overbrace{\prod_p (1 + p^{-s})}^{\text{Euler prod for Dirichlet ser. for } \mu^2}$$ (so already derive $2^{\omega} = 1 * \mu^2$).

$$1 + x = \frac{1-x^2}{1-x}$$

Doing more, $\cancel{\zeta(s)}$ $$\prod_p \left(\frac{1-p^{-2}}{1-p^{-s}}\right) = \zeta(s) / \zeta(2s) \Rightarrow \sum \frac{2^{\omega(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}.$$

So if $F(s) = \sum \frac{2^{\omega(n)}}{n^s}$, then $\zeta(2s) \cdot F(s) = \zeta(s)^2 \Rightarrow g * 2^{\omega} = d$

where g = char. function of squares.

Rk/ If f is <u>completely multiplicative</u>, and $F(s)$ converges absolutely, then:

$$F(s) = \prod_p \left(1 + f(p) p^{-s} + f(p^2) p^{-2s} + \cdots\right) = \prod_p \left(1 - f(p) p^{-s}\right)^{-1}.$$

• Region of Convergence (ROC)

Recall If we have a power series $p(z) = \sum a_n z^n$

then $\exists\, r \in [0,\infty]$ s.t $\begin{cases} p(z) \text{ converges absolutely for } |z|<r \\ p(z) \text{ diverges for } |z|>r. \end{cases}$

Given a D.s. $F(s) = \sum_{n\geq 1} a_n n^{-s}$, define $\sigma_c, \sigma_a \in \mathbb{R} \cup \{\pm\infty\}$ by:

$\sigma_c := \inf\{\sigma : F(\sigma) \text{ converges}\}$. ← abcissa of convergence.

$\sigma_a := \inf\{\sigma : F(\sigma) \text{ converges absolutely}\}$ ← abcissa of absolute convergence.

Clearly, $\sigma_a \geq \sigma_c$. Also, $|n^{-\sigma-it}| = n^{-\sigma}$. so the imaginary part doesn't matter:

<u>Theorem 4.6</u>: Let $s = \sigma + it \in \mathbb{C}$, $F(s)$ a D.s. If $\sigma > \sigma_a$, then $F(s)$ converges abs. If $\sigma < \sigma_a$, then $F(s)$ does not (not absolutely, at least).

Pf $\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right| = \sum_{n=1}^{\infty} \left|\frac{a_n}{n^\sigma}\right|$. Thus $F(\sigma)$ and $F(s)$ either $\begin{cases} \text{both converge abs.} \\ \text{both don't.} \end{cases}$

Next, if $\sigma' < \sigma$, then $\sum_1^\infty \frac{|a_n|}{n^\sigma} < \sum_1^\infty \frac{|a_n|}{n^{\sigma'}}$, so if $F(\sigma')$ converges absolutely, then $F(\sigma)$ does so.

<u>Note</u>: on the vertical line $\sigma = \sigma_a$, either $F(s)$ converges abs. $\forall\, s = \sigma_a + it$, or $F(s)$ does not conv. absolutely for any of the values $s = \sigma_a + it$.

<u>Example</u>:

1) $\zeta(s) = \sum n^{-s}$, $\sigma_a = 1$. Div as $\sigma = 1$, so not abs conv. for $\sigma = 1$.

2) $F(s) = \sum \frac{1}{\log^2(2n)\, n^s}$ abs convergent for $\sigma = 1$ (and $\sigma_a = 1$) ← comparison with the harmonic series.

Define also $\sigma_a = \begin{cases} +\infty & \text{if } F(\sigma) \text{ does not converge absolutely, for all real } \sigma. \\ -\infty & \text{if } F(\sigma) \text{ converges absolutely } \forall \text{ real } \sigma. \end{cases}$

Example:

3) $\sum n^n n^{-s}$ $\quad \sigma_a = +\infty$

4) $\sum n^{-n} n^{-s}$ $\quad \sigma_a = -\infty$

5) f bounded function. Then $\sum \frac{f(n)}{n^s}$ has $\sigma_a \leq 1$. (comparison with $\zeta(s)$)

Example:

Consider $\sum \frac{(-1)^n}{n^s}$. Then obviously $\sigma_a = 1$. But $\sigma_c = 0$.

Theorem 4.7: Suppose $F(s) = \sum f(n) n^{-s}$ converges at $s_0 = \sigma_0 + it_0$. Then it converges at every points with larger real part ($\sigma > \sigma_0$).

Moreover, the convergence is uniform in compact subsets of $\{s : \sigma > \sigma_0\}$.

Proof:

Let $s = \sigma + it$, $\sigma > \sigma_0$. Let $\delta = \sigma - \sigma_0 > 0$.

Let $S(x,y) := \sum_{y < n \leq x} f(n) n^{-s}$, $\quad S_0(x,y) := \sum_{y < n \leq x} f(n) n^{-s_0}$

By partial summation, as $f(n) n^{-s} = \left(f(n) n^{-s_0}\right) n^{s_0 - s}$, so:

$$S(x,y) = \int_y^x t^{s_0 - s} \, dS_0(t,y) = \frac{S_0(x,y)}{x^{s-s_0}} + (s - s_0) \int_y^x S_0(t,y)\, t^{-1-(s-s_0)} dt$$

By Cauchy's criterion, $\forall \varepsilon \; \exists \, y_0 = y_0(\varepsilon)$ so that when $y_0 \leq y < x$; $|S_0(x,y)| \leq \varepsilon$.

If x, y are s.t. $y_0 \leq y < x$, then:

$$|S(x,y)| \leq \frac{\varepsilon}{|x^{s-s_0}|} + |s - s_0| \cdot \varepsilon \underbrace{\int_y^x \left| t^{-1-(s-s_0)} \right| dt}_{\leq \int_y^\infty (\,)\, dt \,=\, \frac{1}{\delta y^\delta}} =$$

$$\leq \varepsilon \left(\frac{1}{x^\delta} + \frac{|s - s_0|}{\delta \cdot y^\delta} \right) \underset{y \geq 1}{\leq} \varepsilon \left(1 + \frac{|s-s_0|}{\delta} \right)$$

Since $1 + \frac{|s-s_0|}{\delta}$ is indep. of x, y, ε, we get $S(x,y)$ satisfies also the Cauchy criterion $\Rightarrow$ ok.

(cont'd)

<u>Uniform convergence</u>: if K is a compact set of $\{s : \sigma > \sigma_0\}$, then for $s \in K$,

have ~~...~~ $|s - s_0| \leq M$, and $|\delta| \geq \eta$.

Thus $1 + \frac{|s-s_0|}{\delta} \leq 1 + \frac{M}{\eta} \Rightarrow$ bounded on K.

<u>Theorem 4.8</u>: Let $F(s)$ be a D.S. If $\sigma > \sigma_c$, then $F(s)$ converges.

If $\sigma < \sigma_c$, $F(s)$ diverges. The convergence is uniform in compacts of $\{s : \sigma > \sigma_c\}$

Furthermore, $\sigma_c \overset{\text{obvious}}{\leq} \sigma_a \leq \sigma_c + 1$.

Pf. If $\sigma > \sigma_c$, then $F\left(\frac{\sigma + \sigma_c}{2}\right)$ converges. By thm 4.7, $F(s)$ converges.

Likewise, if $\sigma < \sigma_c$. If $F(s)$ converges, then $F\left(\frac{\sigma + \sigma_c}{2}\right)$ would also converge,

which contradicts the def. of σ_c.

Just need to prove $\sigma_a \leq \sigma_c + 1$.

If $F(\sigma_0)$ converges, then $|f(n) n^{-\sigma_0}| \to 0$ as $n \to \infty$.

So if $\sigma > \sigma_0 + 1$, then $\displaystyle\sum_{n \leq N} \left|\frac{f(n)}{n^\sigma}\right| = \sum_{n \leq N} \left|\frac{f(n)}{n^{\sigma_0}}\right| \left|\frac{1}{n^{\sigma - \sigma_0}}\right| << \sum_{n \leq N} \frac{1}{n^{\sigma - \sigma_0}} \leq \zeta(\sigma - \sigma_0)$

Therefore $\sigma_a \leq \sigma_c + 1$ (take $\sigma_0 \to \sigma_c^+$, $\sigma \to \sigma_c + 1$).

<u>Example</u>:

$\sum \frac{f(n)}{n^s}$, $f(n) \in \{\pm 1\}$, $\sigma_a = 1$, $\sigma_c \geq 0$.

By appropriate f, can make σ_c any value between 0 and 1. (exercise).

<u>Analyticity</u>:

<u>Thm 4.9</u>: A D.S $F(s) = \sum_{n \geq 1} f(n) n^{-s}$ represents an analytic function in $\{s : s > \sigma_c\}$

In this region, the series may be differentiated termwise, i.e.

$$F'(s) = -\sum_{n=1}^{\infty} (\log n) f(n) n^{-s}$$

Proof: Let $F_N(s) = \sum_{n=1}^{N} f(n) n^{-s}$. Each $F_N(s)$ is entire, and $F_N(s) \to F(s)$ uniformly on compact subsets of $\{s : \sigma > \sigma_c\}$.

By a theorem of Weierstrass, the result follows. (Similar with $F_N'(s) \to F'(s)$)

Rk: σ_c and σ_a for $F'(s)$ are the same for $F(s)$.

Examples:

1) $\zeta(s) = \sum n^{-s}$ is analytic for $\sigma > 1$.

2) $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ is analytic for $\sigma > 1$. $(\sigma_a = 1)$

Also, $\sum_{n=1}^{\infty} \mu(n) n^{-s} = \frac{1}{\zeta(s)}$, so $\zeta(s) \neq 0$ for $\sigma > 1$. (we know that, as the Euler prod converges.)

$$\boxed{RH \Leftrightarrow \sigma_c = \frac{1}{2}}$$

3) If f bounded, then $\sum_{n=1}^{\infty} f(n) n^{-s}$ is analytic for $\sigma > 1$.

4) Let $\sigma > 1$. Then

$\sigma_a = 1$, has $\sigma_c = 0$

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} 2^{1-s} n^{-s} = \sum_{n=1}^{\infty} n^{-s} - 2 \cdot (2n)^{-s} =$$

$$= 1^{-s} + 2^{-s} + 3^{-s} + \cdots \quad - 2\left(2^{-s} + 4^{-s} + \cdots\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} =: G(s)$$

Therefore, $G(s)$ is analytic for $\sigma > 0$. On $\{\sigma > 1\}$, $G(s) = (1 - 2^{1-s})\zeta(s)$

Thus, the series for $G(s)$ provides an analytic continuation of $(1 - 2^{1-s})\zeta(s)$ to the half-plane $\{\sigma > 0\}$.

$\therefore$ Define $\zeta(s) := \frac{G(s)}{1 - 2^{1-s}}$ for $\sigma > 0$, $\overbrace{2^{1-s} \neq 1}^{s=1}$

~~*Proof*: Consider the identity $\sqrt{2}^{(\sqrt{2}+1)} = (\sqrt{2}^{\sqrt{2}})\sqrt{2}$.~~

~~If $\sqrt{2}^{\sqrt{2}}$ is irrational then we are finished. If not, then $\sqrt{2}^{\sqrt{2}}$ is rational. Hence $(\sqrt{2}^{\sqrt{2}})\sqrt{2}$ is irrational, and $\sqrt{2}^{(\sqrt{2}+1)}$ is the example in this case.~~

~~There is also a simple identity by means of which it can be proved that a rational number raised to an irrational power may be irrational. But perhaps the reader would enjoy finding this one himself.~~

~~**References**~~

~~1. R. Kuzmin, On a new class of transcendental numbers, Izv. Akad. Nauk SSSR, Ser. Mat., 7 (1930) 585–597.~~
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A SIMPLE PROOF OF THE FORMULA $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$

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Start with the inequality $\sin x < x < \tan x$ for $0 < x < \pi/2$, take reciprocals, and square each member to obtain

$$\cot^2 x < 1/x^2 < 1 + \cot^2 x.$$

Now put $x = k\pi/(2m+1)$ where k and m are integers, $1 \leqq k \leqq m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}. \tag{1}$$

But since we have

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}, \tag{2}$$

(a proof of (2) is given below) relation (1) gives us

$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \frac{m(2m-1)}{3}.$$

Multiply this relation by $\pi^2/(4m^2)$ and let $m \to \infty$ to obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Proof of (2). By equating imaginary parts in the formula

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = \sin^n\theta(\cot\theta + i)^n$$

$$= \sin^n\theta \sum_{k=0}^{n} \binom{n}{k} i^k \cot^{n-k}\theta,$$

we obtain the trigonometric identity

$$\sin n\theta = \sin^n\theta \left\{ \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - + \cdots \right\}.$$

Take $n = 2m + 1$ and write this in the form

(3) $$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m(\cot^2\theta) \text{ with } 0 < \theta < \frac{\pi}{2},$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - + \cdots .$$

Since $\sin\theta \neq 0$ for $0 < \theta < \pi/2$, equation (3) shows that $P_m(\cot^2\theta) = 0$ if and only if $(2m+1)\theta = k\pi$ for some integer k. Therefore $P_m(x)$ vanishes at the m distinct points $x_k = \cot^2 \pi k/(2m+1)$ for $k = 1, 2, \cdots, m$. These are all the zeros of $P_m(x)$ and their sum is

$$\sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = \binom{2m+1}{3} \Big/ \binom{2m+1}{1} = \frac{m(2m-1)}{3},$$

which proves (2).

NOTE. This paper was translated from a Greek manuscript and communicated to the MONTHLY on behalf of the author by Tom M. Apostol, California Institute of Technology. After this paper was written it was learned that the same proof was discovered independently and published in Norwegian by Finn Holme in Nordisk Matematisk Tidskrift, vol. 18 (1970), pp. 91–92. See also A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions*, vol. II, Holden-Day, San Francisco, 1967, problem 145.

ANOTHER ELEMENTARY PROOF OF EULER'S FORMULA FOR $\zeta(2n)$

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1. Introduction. The classic formula

(1) $$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

which expresses $\zeta(2n)$ as a rational multiple of π^{2n} was discovered by Euler [**2**]. The numbers B_n are Bernoulli numbers and can be defined by the recursion formula

$$B_0 = 1, \ B_n = \sum_{s=0}^{n} \binom{n}{s} B_s \text{ for } n \geqq 2,$$

or equivalently, as the coefficients in the power series expansion

Theorem 4.10 (Uniqueness): Suppose $F(s) = \sum f(n) n^{-s}$, $G(s) = \sum g(n) n^{-s}$

are D.S., and spz they converge for some $s \in \mathbb{C}$.

If $F(s_k) = G(s_k)$ for a sequence of complex numbers $\{s_k\}$ with $\sigma_k \to \infty$

Then: $f(n) = g(n)$ $\forall n \geq 1$.

Pf Let $h(n) = f(n) - g(n)$, $H(s) = \sum_{n\geq 1} h(n) n^{-s}$. Assume n_0 is the smallest integer with $h(n_0) \neq 0$.

Then $H(s) = \dfrac{h(n_0)}{n_0^s} + \sum_{n > n_0} \dfrac{h(n)}{n^s}$

So $h(n_0) = -n_0^{s_k} \sum_{n > n_0} \dfrac{h(n)}{n^{s_k}}$ $(k \geq 1)$.

Let c be larger than the σ_a for F and the σ_a for G.

If k is large, then $\sigma_k = \operatorname{Re} s_k > c$. For such k,

$$|h(n_0)| \leq n_0^{\sigma_k} \sum_{n > n_0} |h(n)| n^{-\sigma_k} \leq (n_0+1)^c \left(\frac{n_0}{n_0+1}\right)^{\sigma_k} \sum_{n > n_0} \frac{|h(n)|}{n^c} \ll \left(\frac{n_0}{n_0+1}\right)^{\sigma_k}$$

$\left(\text{write } n^{-\sigma_k} = \dfrac{1}{n^c} \cdot \dfrac{1}{n^{\sigma_k - c}} \leq \dfrac{1}{n^c} \dfrac{1}{(n_0+1)^{\sigma_k - c}} \right)$

Now let $k \to \infty \Rightarrow \sigma_k \to \infty \Rightarrow |h(n_0)| \leq 0 \Rightarrow h(n_0) = 0 \Rightarrow$!!

Corollary 4.11: Suppose $F(s) = \sum f(n) n^{-s}$, $G(s) = \sum g(n) n^{-s}$, $H(s) = \sum h(n) n^{-s}$, and $H(s) = F(s) \cdot G(s)$ for $\sigma > \sigma_1$, and all three converge ~~uniformly~~ absolutely for $\sigma > \sigma_1$. Then $h = f * g$.

Pf Let $k = f * g$, $K(s) = \sum k(n) n^{-s}$. Then, by thm 4.1, $K(s)$ converges absolutely for $\sigma > \sigma_1$, and $K(s) = F(s) G(s)$. So $K(s) = H(s)$ for $\sigma > \sigma_1$. Hence $k = h$ by thm 4.10.

Application - example :

1) Let $s(n)$ = char. function on squares. Then $\sum \frac{s(n)}{n^s} = \zeta(2s)$

As we saw, $\sum \frac{2^{\omega(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}$ $\qquad \sum d(n) n^{-s} = \zeta(s)^2$

Therefore, $\zeta(s)^2 = \left(\sum 2^{\omega(n)} n^{-s}\right)\left(\sum s(n) n^{-s}\right) \implies \boxed{2^{\omega} * s = d.}$

2) Computing inverses: Fix $k \in \mathbb{N}$, $k \geq 2$, let $f(1)=1$, $f(k)=-1$, $f(n)=0$ o.w.

As $F(s) = \sum f(n) n^{-s} = 1-k^{-s} \Rightarrow \frac{1}{F(s)} = \frac{1}{1-k^{-s}} \underset{\sigma>0}{=} \sum_{n\geq 0} k^{-sn} = \sum_{n\geq 0}\left(\frac{1}{k^s}\right)^n = \sum_{m=1}^{\infty} \frac{g(m)}{m^s}$

(where $g(m)=1$ only at k^{th} powers).

$\therefore$ $g = f^{-1}$

3) Convolution "square-roots"

Suppose that f is completely multiplicative. Then $F(s) = \prod_p (1-f(p)p^{-s})^{-1}$ $\quad (\sigma > \sigma_a^{\#})$ assume it.

Take a "formal square root": as $(1-x)^{1/2} = \sum_{n\geq 0} \binom{-1/2}{n}(-x)^n$,

$F(s)^{1/2} = \prod_p \left(1 + \sum_{n=1}^{\infty} \binom{-1/2}{n}\left(-f(p)p^{-s}\right)^n\right)$ = Euler product of a mult. function g,

with $g(p^n) = \binom{-1/2}{n}(-f(p))^n$. $\rightarrow$ $f = g * g$.

Theorem 5.1: $\zeta(s)$ can be analytically continued to $\sigma > 0$ except for a simple pole at $s=1$.

In this half-plane, $\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\,dx$

Pf When $\sigma > 1$, by partial summation.

$$\zeta(s) = \sum n^{-s} = 1 + \underbrace{\int_1^{\infty} \frac{dx}{x^s}}_{\frac{1}{s-1}} - s \cdot \int_1^{\infty} \frac{\{x\}}{x^{s+1}}\,dx.$$

Note that $\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\,dx$ is abs. and unif. convergent for $\sigma \geq \varepsilon$, $\forall \varepsilon > 0 \Rightarrow$ analytic for $\sigma > 0$.

Thus the RHS is meromorphic on $\sigma > 0$, giving (by uniqueness thm) a continuation to $\{\sigma > 0\}$.

Corollary: In $|s-1| \leq \frac{1}{2}$, then $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$.

Pf $\zeta(s) - \frac{1}{s-1}$ is analytic in $|s-1| \leq \frac{1}{3} \Rightarrow$ has a Taylor expansion about $s=1$.

From lemma 4.1, the first term has to be γ. (we know it for real s!)

Theorem 5.2: $\zeta(1+it) \neq 0$ for real $t \neq 0$.

Pf Note that in a nbhd of $t=0$, this is true, as ζ has a pole at $s=1$.

First we prove that:

(*) $\zeta^3(\sigma) \cdot |\zeta(\sigma+it)|^4 \cdot |\zeta(\sigma+2it)| \geq 1 \quad \forall \sigma > 1$.

Trick: use that we have an Euler product: $s = \sigma + it$:

$$\log \zeta(s) = -\sum_p \log(1-p^{-s}) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_p \sum_{m=1}^{\infty} \frac{e^{-itm\log p}}{mp^{m\sigma}}$$

Take real parts:

$$\log|\zeta(s)| = \sum_p \sum_m \frac{\cos(mt\log p)}{mp^{m\sigma}}$$

Thus:

$$3\log\zeta(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+it)| = \sum_p \sum_m \frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}}$$

$$= \sum_p \sum_m \frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}}.$$

Since $3+4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0$, so done.

Now, to get a contradiction, suppose that $\zeta(1+it)=0$, $t \neq 0$. Let $\sigma > 1$.

By (*), $\left((\sigma-1)\zeta(\sigma)\right)^3 \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\zeta(\sigma+2it)| \geq \frac{1}{\sigma-1}$.

As $\sigma \to 1^+$, $(\sigma-1)\zeta(\sigma) \to 1$ by thm 5.1; $|\zeta(\sigma+2it)| \to |\zeta(1+2it)| \in \mathbb{C}$,

and $\frac{\zeta(\sigma+it)}{\sigma-1} = \frac{\zeta(\sigma+it)-\zeta(1+it)}{\sigma-1} \to \zeta'(1+it) \in \mathbb{C}$.

But RHS ~~has a~~ $\to \infty$ when $\sigma \to 1^+$, so contradiction.

Theorem 5.3 (Ingham, 1935):

Suppose $|f(n)| \leq 1$, $F(s) = \sum f(n) n^{-s}$ is analytic for $\sigma > 1$.

If $F(s)$ can be analytically continued to $\sigma \geq 1$, then the series $\sum f(n) n^{-s}$ converges to $F(s)$ for $\underline{\underline{\sigma \geq 1}}$.

Remark: Thm 5.3 $\Rightarrow$ PNT: $f(n) := \mu(n)$. Then $F(s) = \sum \mu(n) n^{-s} = \frac{1}{\zeta(s)}$ $(\sigma > 1)$.

Know $\zeta(1+it) \neq 0$ $\forall t \in \mathbb{R}$.

Therefore, $\zeta(s) \neq 0$ for $\sigma \geq 1 \Rightarrow \frac{1}{\zeta(s)}$ is analytic for $\sigma \geq 1$.

So $\sum \frac{\mu(n)}{n} = 0. \Rightarrow$ PNT by Thm (3.11).

Pf of 5.3 (proof due to D. J. Newman, 1980):

Fix w with $\mathrm{Re}\, w \geq 1$. So $F(z+w)$ is analytic for $\mathrm{Re}\, z \geq 0$.

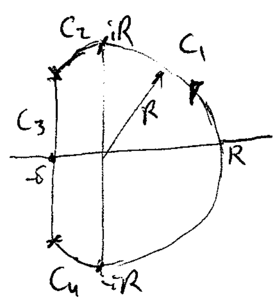
Let $R \geq 1$, $I_R :=$ segment $-iR$ to $+iR$.

Since $F(z+w)$ is analytic on I_R, then $F(z+w)$ is analytic on some rectangle

$$I_R(\eta) = \{ z : |\mathrm{Re}\, z| \leq \eta, \ |\mathrm{Im}\, z| \leq R \}. \qquad (\eta = \eta(R)).$$

Since $I_R(\eta)$ is compact, $\exists M$ s.t. $\max\limits_{z \in I_R(\eta)} |F(z+w)| = M$

Choose δ, $0 < \delta < \eta$ (will fix it later), and form a contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$.



Let $N \in \mathbb{N}$, $N > \frac{1}{\delta}$. By Residue Thm,

$$(1) \quad \oint_C F(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz = 2\pi i \, F(w).$$

Note: N^z and $\frac{z}{R^2}$ don't affect the residue, but will aid the estimation on C.

Let $S_N(s) := \sum_{n=1}^{N} f(n) n^{-s}$, $T_N(s) = \sum_{n=N+1}^{\infty} f(n) n^{-s}$ (tail).

$\tilde{C}_1$ is the reflection of C_1 along Im axis.

S_N is entire, so $\oint_{C_1} S_N(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz = 2\pi i \, S_N(w) - \oint_{\tilde{C}_1} S_N(z+w) N^z \left(\frac{1}{z} + \frac{z}{R^2} \right) dz$

$= 2\pi i S_N(w) - \int_{C_1} S_N(w-z) N^{-z} \left(\frac{1}{z} + \frac{z}{R^2} \right) dz \qquad (2).$

(cont pf)

On C_1, $F(z+w) = S_N(z+w) + T_N(z+w)$.

Substract (2) from (1), to get:

$$(3)\ 2\pi i\left(F(w) - S_N(w)\right) = \oint_{C_1}\left(T_N(z+w)N^z - \frac{S_N(w-z)}{N^z}\right)\left(\frac{1}{z}+\frac{z}{R^2}\right)dz + \oint_{C_2\cup C_3\cup C_4} F(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz$$

The goal is to show that LHS of (3) $\to 0$ when $N\to\infty$. We'll show that RHS $\to 0$.

Write $x = \mathrm{Re}(z)$.

(4) On $|z| = R$, $\frac{1}{z} + \frac{z}{R^2} = \frac{2x}{R^2}$

(5) If $x = -\delta$, $|z| \leq R$, $\left|\frac{1}{z} + \frac{z}{R^2}\right| \leq \left|\frac{1}{z}\right| + \left|\frac{z}{R^2}\right| \leq \frac{1}{\delta} + \frac{1}{R}$

(6) If $x > 0$, $|T_N(z+w)| \leq \sum_{N+1}^{\infty} \frac{1}{n^{x+1}}$ as $|\zeta(n)| \leq 1$.

So $|T_N(z+w)| \leq \int_N^\infty \frac{dt}{t^{x+1}} = \frac{1}{xN^x}$ (Thm 2.18).

(7) If $x > 0$, $|S_N(w-z)| \leq \sum_{n=1}^{N} n^{x-1} \leq N^{x-1} + \int_0^N t^{x-1}\,dt = N^{x-1} + \frac{N^x}{x}$

By (4), (6), (7), if $z \in C_1$ then:

$$\left|\left(T_N(w+z)N^z - \frac{S_N(w-z)}{N^z}\right)\left(\frac{1}{z}+\frac{z}{R^2}\right)\right| \leq \left|\frac{1}{x} + \frac{1}{N} + \frac{1}{x}\right| \cdot \frac{2x}{R^2} \leq \frac{4}{R^2} + \frac{2x}{R^2 N} \underset{x\leq R}{\leq} \frac{4}{R^2} + \frac{2}{RN}$$

Thus, $\left|\int_{C_1}(\cdots)\,dz\right| \leq \pi R\left(\frac{4}{R^2} + \frac{2}{RN}\right) = \frac{4\pi}{R} + \frac{2\pi}{N}$.

By (4) again, the length of each of arcs $\leq 2\delta$ if $\delta << R$.

$$\left|\int_{C_2\cup C_4} F(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz\right| \leq 2\delta\cdot 2\cdot M\cdot 1\cdot\frac{2x}{R^2} \underset{-\delta<x<0}{\leq} \frac{8\delta^2 M}{R^2}$$

By (5), $\left|\int_{C_3} F(z+w)N^z\left(\frac{1}{z}+\frac{z}{R^2}\right)dz\right| \leq 2R\cdot M\cdot N^{-\delta}\left(\frac{1}{\delta}+\frac{1}{R}\right) \leq \frac{4RM}{\delta N^\delta}$ assume $\delta < \frac{1}{R}$

We get $|F(w) - S_N(w)| \leq \left(\frac{2}{R} + \frac{1}{N}\right) + \frac{8M}{R^2}\delta^2 + \frac{RM}{\delta N^\delta}$. Choose ε, take $R = \frac{10}{\varepsilon}$, choose δ s.t. $\frac{8M}{R^2}\delta^2 \leq \frac{\varepsilon}{10}$

$|(\ )| \leq \frac{\varepsilon}{5} + \frac{1}{N} + \frac{\varepsilon}{10} + \frac{RM\delta^{-1}}{N^\delta}$. If N large, RHS $< \varepsilon$. $\Rightarrow$ VV.

(5.3).

The connection between the additive & multiplicative structures of the Riemann Zeta function is the logarithmic derivative:

$$\frac{-\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) = \sum_p \frac{p^{-s}\log p}{1-p^{-s}} = \sum_p \log p \sum_{\alpha=1}^{\infty} p^{-\alpha s} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} =$$

$$= \int_1^{\infty}\frac{d\,\psi(t)}{t^s} \qquad (\mathrm{Re}(s) > 1).$$

We know that $\zeta(s)$ has a simple pole with residue 1 at $s=1$, and is analytic elsewhere in $\mathbb{C}$.

Thus $\dfrac{-\zeta'(s)}{\zeta(s)} - \dfrac{1}{s-1}$ is meromorphic, and has simple poles at zeros of $\zeta(s)$.

Thus the knowledge of zeros of $\zeta(s)$ is important for the study of the distribution of primes.

- Relation between better zero-free regions and P.N.T error term:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\text{Upper bound for } \zeta(s) \text{ near the line } \sigma=1} \arrow[r, Rightarrow, "\Uparrow \text{ contour integration of } \frac{\zeta'}{\zeta}"'] \arrow[d, Rightarrow, "\leftarrow \text{Landau Method, 3-4-1 ineq.}"] & \boxed{\text{Zero density bounds}} \arrow[d, Rightarrow, "\leftarrow \text{explicit formulas}"] \\
\boxed{\text{Zero-free regions for } \zeta(s) \text{ in critical strip}} \arrow[d, Rightarrow, "\leftarrow \text{Perron formula}"] & \boxed{\text{Lower bounds for primes in short intervals}} \\
\boxed{\text{PNT error term}} &
\end{tikzcd}$$

Lemma 5.4: Suppose $\psi(x) = x + O(E(x))$ where E satisfies:

- $E(x) \geq \sqrt{x}$
- $E(x) \nearrow$
- $\frac{E(x)}{x} \searrow$

Then: $\theta(x) = x + O(E(x))$ and $\pi(x) = li(x) + O\left(\frac{E(x)}{\log x}\right)$

(will skip to 32)

Remarks:

The previous lemma is versatile!

Some common PNT error terms: $O(x)$, $\frac{x}{(\log x)^c}$ $(c>0)$,

$x e^{-c\sqrt{\log x}}$, $x e^{-(\log x)^{3/5}(\log\log x)^{-1/5}}$.

Pf (of lemma): $\theta(x) = \psi(x) + O(\sqrt{x})$ $\Rightarrow$ result, as $E(x) \gg \sqrt{x}$.

$$\pi(x) = \int_{2^-}^{x} \frac{d\theta(t)}{\log t} = \frac{\theta(x)}{\log x} + \int_{2^-}^{x} \frac{\theta(t)}{t\log^2 t}\,dt = \frac{x + O(E(x))}{\log x} + \int_{2^-}^{x} \frac{t + O(E(t))}{t\log^2 t}\,dt =$$

$$= \frac{x}{\log x} + \int_{2^-}^{x} \frac{dt}{\log^2 t} + \frac{O(E(x))}{\log x} + \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^{x}\right)\left(\frac{O(E(t))}{t\log^2 t}\,dt\right) =$$

$$= li(x) + O\left(\frac{E(x)}{\log x}\right) + (\text{integrals}).$$

• $\displaystyle\int_2^{\sqrt{x}} \frac{E(t)}{t\log^2 t}\,dt \ll \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} \ll \frac{\sqrt{x}}{\log^2 x}$.

By L'Hopital, $\displaystyle\frac{\int_2^{u} \frac{dt}{\log^2 t}\,dt}{\frac{u}{\log^2 u}} \to 1$ as $u \to \infty$.

• $\displaystyle\int_{\sqrt{x}}^{x} \frac{E(t)}{t\log^2 t}\,dt \leq E(x)\int_{\sqrt{x}}^{x} \frac{dt}{t \log^2 t} \ll E(x)\int_{\sqrt{x}}^{\infty} \frac{dt}{t\log^2 t} = E(x)\left|\frac{1}{\log t}\right|_{\sqrt{x}}^{\infty} \ll \frac{E(x)}{\log x}$

Thm 5.5.

(i) In the region $\frac{1}{2} \leq \sigma \leq 1$, $|t| \geq \frac{1}{2}$, $s = \sigma + it$ we have $|\zeta(s)| \leq |t|^{1-\sigma}(\log|t| + 12)$

(ii) In the region $\sigma \geq 1$, $|t| \geq \frac{1}{2}$ we have $|\zeta(s)| \leq \log|t| + 26$

Pf

Extend ζ into the critical strip:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \int_{N^+}^{\infty} x^{-s}\,d[x] + \int_N^\infty x^{-s}\,ds = \sum_{n=1}^{N} \frac{1}{n^s} + x^{-s}\left[\cancel{[x]-x}\right]_N^\infty + s\int_N^\infty x^{-s-1}([x]-x)\,dx$$

$$= \sum_{n=1}^{N} \frac{1}{n^s} - s\int_N^\infty x^{-s-1}\{x\}\,dx + \frac{N^{1-s}}{s-1} \quad (\text{works in the critical strip}).$$

WLOG, assume $t \geq 0$ (as $\zeta(\bar{s}) = \overline{\zeta(s)}$).

(Cont pf)

(i) $\left|\zeta(\sigma+it)\right| \leq \sum_{n=1}^{N} \frac{1}{n^{\sigma}} + |s| \int_N^{\infty} x^{-s-1}\,dx + \frac{N^{1-\sigma}}{|s-1|} \leq \max(1, N^{1-\sigma}) \cdot \sum_{n=1}^{N} \frac{1}{n} + |s| \frac{N^{-\sigma}}{\sigma} + \frac{N^{1-\sigma}}{|s-1|}$

If $t \geq 1$, $\sigma \geq \frac{1}{2}$ then:

$$|\zeta(s)| \leq \max(1, N^{1-\sigma})(1+\log N) + 2|s|N^{-\sigma} + N^{1-\sigma} \quad (*)$$

when $\frac{1}{2} \leq \sigma \leq 1$, take $N := \lfloor t \rfloor$, then $\frac{t}{2} \leq N \leq t$ and so

$$|\zeta(s)| \leq t^{1-\sigma}(1+\log t) + 2(t+1)\left(\frac{t}{2}\right)^{-\sigma} + t^{1-\sigma} \leq t^{1-\sigma}(1+\log t + 8) \leq t^{1-\sigma}(\log t + 10)$$

when $1 \leq \sigma \leq 2$, take $N = \lfloor t \rfloor$. By $(*)$:

$$|\zeta(s)| \leq 1 + \log t + \underbrace{N^{1-\sigma}}_{\leq 1} + \underbrace{2(t+2)}_{\leq 3t}\left(\frac{t}{2}\right)^{-\sigma} \leq \log t + 26$$

If $\sigma \geq 2$,

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \leq \zeta(2) \leq 2.$$

when $\frac{1}{2} \leq \sigma \leq 2$ and $\frac{1}{2} \leq |t| \leq 1$, (α) with $N=1$ (or lemma 5.1) implies:

$$|\zeta(s)| \leq 1 + \frac{1}{|s-1|} + |s| \int_1^{\infty} \frac{dt}{t^{\sigma+1}} \leq 1 + 2 + \sqrt{5} \cdot 2 \leq 8.$$

Remarks:

- If $\sigma \geq 1 - \frac{c}{\log t}$, and $t \geq 2$, then $|\zeta(s)| \leq |t|^{1-\sigma}(\log|t| + 12) \leq e^{c}(\log t + 12)$
  $\ll \log t$.

- Best bounds known:

(Vinogradov - Korobov, 1958) : $|\zeta(1+it)| \ll (\log|t|)^{2/3}$, $|t| \geq 2$.

R.H. $\Rightarrow$ $|\zeta(1+it)| \ll \log\log|t|$ $(|t| \geq 10)$.

(comes from 29)

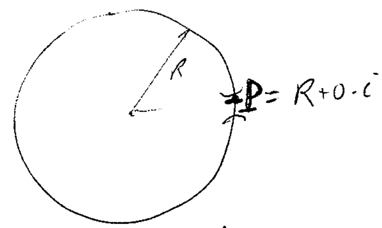
Lemma 5.6 (Heath-Brown, 1989):
Suppose f is analytic in $|s-s_0| \le R$, $f(s_0) \neq 0$ and $f(s) \neq 0$ on $|s-s_0| = R$.
Let $\rho_k = s_0 + r_k e^{i\theta_k}$ be the zeros inside the circle, each with multiplicity n_k. Then:

$$-\operatorname{Re}\left(\frac{f'}{f}(s_0)\right) = -\frac{1}{\pi R}\int_0^{2\pi} (\cos\theta) \log|f(s_0+Re^{i\theta})|\, d\theta + \sum_k n_k\left(\frac{1}{r_k} - \frac{r_k}{R^2}\right)\cos\theta_k$$

Pf/ WLOG, $s_0 = 0$.

Let $$I = \frac{1}{2\pi i}\oint_{|z|=R}\left(\frac{1}{z} - \frac{z}{R^2}\right)\frac{f'}{f}(z)\,dz$$

By the residue theorem, we get



$$(1) \quad I = \frac{f'}{f}(0) + \sum_k n_k\left(\frac{1}{\rho_k} - \frac{\rho_k}{R^2}\right)$$

Cutting the circle at P, call the new contour $\mathcal{D}$. Then f has an analytic logarithm on $\mathcal{D}$.
(The two ends of $\mathcal{D}$ meet at P, and the two endpoint values of $\log f(s)$ may be different).
By integration by parts,
$$I = \frac{1}{2\pi i}\oint_{\mathcal{D}}\left(\frac{1}{z} - \frac{z}{R^2}\right)\frac{f'}{f}(z)\,dz = \cancel{\frac{1}{2\pi i}\left(\left(\frac{1}{z} - \frac{z}{R^2}\right)\log f(z)\Big|_{\mathcal{D}}\right)}^{\,0} + \frac{1}{2\pi i}\oint_{\mathcal{D}}\left(\frac{1}{z^2} + \frac{1}{R^2}\right)\log f(z)\,dz$$

as at $z = R$, $\frac{1}{z} - \frac{z}{R^2}$ vanishes!

Let $z = Re^{i\theta}$, $0 \le \theta \le 2\pi$. Then

$$(2) \quad I = \frac{1}{2\pi R}\int_0^{2\pi} e^{i\theta}\left(e^{-2i\theta}+1\right)\log\left(f(Re^{i\theta})\right)d\theta = \frac{1}{\pi R}\int_0^{2\pi}\cos\theta \log f(Re^{i\theta})\,d\theta$$

Taking real parts of (1) and (2), we get the result. //

Lemma 5.8: If $\sigma > 1$, then $-\frac{\zeta'}{\zeta}(\sigma) < \frac{1}{\sigma-1}$

Pf Write $a_m := \sum_{n=m+1}^{\infty} n^{-\sigma}$. So $-\zeta'(\sigma) = \sum_{n=1}^{\infty} (\log n) n^{-\sigma} = \sum_{n=1}^{\infty} (\log n)(a_{n-1} - a_n) =$

$$= \sum_{m=1}^{\infty} a_m \left(\log(m+1) - \log(m)\right).$$

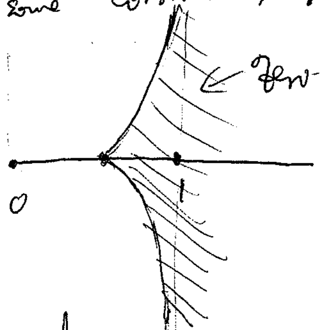
Notice $\log(m+1) - \log(m) = \int_m^{m+1} \frac{dt}{t} \leq \frac{1}{m}$, so:

$$-\zeta'(\sigma) \leq \sum_{m=1}^{\infty} \frac{a_m}{m}, \quad a_m = \sum_{n=m+1}^{\infty} n^{-\sigma} \leq \int_m^{\infty} \frac{dt}{t^{\sigma}} = -\frac{m^{1-\sigma}}{1-\sigma}$$

Therefore, $-\zeta'(\sigma) \leq \sum_{m=1}^{\infty} \frac{-1}{1-\sigma} \frac{1}{m^{\sigma}} = \frac{-1}{1-\sigma} \zeta(\sigma)$.

Theorem 5.7 (de la Vallée Poussin, 1899):

For some constant $A > 0$, $\zeta(\sigma + it) \neq 0$ for $\sigma \geq 1 - \frac{A}{\log(|t|+10)}$



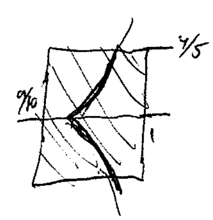
Pf We already know $\zeta(\sigma+it) \neq 0$ for $\sigma \geq 1$.

Also, by Lemma 5.1, $\zeta(s) = s\left(\frac{1}{s-1} - \int_1^{\infty} \frac{\{x\} dx}{x^{s+1}}\right) \Rightarrow$

$$|\zeta(s)| = \left| s\left(\frac{1}{s-1} - \int_1^{\infty} \frac{\{x\} dx}{x^{s+1}}\right)\right| \geq |s| \left(\frac{1}{|s-1|} - \int_1^{\infty} \frac{dx}{x^{\sigma+1}}\right) \geq |s| \left(\frac{1}{|s-1|} - \frac{1}{\sigma}\right)$$

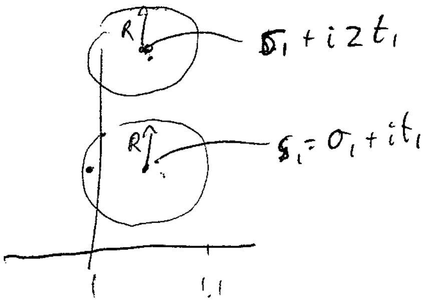
proving that $\zeta(\sigma+it) \neq 0$ if $\frac{9}{10} \leq \sigma \leq 1$, $|t| \leq \frac{4}{5}$.

So take $A \leq \frac{\log 10}{\log 10}$, and this takes care of $|t| \leq \frac{4}{5}$,



Since $\zeta(\bar{s}) = \overline{\zeta(s)}$, WLOG can assume $t > \frac{4}{5}$.

Take now $1 < \sigma_1 \leq 1.1$, $t_1 \geq 4/5$, $R \leq \frac{3}{10}$; $s_1 := \sigma_1 + i t_1$



If $s = s_1 + Re^{i\theta}$, the $s = \sigma_1 + R\cos\theta \geq 1 - R$.

Check ↓

By thm 5.5, $\left|\zeta(s_1 + Re^{i\theta})\right| \leq \max(1, t^R)\left(\log(|t|+R) + 26\right) \leq 15\, t^R \log(|t|+10)$

We'll use this only for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. (left-part of circle). (1)

For the other θ, $\mathrm{Re}(s_1 + Re^{i\theta}) \geq 1 + R\cos\theta =: \sigma_2$.

$$\left|\zeta(s_1 + Re^{i\theta})\right|^{-1} \leq \sum_{n=1}^{\infty} \left|\frac{\mu(n)}{n^{\sigma_2}}\right| \leq \zeta(\sigma_2) = 1 + \frac{1}{\sigma_2 - 1} - \sigma_2 \int_1^\infty \{t\} t^{-\sigma_2 - 1}\, dt \leq 1 + \frac{1}{\sigma_2 - 1} = \frac{\sigma_2}{\sigma_2 - 1}$$

$$\leq \frac{1+R}{R\cos\theta}$$

So $\left|\zeta(s_1 + Re^{i\theta})\right| \geq \dfrac{R\cos\theta}{1+R}$ for $\theta \leq \frac{\pi}{2}$, $\theta \geq \frac{3\pi}{2}$. (2)

Use (1) & (2) to show: $-\int_0^{2\pi} \cos\theta \log\left|\zeta(s_1 + Re^{i\theta})\right| d\theta \leq \cancel{\log\left(15 t^R \log(|t|+10)\right) \cdot \int_{\pi/2}^{3\pi/2}}$

$$\leq \log\left(15 t^R \log(|t|+10)\right) \cdot \int_{\pi/2}^{3\pi/2} -\cos\theta\, d\theta + \int_{-\pi/2}^{\pi/2} \cos\theta \log\left(\frac{1+R}{R\cos\theta}\right) d\theta =$$

$$\cancel{(}\, 2 \cdot \log\left(15 t^R \log(t+10)\right) + 2\log\left(\frac{1+R}{R}\right) \cdot 2(1 - \log 2)\Big) = 2\log\left(\frac{1+R}{R} 15 t^R \log(t+10)\right) + 2(1 - \log 2)$$

Now use thm 5.6. Let $\rho_k = s_1 + r_k e^{i\theta_k}$ be the zeros of $\zeta(s)$ inside the circle $|s - s_1| = R$. Assume there are no zeros on $|s - s_1| = R$ (finite #, so change R).

<u>Then</u>: $-\mathrm{Re}\, \dfrac{\zeta'}{\zeta}(s_1) \leq \dfrac{2}{\pi R} \log\left(\dfrac{1+R}{R} 15 t^R \log(t+10)\right) + \dfrac{\cancel{2}}{\pi R} \cdot 1 + \sum_k n_k \left(\dfrac{1}{r_k} - \dfrac{r_k}{R^2}\right)\cos\theta_k$

$$= \frac{2}{\pi}\log t + \frac{2}{\pi R}\log\left(21\, \frac{1+R}{R} \log(t+10)\right) + \sum n_k \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right)\cos\theta_k$$

By the 3-4-1 inequality, if $\sigma > 1$ then:

$$-\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+2it) - 4\,\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it) - 3\,\frac{\zeta'}{\zeta}(\sigma) = \sum \frac{\Lambda(n)}{n^\sigma}\Big(\cos(2t\log n) + 4\cos(t\log n) + 3\Big) \geq 0$$

In (1), notice that $\mathrm{Re}(\rho_k) \leq 1$ then $\pi/2 < \theta_k < \frac{3\pi}{2}$, so:

$\left(\frac{1}{r_k} - \frac{r_k}{R^2}\right)\cos\theta_k < 0$. For $\sigma_0 + it$, have $\theta_k = \pi$, $r_k = r$, so get $-\left(\frac{1}{r} - \frac{r}{R^2}\right) \leq -\frac{15}{16r}$ (3)

So can ignore all the extra zeros that might be in the disk: suppose $\zeta(\sigma_0 + it) = 0$,

with $t \geq \frac{4}{5}$, $\frac{19}{20} \leq \sigma_0 < 1$.

Choose $R \leq \frac{3}{10}$, $1 < \sigma_1 \leq 1.1$, so that $r := \sigma_1 - \sigma_0 \leq \frac{R}{4}$. (4)

Use (1) with $s_1 = \sigma_1 + it$, and with $s_1 = \sigma_1 + 2it$, and combine this with (2) and (3), to get:

$$0 \leq -3\frac{\zeta'}{\zeta}(\sigma_1) - 4\,\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma_1+it) - \mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma_1+2it) \Rightarrow$$

$$0 \leq -3\frac{\zeta'}{\zeta}(\sigma_1) + \frac{10}{\pi}\log(2t) + \frac{10}{\pi R}\log\left(21\cdot\frac{1+R}{R}\log(2t+10)\right) - \frac{15}{4r}$$

Let $L = \log(2t+10)$. By (5.8),

$$0 \leq \frac{3}{\sigma_1 - 1} + \frac{10}{\pi}L + \frac{10}{\pi R}\log\left(21\cdot\frac{1+R}{R}L\right) - \frac{15}{4(\sigma_1-\sigma_0)}$$

Let now $R = \frac{3}{10}$. Since $L \geq \log\left(2\cdot\frac{4}{5}+10\right) > 2.3$,

$$0 \leq \frac{3}{\sigma_1-1} - \frac{15/4}{\sigma_1-\sigma_0} + \frac{10}{\pi}L + 10.2\log L + 51. \leq \frac{3}{\sigma_1-1} - \frac{3.75}{\sigma_1-\sigma_0} + 30L \quad (7).$$

Write $\sigma_0 = 1 - \frac{B}{L}$, $\sigma_1 = 1 + \frac{C}{L}$, with $B, C > 0$, and get

$$0 \leq \frac{3}{C} - \frac{3.75}{B+C} + 30 \Rightarrow B \geq \frac{C(0.75 - 30C)}{3 + 30C}$$

The optimal C is $\sim C = \frac{1}{80}$. Then check $\sigma_1 < 1.1$ and (4) holds,

and thus $B \geq \frac{1}{720}$. Thus $\sigma_0 \leq 1 - \frac{1/720}{\log(2t+10)} \leq 1 - \frac{A}{\log(t+10)}$ $(t \geq 4/5)$

- Zeros of $\zeta(s) \longleftrightarrow$ error term in PNT

Suppose $\psi(x) = x + O(x^{\alpha})$ for $\alpha < 1$.

Consider $F(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)-1}{n^s} = -\frac{\zeta'}{\zeta}(s) - \zeta(s) \qquad (\sigma > 1)$.

Note: the RHS is analytic at $s=1$

Let $A(x) = \psi(x) - \lfloor x \rfloor = \sum_{n \leq x} \Lambda(n) - 1 = O(x^{\alpha})$ (by hypothesis).

By partial summation (see hw#5, prob 1), for $\sigma > 1$:

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)-1}{n^s} = s \cdot \int_1^{\infty} A(x)\, x^{-1-s}\, dx.$$

Since $A(x) = O(x^{\alpha})$, RHS converges (and is analytic for $\sigma > \alpha$).

(as $\sigma \geq \alpha + \varepsilon \Rightarrow \int_1^{\infty} |A(x)\, x^{-1-s}|\, dx \ll \int_1^{\infty} x^{-1-\sigma+\alpha}\, dx \leq \int_1^{\infty} x^{-1-\varepsilon}\, dx = \frac{1}{\varepsilon}$.

Thus does $-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s)$ has analytic continuation to $\sigma > \alpha$.

In fact, $\sum_{n=1}^{\infty} \frac{\Lambda(n)-1}{n^s}$ converges for $\sigma > \alpha \Rightarrow$ no zeros of $\zeta(s)$ with $Re(s) > \alpha$.

(also one can prove essentially the converse of it).

Moreover, suppose $\zeta(\sigma + it) = 0$ with $\sigma < 1$. Then $\forall \varepsilon > 0$, $\psi(x) = x + O(x^{\sigma - \varepsilon})$ is false

It is known (Riemann) that $\zeta(\frac{1}{2} + i\gamma) = 0$ for some $\gamma \in \mathbb{R}$. So $\forall \varepsilon > 0$ $x + O(x^{\frac{1}{2} - \varepsilon})$ is false.

Goal: use the fact that $\zeta(\sigma + it) \neq 0$ for $\sigma \geq 1 - f(t)$, for $f(t) \to 0$

& show that $\psi(x) = x + O(x \cdot \mathcal{E}(x))$, with $\mathcal{E}(x) \to 0$

If $\phi : [1,\infty) \to \mathbb{C}$, then the <u>Mellin transform</u> of ϕ is given by

$$(M) \quad \hat{\phi}(s) = \int_1^\infty \phi(x)\, x^{-s-1}\, dx \qquad \text{(wherever it makes sense).}$$

Substitute $x = e^u$, $\theta(u) = \phi(e^u)$ $\longrightarrow$ $\hat{\phi}(s) = \int_0^\infty \theta(u) e^{-us}\, du$ = Laplace transform of θ.

If (M) converges at $s = \sigma + it$, let $\xi(u) = \begin{cases} 0 & u < 0 \\ \phi(u) e^{-u\sigma} & u \geq 0 \end{cases}$

Then $\hat{\phi}(s) = \int_{-\infty}^{\infty} \xi(u)\, e^{-iut}\, du$ = Fourier transform of ξ.

Mellin transforms also σ_c and σ_a. In fact, Dirichlet series are special cases of Mellin transforms!

<u>Theorem 4.12</u>: Let $K(x) = \sum_{n \leq x} f(n)$, $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$. Then for $\sigma > \max(0, \sigma_c)$,

then $F(s) = s \int_1^\infty K(x)\, x^{-s-1}\, dx$ (ie $\frac{F(s)}{s}$ = Mellin transform of $K(x)$.)

Pf: if $\sigma > \sigma_c$, then $F(s)$ converges by Thm 4.7. Since $\sigma > 0$, the result follows by HW #5, pb 1. //

<u>Examples</u>:

1) $\zeta(s) = \sum \frac{1}{n^s}$. $K(x) = \lfloor x \rfloor$, if $\sigma > 1$ then $\zeta(s) = s \int_1^\infty \lfloor x \rfloor x^{-s-1}\, dx$.

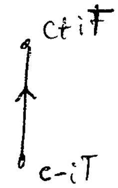
writing $\lfloor x \rfloor = x - \{x\}$, we recover $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\} x^{-s-1}\, dx$.

2) $\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s} = s \int_1^\infty M(x)\, x^{-s-1}\, dx$, for $M(x) = \sum_{n \leq x} \mu(n)$. $(\sigma > 1)$

3) $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} = s \int_1^\infty \psi(x)\, x^{-s-1}\, dx$

Inversion Formulas:

Define, for $c>0$, $I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds$



($T\in\mathbb{R}_{>0}$).

Lemma 4.14: Let $c>0, T>0, y>0$. Let $\delta(y) := \begin{cases} 1 & y>1 \\ \frac{1}{2} & y=1 \\ 0 & y<1 \end{cases}$

Then $|I(y,T)-\delta(y)| \leqslant \begin{cases} \frac{c}{\pi T} & y=1 \\ \frac{y^c}{\pi T|\log y|} & y\neq 1 \end{cases}$

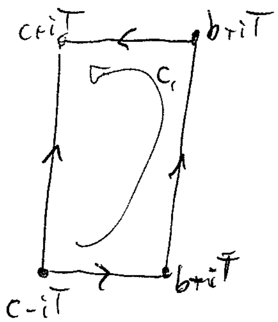
Pf

<u>If $y=1$</u>, then $I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{ds}{s} \overset{\text{away from the pole}}{=} \frac{1}{2\pi i}\left|\log(c+iT)-\log(c-iT)\right| = \frac{1}{2\pi i}\left|\log\left(\frac{c+iT}{c-iT}\right)\right| =$

$= \frac{1}{2\pi}\arg\left(\frac{c+iT}{c-iT}\right) = \frac{1}{\pi}tg^{-1}\frac{T}{c} = \frac{1}{2}-\frac{1}{\pi}tg^{-1}\frac{c}{T}$

Now, just note that $tg^{-1}\frac{c}{T}\leqslant\frac{c}{T}$ (as $(tg^{-1}(x))' \leqslant 1\ \forall x$).

<u>If $0<y<1$:</u>



So $I(y,T) = \frac{1}{2\pi i}\int_{C_1}\frac{y^s}{s}\,ds$

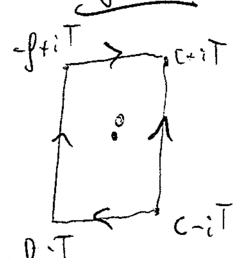
In the horizontal segments, $|s|\geqslant T$.

& the $|\,|$ of the integral on the horiz. $\leqslant 2\left|\frac{1}{2\pi i}\right|\int_c^b \frac{y^u}{T}\,du \leqslant \frac{1}{T\pi}\int_c^\infty y^u\,du$

$= \frac{y^c}{\pi T(-\log y)}$ ✓.

Also, $\left|\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{y^s}{s}\,ds\right| \leqslant \frac{2T}{2\pi}\cdot\frac{y^b}{b}$. Let $b\to\infty$. So it goes to 0. so ✓.

<u>If $y>1$</u>: then we similarly deform the contour to C_2, where $\beta>0$.



So $I(y,T) = \underbrace{\operatorname{Res}_{s=0}\frac{y^s}{s}}_{1} + \int_{C_2}\frac{y^s}{s}\,ds$ $\leqslant \frac{y^c}{2\pi T\log y}$ as $\beta\to\infty$

Note then if c, y are fixed, $I(y,T) \longrightarrow \sigma(y)$ as $T \to \infty$.

Put $y = \frac{x}{n}$. Then $I\left(\frac{x}{n}, \infty\right) = \begin{cases} 1 & n<x \\ 1/2 & n=x \\ 0 & n>x \end{cases}$

If we multiply by $f(n)$ and sum:

$$K(x) = \sum_{n \leq x} f(n) \approx \sum_{n=1}^{\infty} I\left(\frac{x}{n}, \infty\right) f(n) \approx \sum_{n=1}^{\infty} I\left(\frac{x}{n}, T\right) f(n) = \sum_{n=1}^{\infty} \frac{f(n)}{2\pi i} \int_{c-iT}^{c+iT} \frac{(x/n)^s}{s} ds$$

$$= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \sum \frac{f(n)}{n^s} ds = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s} ds$$

<u>Theorem 4.16</u> (Perron's inversion formula)

Let $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ with $\sigma_a < \infty$, let $K(x) = \sum_{n \leq x} f(n)$, and $K^*(x) := \frac{1}{2}\left(K(x^-) - K(x^+)\right)$

(note $K^*(x) = \begin{cases} K(x) & x \notin \mathbb{N} \\ K(x) - \frac{1}{2} f(x) & x \in \mathbb{N} \end{cases}$)

Then (1) $K^*(x) = \frac{1}{2\pi i} \int_{(c)} F(s) \frac{x^s}{s} ds \quad (x > 1,\ c > \sigma_a)$ where $\int_{(c)} = \lim_{T \to \infty} \int_{c-iT}^{c+iT}$

Furthermore, for $T > 0$, (2) $\left| K^*(x) - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s} ds \right| \leq \frac{x^c}{\pi T} \sum_{\substack{n=1 \\ n \neq x}}^{\infty} \frac{|f(n)|}{n^c \left|\log \frac{x}{n}\right|} + \frac{c}{\pi T} |f(x)|$

$= 0$ if $x \notin \mathbb{N}$

Pf First, note that (2) $\Rightarrow$ (1) by taking $T \to \infty$.

Also, $\sum_{\substack{n=1 \\ n \neq x}}^{\infty} \frac{|f(n)|}{n^c \left|\log \frac{x}{n}\right|} \leq O_x\left(\sum_{n=1}^{\infty} \frac{|f(n)|}{n^c}\right) = O_{x,f}(1)$, so it makes sense.

(2) follows directly from the previous lemma and the fact that:

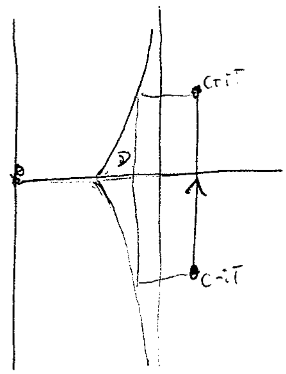
$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) \frac{x^s}{s} ds = \sum_{n=1}^{\infty} I\left(\frac{x}{n}, T\right) f(n) = \sum_{n=1}^{\infty} \left(\sigma\left(\frac{x}{n}\right) + R(n)\right) f(n),$$

where by lemma 4.14, $|R(n)| \leq \begin{cases} \frac{c}{\pi T} & \text{if } n = x \\ \frac{1}{\pi T \left|\log \frac{x}{n}\right|} \left(\frac{x}{n}\right)^c & n \neq x \end{cases}$

Now just note $\sum_{n=1}^{\infty} \sigma\left(\frac{x}{n}\right) f(n) = K^*(x)$.

Application: Let $F = \psi$.

Then $\psi^*(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds$ + "error term". $\quad (c>1)$.



Deform the contour to D, that is still in the zero-free region.
We will get an extra term for the pole at $s=1$, of residue x.

$$\Rightarrow \psi^*(x) = x + \frac{1}{2\pi i}\int_D -\frac{\zeta'}{\zeta}(x)\frac{x^s}{s}\,dx + \text{"error term"}.$$

We need estimates for $\frac{\zeta'(s)}{\zeta(s)}$.

Lemma 5.10 (Borel-Caratheodory Lemma):
Suppose f is analytic on a disk $D = \{z : |z-z_0| \leq R\}$.
Let $A := \max_{z\in D} \operatorname{Re}(f(z)) > 0$ (assume). Assume $f(z_0)=0$. Then, for $|z-z_0| = r < R$,

$$|f(z)| \leq \frac{2Ar}{R-r}$$

Pf/ WLOG $z_0 = 0$. Consider the function $h(z) := \frac{f(z)}{2A - f(z)}$

$h(z)$ is analytic on D, as $\operatorname{Re} f(z) \neq 2A$ on D.

Also, $\operatorname{Im}(2A - f(z)) = -\operatorname{Im} f(z)$, and $\operatorname{Re}(2A - f(z)) \geq A \geq \operatorname{Re} f(z)$. Thus $|2A - f(z)| \geq |f(z)|$.

So $|h(z)| \leq 1$ for $z \in D$.

Consider now $g(z) := \begin{cases} \frac{h(z)}{z}, & z\in D, z\neq 0 \\ h'(0) & z = 0 \end{cases}$ (so g continuous)
So g is analytic on D.

On the boundary ∂D ($|z|=R$), $|g(z)| = \left|\frac{h(z)}{z}\right| \leq \frac{1}{R}$.

By the maximum modulus pple, $|g(z)| \leq \frac{1}{R}$ on D, i.e. $|h(z)| \leq \frac{|z|}{R}$ on D

Thus $|f(z)| = |h(z)| \cdot |2A - f(z)| \leq \frac{r}{R}\cdot(2A + |f(z)|)$ if $|z|=r$. Solve for $|f(z)|$ and done

Theorem 5.11: Suppose $A \leq \frac{1}{10}$ and $\zeta(s) \neq 0$ for $\sigma \geq 1 - \frac{A}{L}$, where $L = \log(|t|+10)$

Then, in the region $\sigma \geq 1 - \frac{A}{10L}$,

$$\left| -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right| \ll_A L \log L.$$

actually one can prove $\ll_A L$.

Proof

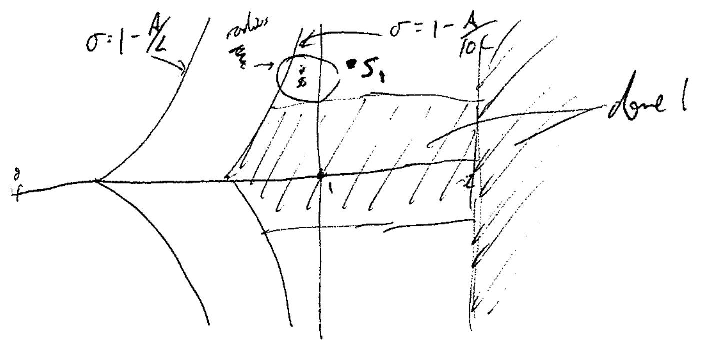
First, if $\sigma \geq 2$, then $\left| -\frac{\zeta'}{\zeta}(\sigma+it) - \frac{1}{s-1} \right| \leq -\frac{\zeta'}{\zeta}(\sigma) + \left|\frac{1}{s-1}\right| < \frac{1}{\sigma-1} + 1 \leq 2.$

So can assume $\sigma < 2$.

Also, $-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}$ is analytic for $\left(1 - \frac{A}{10L} \leq \sigma \leq 2,\ |t| \leq 10\right)$.

$\therefore$ takes a maximum value in this region, so also done.

Also WLOG, assume $t \geq 10$ ($t \geq 0$, by symmetry), and $2 \leq \sigma \leq 1 - \frac{A}{10L}$



Let $s = \sigma + it$, let $\xi := \frac{A}{10L}$, and $s_1 := s + 2\xi = \sigma_1 + it$, $\sigma_1 \geq 1 + \frac{A}{10L}$

By Cauchy's integral theorem:

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{2\pi i} \oint_{|z-s|=\xi} \frac{\log \zeta(z)}{(z-s)^2}\, dz = \frac{1}{2\pi i} \oint_{|z-s|=\xi} \frac{\log\left(\frac{\zeta(z)}{\zeta(s_1)}\right)}{(z-s)^2}\, dz$$

ζ has analytic log!   added a constant

call it $f(z)$, choose branch of log s.t. $f(s_1) = 0$.

Thus, taking modulus:

$$(1)\quad \left| -\frac{\zeta'}{\zeta}(s) \right| \leq \frac{1}{\xi} \max_{|z-s|=\xi} \left| \log \frac{\zeta(z)}{\zeta(s_1)} \right|$$

Let $R := 5\xi$. For $z \in D = \{z : |z - s_1| \leq R\}$, f is analytic,

and since $\mathrm{Re}\, z \geq \mathrm{Re}\, s - 3\xi \geq 1 - \frac{A}{2L} \geq 1 - \frac{A}{\log(|\mathrm{Im}\, z| + 10)}$

(cont' pf)

For $|z-s| = \xi_1$, $|z - s_1| \leq \frac{3}{5}R$, so by lemma 5.10,

(2) $|f(z)| \leq 3 \max_{|z-s|=\xi} \mathrm{Re}\, f(z)$

By thm 5.5, $\begin{cases} |\zeta(z)| \ll_A L & \text{in there, since } \mathrm{Re}\, s - \xi \geq 1 - A/L. \\ \left|\frac{1}{\zeta(s_1)}\right| \leq \zeta(\sigma_1) \ll \frac{1}{\sigma_1 - 1} \ll L. \end{cases}$

So $|f(z)| \ll \log L$. Combining this with (1) and (2), we prove the thm. //

Theorem 5.12: (de la Vallée Poissin, 1899):
For a positive constant c, $\psi(x) = x + O\left(x e^{-c\sqrt{\log x}}\right)$
and $\pi(x) = li(x) + O\left(x e^{-c\sqrt{\log x}}\right)$

Pf The 2^{nd} assertion follows from the first by lemma 5.4.
WLOG, $x \in \mathbb{N}$, $x \geq 2$. (since $\psi(x) = \psi(\lfloor x \rfloor) \overset{\text{if we prove it}}{=} \lfloor x \rfloor + O\left(\lfloor x \rfloor e^{-c\sqrt{\log \lfloor x \rfloor}}\right) = x + O\left(xe^{-c\sqrt{\log x}}\right)$).
Choose now $T \geq 1$, and let $c_1 = 1 + \frac{1}{\log x}$. By thm 4.16,

$$|\psi^*(x) - I| \leq \frac{x^{c_1}}{\pi T} \sum_{\substack{n=1 \\ n \neq x}}^{\infty} \frac{\Lambda(n)}{n^{c_1}|\log \frac{x}{n}|} + \frac{c}{\sigma T}\Lambda(x) \quad \text{where } I = \int_{c_1 - iT}^{c_1 + iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds$$

(Since $\psi^*(x) = \psi(x) + \frac{1}{2}\Lambda(x) = \psi(x) + O(\log x)$) ~~and also~~

Also, $x^{c_1} = e \cdot x$. Thus,

(1) $|\psi(x) - I| \ll \log x + \frac{x}{T} \cdot S$; where $S = \sum_{\substack{n=1 \\ n \neq x}}^{\infty} \frac{\Lambda(n)}{n^{c_1}(\log \frac{x}{n})}$

The terms in S with $\underbrace{n \leq \frac{x}{2} \text{ or } n \geq \frac{3}{2}x}_{|\log \frac{x}{n}| \gg 1}$ contribute to S as: $\ll \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{c_1}}$:

$= -\frac{\zeta'}{\zeta}(c_1) \leq \frac{1}{c_1 - 1} = \log x$. (by lemma 5.8).

If $\frac{x}{2} < n < \frac{3}{2}x$, $n^{c_1} \asymp x$ and $\Lambda(n) \ll \log n \ll \log x$.

When $x < n < \frac{3}{2}x$, $|\log \frac{x}{n}| = \log \frac{n}{x} = -\log\left(1 - \frac{n-x}{n}\right) \geq \frac{n-x}{n} \geq \frac{n-x}{2x}$

Similarly, when $\frac{x}{2} < n < x$, $\left|\log \frac{x}{n}\right| = \log \frac{x}{n} = -\log\left(1 - \frac{x-n}{x}\right) \geq \frac{x-n}{x}$.

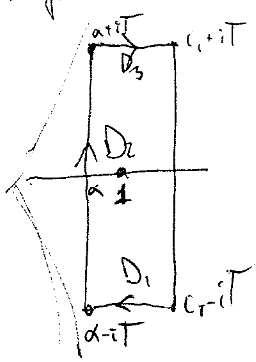
Therefore, $S \ll \log x + \frac{\log x}{x}\left[x \sum_{\frac{x}{2} \le n \le x-1} \frac{1}{x-n} + x \sum_{x+1 \le n \le \frac{3}{2}x} \frac{1}{n-x} \right] \ll$

$$\ll \log x + \log x \left(\sum_{m \le \frac{x}{2}} \frac{1}{m} \right) \ll \log^2 x$$

Therefore, by (1):

$$|\psi(x) - I| \ll \log x + \frac{x \log^2 x}{T} \qquad (2).$$

Deform now the contour $c_1 - iT \longrightarrow c_1 + iT$ to $D_1 \cup D_2 \cup D_3$:



where $\alpha = 1 - \frac{A}{10L}$, $L = \log(T+10)$ (A the ct. from Thm 5.7).

By the residue thm, $I = x + \int_{D_1 \cup D_2 \cup D_3} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds$.

By Thm 5.11, on $D_1 \cup D_2 \cup D_3$, $\left|-\frac{\zeta'}{\zeta}(s)\right| \le \left|-\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}\right| + \left|\frac{1}{s-1}\right| \ll L \log L + L \overset{\text{crude}}{\ll} L^2$

On the horiz. segments, $\left|\frac{x^s}{s}\right| \le \frac{x^{c_1}}{T} \ll \frac{x}{T}$

on D_2, $\left|\frac{x^s}{s}\right| = \frac{x^\alpha}{|s|} \ll \frac{x^\alpha}{1+|t|}$ $(s = \alpha + it)$.

Thus $|I - x| \ll L^2 \left(\frac{x}{T}(c_1 - \alpha) + \int_{-T}^{T} \frac{x^\alpha dt}{1+|t|} \right) \ll L^2 \left(\frac{x}{T} + x^\alpha \overbrace{\log T}^{\le L} \right) \ll L^3 \left(\frac{x}{T} + x^\alpha \right)$

Suppose now $1 \le T \le x$, so $L \ll \log x$. Since $T \asymp e^L$, by (2) we obtain:

$$|\psi(x) - x| \ll \log x + \frac{x \log^2 x}{e^L} + \log^3 x \left(\frac{x}{e^L} + x^{1 - \frac{A}{10L}} \right) \ll x \log^3 x \left(\frac{1}{e^L} + x^{-\frac{A}{10L}} \right)$$

The RHS can be minimized at $L^2 = \frac{A}{10} \log x \Rightarrow |\psi(x) - x| \ll x \log^3 x \, e^{-\sqrt{\frac{A}{10} \log x}} \ll x e^{-\sqrt{\epsilon \log x}}$

$\ll x e^{-\frac{1}{2}\sqrt{\frac{A}{10}}\sqrt{\log x}}$.

Remarks:

- The best known error term is given by $\psi(x) = x + O\left(x e^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}}\right)$ (due to Korobov - Vinogradov, 1958).
- RH $\Rightarrow \psi(x) = x + O(\sqrt{x}\log x)$ (Cramer, 1920's).

  Also, $\psi(x) = x + o(\sqrt{x}\log\log\log x)$ is false. (Littlewood, 1914).

- One can check (hw set 7) that $\zeta(s)$ is analytic in $\mathbb{C}\setminus\{1\}$, with a simple pole at $s=1$.

Functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\,\Gamma(1-s)\,\zeta(1-s) \quad \text{(Riemann, 1859).}$$

This implies that $\zeta(s)$ has "trivial" zeros at $-2k$ ($k\in\mathbb{N}$) (as $\Gamma(-2k)=\infty$).

Since we know that there are no zeros for $\mathrm{Re}(s)\geq 1$, then there are no zeros on $\mathrm{Re}(s)\leq 0$ ~~for~~ except for the trivial ones, that correspond to the poles of Γ.

Symmetric version of f'l equation. Let $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

$\cancel{\xi(s) = \frac{1}{2}s(s-1)\pi^{s/2}}$

Then $\xi(s)$ is entire, and $\xi(1-s) = \xi(s)$.

Being entire, ξ has a Hadamard product: $\xi(s) = \frac{1}{2}e^{cs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}$ where ρ runs over the nontrivial zeros of $\zeta(s)$ (with multiplicity) and c is a constant.

Also, $\frac{\zeta'}{\zeta}(s) = c + \frac{\log\pi}{s} - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(1+\frac{s}{2}\right) + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$

$\checkmark$

The Gamma-function.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad (\mathrm{Re}\, z > 0).$$

- By integration by parts, $\Gamma(n+1) = n!$, Also, it is true that $\Gamma(z+1) = z\Gamma(z)$.
- Use analytic continuation to extend it to $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$.

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\, n^s}{s(s+1)\cdots(s+n)}.$$

$\frac{1}{\Gamma(s)}$ is entire, and $\frac{1}{\Gamma(s)} = s e^{\underset{\text{the Euler-Mascheroni constant}}{\gamma} s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$

Also, $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ and $\Gamma(2s) = 2^{2s-1} \pi^{-\frac{1}{2}} \Gamma(s)\Gamma(s+\frac{1}{2})$

due to Legendre, called "duplication formula".

Explicit formula

$$\psi^*(x) = x - \underbrace{\sum_\rho \frac{x^\rho}{\rho}}_{\text{nontrivial zeros w/ multiplicity}} - \underbrace{\frac{\zeta'}{\zeta}(0)}_{\text{pole } s=0} - \underbrace{\frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)}_{\text{residues at trivial zeros}} \qquad (x > 1).$$

This is proven by deforming the contour of ψ^* to "$-\infty - iT$" $\to$ "$-\infty + iT$"

(Also, $\sum_\rho$ means $\lim_{T\to\infty} \sum_{|\mathrm{Im}\,\rho| \le T}$ (symmetric limit)).

(note that, as $\psi^*(x)$ is not continuous, $\sum_\rho$ is infinite!).

Let $N(T) = |\{\rho : 0 < \mathrm{Im}\,\rho \le T\}| \underset{\text{contour integration}}{=} \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$ really good estimate!

Define $N(\sigma, T) := |\{\rho : 0 < \mathrm{Im}\,\rho \le T,\ \mathrm{Re}\,\rho > \sigma\}|$.

It's known that $\forall \sigma > \frac{1}{2}$, $\exists\, c = c(\sigma) < 1$, s.t. $N(\sigma, T) \ll T^c = o(N(T))$.

Known: all zeros with $0 < \mathrm{Im}\,\rho < 5.7 \cdot 10^{10}$ have real part $\frac{1}{2}$.

- There are ∞-many ρ with $\mathrm{Re}\,\rho = \frac{1}{2}$ (Hardy, 1915).
- Let $N_0(T) := |\{\rho : 0 < \mathrm{Im}\,\rho \le T;\ \mathrm{Re}\,\rho = \frac{1}{2}\}|$. Then (by Selberg, et al):

$$N_0(T) \ge 0.4 \cdot N(T) \text{ for large } T.$$

- Distribution of imaginary parts

Conj: all zeros are simple.

Conj: imaginary parts are linearly-independent over $\mathbb{Q}$.

Conj: gaps between imaginary parts match the GUE distribution ($\sim$ eigenvalues of Hermitian matrices). $\leftarrow$ connection with physics!

Pt 3: Primes in Arithmetic progressions.

Theorem 6.1 (Dirichlet, 1837). Let $1 \leq a \leq q$, $(a,q) = 1$. Then there are ∞ly many primes $p \equiv a \pmod q$.

Def: A Dirichlet character χ is a completely multiplicative function $\chi : \mathbb{N} \to \mathbb{C}$ satisfying, for some integer $k \in \mathbb{N}$:

i) $\chi(n) = 0$ if $(n,k) > 1$.

ii) $\chi(n) = \chi(n+k)$ for all $n \in \mathbb{N}$.

iii) not identically 0.

Equivalently, it's a homomorphism $(\mathbb{Z}/k\mathbb{Z})^\times \to \mathbb{C}$.

Facts:

- $\chi(1) = 1$
- If $(a,k) = 1$, then by Euler's theorem $a^{\phi(k)} \equiv 1 \ (k) \rightarrow \chi(a^{\phi(k)}) = 1$.

Also, $\chi(a^{\phi(k)}) = \chi(a)^{\phi(k)}$. So $\chi(a)$ has order $| \phi(k)$ (or is a $\phi(k)$th root of 1).

Examples:

1) $\chi(n) = \begin{cases} 1 & (n,k) = 1 \\ 0 & (n,k) > 1 \end{cases}$ (trivial/principal character mod k).

2) p prime. Then $\chi(n) = \left(\frac{n}{p}\right)$

3) If $(\mathbb{Z}/k\mathbb{Z})^\times$ is cyclic (i.e. $k = 2, 4, p^a, 2p^a$) then let g be a generator (primitive root)

Then every a, $1 \leq a \leq k$, $(a,k) = 1$ can be written as $a \equiv g^\ell \pmod k$ for some ℓ.

$\chi(a) = \chi(g^\ell) = \chi(g)^\ell \Rightarrow \chi$ is determined by $\chi(g)$.

$\downarrow$

As g has order $\phi(k)$, then $\chi(g)$ must be a $\phi(k)^{th}$ root of 1,

So the possible values of $\chi(g)$ are $e^{2\pi i m/\phi(k)}$, $m \in \{0, \dots \phi(k)-1\}$.

Hence: There are $\phi(k)$ characters modulo k (proof in general: later)

and so (non-canonically) $\widehat{(\mathbb{Z}/k\mathbb{Z})^*} \cong (\mathbb{Z}/k\mathbb{Z})^\times$ grp of characters.

The group of characters mod k (denoted C_k) forms a convenient basis for U_k, the vector-space of arithmetic functions that are periodic modulo k and supported on $\{n : (n,k)=1\}$.

Another basis for U_k is $\{e_{k\ell} : 1 \leq \ell \leq k, (\ell,k)=1\}$ where $e_{k\ell}(n) = \begin{cases} 1 & n \equiv \ell \pmod k \\ 0 & \text{else} \end{cases}$

Thm 6.2 (orthogonality):

Let χ_1, χ_2 be two Dirichlet characters mod k. Then $\displaystyle\sum_{n=0}^{k-1} \chi_1(n)\overline{\chi_2(n)} = \begin{cases} \phi(k) & \chi_1 = \chi_2 \\ 0 & \text{else} \end{cases}$

Pf: if $\chi_1 = \chi_2$ then it's trivial.

if $\chi_1 \neq \chi_2$, $\exists\, m$, $(m,k)=1$ s.t. $\chi_1(m) \neq \chi_2(m)$. Let $S := \displaystyle\sum_{n=0}^{k-1} \chi_1(n)\overline{\chi_2(n)}$

So $S = \displaystyle\sum_{\ell=0}^{k-1} \chi_1(m\ell)\overline{\chi_2(m\ell)} = \chi_1(m)\overline{\chi_2(m)} \cdot S \Rightarrow S\left(1 - \overline{\chi_1(m)\overline{\chi_2(m)}}\right) = 0 \Rightarrow S = 0$ //

Denote by χ_0 the principal (trivial) character.

Set $\chi_2 = \chi_0$ in thm (6.2). Then $\displaystyle\sum_{n=0}^{k-1} \chi_1(n) = \begin{cases} \phi(k) & \chi_1 = \chi_0 \\ 0 & \text{else} \end{cases}$ (Corollary 6.3)

Thm 6.4 (Linear independence):

Let $\chi_1, \dots, \chi_r$ be distinct Dirichlet characters mod k.

If $a_1, \dots, a_r \in \mathbb{C}$. Then $\displaystyle\sum_{i=1}^{r} a_i \chi_i(n) = 0 \ \forall n \Leftrightarrow a_1 = \dots = a_r = 0$.

Pf Suppose $\sum_{i=1}^{r} a_i \chi_i(n) = 0 \ \forall n$. Let $1 \leq s \leq r$. Multiply by $\overline{\chi_s(n)}$ on both sides and sum on n: $0 = \displaystyle\sum_{n=0}^{k-1} \overline{\chi_s(n)} \sum_{i=1}^{r} a_i \chi_i(n) = \sum_{i=1}^{r} a_i \sum_{n=0}^{k-1} \chi_i(n)\overline{\chi_s(n)} = a_s \phi(k)$

//

Corollary: There are $\le \phi(k)$ Dirichlet characters mod k.

Pf/ $\dim(\mathcal{U}_k) = \phi(k)$, and the Dirichlet characters mod k are l.i. //

Operations: Given χ mod k, define $\bar{\chi}$ by $\bar{\chi}(n) := \overline{\chi(n)}$.

multiply characters $(\chi_1 \chi_2)(n) := \chi_1(n)\chi_2(n)$.

Example:

$k=4$: $\chi_0(1)=1$, $\chi_0(3)=1$ $\quad (\chi_0 = \chi_1^2)$.
$\chi_1(1)=1$, $\chi_1(3)=-1$

$k=5$: ~~$\chi_0(1)=1$~~

	1	2	3	4
χ_0	1	1	1	1
χ_1	1	-1	-1	1
χ_2	1	i	$-i$	-1
χ_3	1	$-i$	i	-1

$k=8$: Since $3^2 \equiv 5^2 \equiv 5^2 \equiv 7^2 \equiv 1 \pmod 8$, then $\chi(n) \in \{1,-1\}$.

Also, as $\chi(7) = \chi(3\cdot 5) = \chi(3)\cdot\chi(5)$

	1	3	5	7
χ_0	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	$+1$	-1
χ_3	1	-1	-1	1

Notation: $Z_k := (\mathbb{Z}/k\mathbb{Z})^\times$.

Theorem 6.5: The set C_k of Dirichlet characters modulo k forms an abelian group under multiplication, with identity χ_0 and $\chi^{-1} = \bar{\chi}$.

Furthermore, $C_k \cong Z_k$.

Pf/ The only nontrivial property is $C_k \cong Z_k$. Write $k = 2^a p_1^{e_1} \cdots p_r^{e_r}$, p_i odd primes.

First, if $a=0$, then $Z_k \cong \mathcal{C}_{\phi(p_1^{e_1})} \times \cdots \times \mathcal{C}_{\phi(p_r^{e_r})}$

if $a=1,2$ then $Z_k \cong \mathcal{C}_{2^{a-1}} \times \mathcal{C}_{\phi(p_1^{e_1})} \times \cdots \times \mathcal{C}_{\phi(p_r^{e_r})}$

if $a > 2$ then $Z_{2^a} \cong \mathcal{C}_2 \times \mathcal{C}_{2^{a-2}}$

In all cases, $\exists$ integers $m_1, \dots, m_s$ with $m_1 \cdot \dots \cdot m_s = \phi(k)$ and $Z_k \simeq C_{m_1} \times \dots \times C_{m_s}$

Hence there are integers $g_1, \dots, g_s$ (generators) so that every element of $n \in Z_k$ has a (unique) representation as $n = g_1^{\alpha_1} \cdots g_s^{\alpha_s}$, $0 \leq \alpha_j \leq m_j - 1$.

For any $\chi \in C_k$, χ is determined by $\chi(g_1), \dots, \chi(g_s)$.

Since $g_j^{m_j} = 1$, $\chi(g_j) = e^{2\pi i \ell_j / m_j}$ for $0 \leq \ell_j \leq m_j - 1$.

Every ~~such~~ choice of $\ell_1, \dots, \ell_s$ produces a character $\chi \in C_k$, which will be distinct to the others.

This shows $|C_k| = \phi(k)$. The map $g_1^{\ell_1} \cdots g_s^{\ell_s} \longleftrightarrow \chi$ gives the isomorphism. //

<u>Corollary</u>: C_k is a basis for U_k.

Pf They are a linearly independent set of ~~$\phi(k)$ functions~~ ($\dim U_k$) functions //

<u>Recall</u>: $e_{k,\ell}$ = function in U_k with $e_{k,\ell}(n) = \begin{cases} 1 & n \equiv \ell \ (k) \\ 0 & \text{else} \end{cases}$

<u>Theorem 6.6</u>: If $k \geq 1$, $(\ell, k) = 1$, then: $$e_{k,\ell} = \frac{1}{\phi(k)} \sum_{\chi \in C_k} \overline{\chi}(\ell) \cdot \chi$$

Pf Since C_k and $\{e_{k,\ell} : 1 \leq \ell \leq k, (\ell, k) = 1\}$ are bases of U_k, then there are constants $a_1, \dots, a_{\phi(k)}$ s.t. $e_{k,\ell} = \sum_{j=0}^{\phi(k)-1} a_j \chi_j$, $C_k = \{\chi_0, \dots, \chi_{\phi(k)-1}\}$

Multiply by $\overline{\chi_h}(n)$, and sum over n, $1 \leq n \leq k$ using Thm 6.2:

$$\underbrace{\sum_{n=1}^{k} e_{k,\ell}(n) \overline{\chi_h}(n)}_{= \overline{\chi_h}(\ell)} = \sum_{j=0}^{\phi(k)-1} a_j \left(\sum_{n=1}^{k} \overline{\chi_h}(n) \chi_j(n) \right) = a_h \cdot \phi(k) \Rightarrow a_h = \frac{1}{\phi(k)} \overline{\chi_h}(\ell).$$

//

Corollary: Let f be an arithmetic function. Let $k \geq 1$, $(\ell, k) = 1$.

Then $$\sum_{\substack{y < n \leq x \\ n \equiv \ell \pmod{k}}} f(n) = \frac{1}{\phi(k)} \sum_{\chi \in C_k} \overline{\chi}(\ell) \sum_{y < n \leq x} \chi(n) f(n)$$

Corollary 6.7: If $n \in \mathbb{N}$, then $$\sum_{\chi \in C_k} \chi(n) = \begin{cases} \phi(k) & \text{if } n \equiv 1 \ (k) \\ 0 & \text{else} \end{cases}$$

Pf/ Put $\ell = 1$ in the previous theorem (so $\overline{\chi}(\ell) = 1$) and evaluate at n. //

Dirichlet L-functions

Let $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ be the Dirichlet L-function attached to χ $(\sigma > 1)$

Also, $\sigma_a = 1$ (as for $n \equiv 1 \bmod k$, $\chi(n) = 1$, and $\sum_{n \equiv 1 \pmod{k}} \frac{1}{n} = \sum_{m=1}^{\infty} \frac{1}{km+1}$ diverges.

Euler product: $L(s,\chi) = \prod_p \left(1 - \chi(p) p^{-s}\right)^{-1}$. $(\sigma_a = 1)$.

If $\chi = \chi_0$, then $\sigma_c = 1$

If $\chi \neq \chi_0$, then $\sigma_c = 0$ (will see that).

For $\sigma > 1$: $\frac{-L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s}$.

Also, $+\log(L(s,\chi)) = \sum_p \sum_{m=1}^{\infty} \frac{\chi(p)^m}{m p^{ms}}$

So what is σ_c? ~~If $\chi = \chi_0$~~

- Principal character (mod k): $L(s,\chi_0) = \prod_{p \nmid k} (1 - p^{-s})^{-1} = \zeta(s) \cdot \prod_{p \mid k} (1 - p^{-s})$ $\Rightarrow \sigma_c = 1$.

(note that $\operatorname{res}_{s=1} L(s,\chi_0) = \prod_{p \mid k}\left(1 - \frac{1}{p}\right) = \frac{\phi(k)}{k}$)

- Non-principal characters:

Let χ be a non-principal character mod k.

By (Cor 6.3), $\sum_{n=1}^{k} \chi(n) = 0$. 2

So let $S(x) = \sum_{n \le x} \chi(n)$, then $|S(x)| = \left| \sum_{k[\frac{x}{k}] < n \le x} \chi(n) \right| \le x - k\left[\frac{x}{k}\right] \le k$.

For $N \ge 1$, partial summation

$$\sum_{n=1}^{N} \chi(n) n^{-s} = \frac{S(N)}{N^s} + s\int_1^N S(t) t^{-s-1}\, dt.$$

If $\sigma > 0$, $\frac{S(N)}{N^s} \xrightarrow[N\to\infty]{} 0$, and also the integral converges (because S is bounded).

Thus $\sigma_c \le 0$. (and for $\sigma > 0$, $L(s,\chi) = s \cdot \int_1^\infty S(t) t^{-s-1}\, dt$)

(we haven't used analytic continuation, just the series itself converges!)

<u>Consequences</u>: All Dirichlet L-function $L(s,\chi)$ can be analytically continued to $\sigma > 0$ except for a simple pole of $L(s,\chi_0)$. (for each k).

<u>Example</u>: $k=4$. $L(s,\chi_0) = \sum_{n \text{ odd}} n^{-s} = \zeta(s)(1-2^{-s})$

$$L(s,\chi_1) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots = \prod_{p\equiv 1(4)} (1-p^{-s})^{-1} \prod_{p \equiv 3(4)} (1+p^{-s})^{-1}$$

Note that the Euler product is guaranteed only to converge for $\sigma > \sigma_a$.

. Recall Euler's proof of the infinitude of primes: For $\sigma > 1$,

$$\log \zeta(\sigma) = \sum_p \left(\frac{1}{p^\sigma} + \frac{1}{2p^{2\sigma}} + \cdots \right) = \sum_p \frac{1}{p^\sigma} + O(1)$$

doesn't depend on σ

Since $\zeta(\sigma) \xrightarrow[\sigma\to 1]{} \infty$, then RHS $\to \infty$. Hence $\sum_p \frac{1}{p} \to \infty \Rightarrow \infty$-many primes.

For primes in an arithmetic progression: $p \equiv \ell \pmod k$. For $\sigma > 1$

$$\sum_{p\equiv \ell \,(\mathrm{mod}\, k)} \frac{1}{p^\sigma} = \frac{1}{\phi(k)} \sum_{\chi \in C_k} \bar{\chi}(\ell) \sum_p \frac{\chi(p)}{p^\sigma} = \frac{1}{\phi(k)} \sum_{\chi \in C_k} \bar{\chi}(\ell) \Bigg(\log L(\sigma,\chi) - \overbrace{\sum_{\substack{m\ge 2 \\ p}} \frac{\chi(p)^m}{m p^{m\sigma}}}^{O(1)}\Bigg)$$

↓

Thus, we obtain $\log L(\sigma,\chi) = \sum_p \frac{\chi(p)}{p^\sigma} + O(1)$, and so:

$$(3): \sum_{p\equiv \ell (\mathrm{mod}\, k)} \frac{1}{p^\sigma} = \frac{\log L(\sigma,\chi_0)}{\phi(k)} + \frac{1}{\phi(k)} \sum_{\substack{\chi\in C_k\\ \chi\neq\chi_0}} \bar{\chi}(\ell)\log L(\sigma,\chi) + O(1)$$

As $\sigma \to 1^+$, $L(\sigma,\chi_0)\to\infty$

Assume that for $\chi\neq\chi_0$, $L(1,\chi)\neq 0$. Then as LHS $\to\infty$ as $\sigma\to 1$,

then RHS $\to\infty$ $\Rightarrow$ ~~log~~ $L(\sigma,\chi_0)\to\infty$ $\Rightarrow$ infinitely many primes $p\equiv \ell(k)$.

So we need to prove $L(1,\chi)\neq 0$ $(\chi\neq\chi_0)$.

Recall that $L(1,\chi) = \int_1^\infty S(t)t^{-2}\,dt$ $\leftarrow$ not useful for what we want.

<u>Lemma 6.8</u>: If $\sigma>1$, then $\prod_{\chi\in C_k} L(\sigma,\chi) > 1$

(<u>Note</u>: $\prod_{\chi\in C_k} L(\sigma,\chi)$ is real since $L(\sigma,\chi)L(\sigma,\bar\chi)$ is real)

Pf Let $g(n) = \begin{cases} \frac{1}{m} & \text{if } n=p^m, p \text{ prime} \\ 0 & \text{else}\end{cases}$

By Thm 6.7, $\log \prod_{\chi\in C_k} L(\sigma,\chi) = \sum_{\chi\in C_k} \log L(\sigma,\chi) = \sum_\chi \sum_{n=1}^\infty \frac{g(n)\chi(n)}{n^\sigma} =$

$= \sum_{n=1}^\infty \frac{g(n)}{n^\sigma}\sum_\chi \chi(n) = \phi(k)\sum_{n\equiv 1(k)} \frac{g(n)}{n^\sigma} > 0$ //

~~We say χ is real that $\chi=\bar\chi$ (χ com~~ Say that χ is real if $\chi=\bar\chi$ (χ complex o.w.)

<u>Thm</u>: if χ is complex, then $L(1,\chi)\neq 0$.

Pf If $L(1,\chi)=0$, then $L(1,\bar\chi)=0$. Both $L(s,\chi)$, $L(s,\bar\chi)$ are analytic ($\Rightarrow$ cont) on a nbhd of $s=1$, and each has a 0 of order ≥ 1.

Also, $L(s,\chi_0)$ has a <u>simple</u> pole at $s=1$. Hence $L(\sigma,\chi)L(\sigma,\bar\chi)L(\sigma,\chi_0) \xrightarrow[\sigma\to 1]{} 0 \Rightarrow$!! with prev. Lemma. //

Theorem 6.10 (Landau's oscillation theorem):

Let f be real valued, $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ with $-\infty < \sigma_c < \infty$.

If $F(s)$ can be continued analytically to a region containing the point σ_c, then $\begin{cases} f(n) < 0 \text{ for } \infty\text{ly many } n \text{, and} \\ f(n) > 0 \text{ for } \infty\text{ly many } n. \end{cases}$

Remarks: • $F(s)$ is analytic for $\sigma > \sigma_c$ by Thm 4.9. So we only need to be able to continue it to a nbhd containing σ_c.

• We only need to prove it for $f(n) \geq 0$ (then change $f \to -f$ and get it).

• Contrapositive: If $f(n)$ is of one sign eventually, then $F(s)$ has a singularity at $s = \sigma_c$.

Pf Spse $\forall n \geq N$, $f(n) \geq 0$. Assume also that $F(s)$ is analytic in a nbhd of σ_c $\{s : |s - \sigma_c| < \delta\}$. Let $\beta := \sigma_c + \frac{\delta}{4}$.

By assumption, $F(s)$ has a Taylor series about β, of radius $\geq \frac{3}{4}\delta$.

$$F(s) = \sum_{j=0}^{\infty} \frac{F^{j}(\beta)(s-\beta)^j}{j!}$$

Since $\beta > \sigma_c$, we have $(-1)^j F^{(j)}(\beta) = \sum_{n=1}^{\infty} \frac{f(n)(\log n)^j}{n^\beta} \geq \sum_{n=1}^{M} \frac{f(n)(\log n)^j}{n^\beta}$ $\underline{\underline{\forall M \geq N}}$

Let $s_0 = \sigma_c - \delta/4$. Then: $\left(|s_0 - \beta| < \frac{3}{4}\delta\right)$.

$$F(s_0) = \sum_{j=0}^{\infty} \frac{F^{(j)}(\beta)(s_0-\beta)^j}{j!} \geq \sum_{j=0}^{\infty} \frac{(-1)^j(s_0-\beta)^j}{j!} \sum_{n=1}^{M} \frac{f(n)(\log n)^j}{n^\beta} = \sum_{j=0}^{\infty} \frac{(\beta-s_0)^j}{j!} \sum_{n=1}^{M} \frac{f(n)(\log n)^j}{n^\beta}$$

$$= \sum_{n=1}^{M} \frac{f(n)}{n^\beta} \sum_{j=0}^{\infty} \frac{((\beta-s_0)\log n)^j}{j!} = \sum_{n=1}^{M} \frac{f(n)}{n^\beta} n^{\beta-s_0} = \sum_{n=1}^{M} f(n) n^{-s_0}$$

Thus, $\overset{\text{finite!}}{F(s_0)} \geq \sum_{n=1}^{M} \frac{f(n)}{n^{s_0}}$ for every $M \geq N$.

$\Rightarrow$ partial sums of the series are bounded + mon. increasing $\Rightarrow$ convergent $\Rightarrow$!! //

Examples:

- $f(n)=1$, $\sigma_c=1$. (6.10) $\Rightarrow$ have a singularity at $s=1$.
- $f(n)=(-1)^{n+1}$. $F(s)=1-2^{-s}+3^{-s}\dots=(1-2^{1-s})\zeta(s)$. $\sigma_c=0$.

  And $F(s)$ is analytic at $s=0$. The thm says that it is oscilatory (and it is!).
- $f(n)=\dfrac{1}{\log^2(2n)}$, $\sigma_c=1$. $F(s)=\displaystyle\sum_{n=1}^{\infty}\frac{1/\log^2(2n)}{n^s}$.

  $f(n)>0\ \forall n \Rightarrow (6.10) \Rightarrow F(s)$ has a singularity at $s=1$.

  But: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\log^2(2n)}<\infty$! $\leftarrow$ will have an essential singularity (not a pole).

Theorem 6.11: If χ is a nonprincipal real character, then $L(1,\chi)\neq 0$.

Pf Say $\chi\in C(k)$ (char mod k).

Let $G(s)=\dfrac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}$. ($\chi_0$ ppal char mod k).

Here $L(s,\chi_0)=\zeta(s)\cdot\prod_{p|k}(1-p^{-s})$.

Assume (to contradict) that $L(1,\chi)=0$.

Then the numerator of $G(s)$ is analytic for $\sigma>0$.

The denominator of $G(s)$ is analytic for $\sigma>\frac{1}{2}$, and also

we have $G(\sigma)\xrightarrow[\sigma\to\frac{1}{2}^+]{}0$

For $\sigma>1$, have Euler products: $G(s)=\displaystyle\prod_p\frac{(1-\chi(p)p^{-s})^{-1}(1-\chi_0(p)p^{-s})^{-1}}{(1-\chi_0(p)p^{-2s})^{-1}}=$

$$=\prod_{p\nmid k}\frac{(1-p^{-2s})}{(1-p^{-s})(1-\chi(p)p^{-s})}\cdot\prod_{p|k}1=\prod_{p\nmid k}\frac{1+p^{-s}}{1-\chi(p)p^{-s}}=\prod_{p \text{ s.t. } \chi(p)=1}\frac{1+p^{-s}}{1-p^{-s}}$$

$\downarrow$

By expanding the denominator,

$$G(s) = \prod_{\chi(p)=1} (1+p^{-s})(1+p^{-s}+p^{-2s}+\dots) = \prod_{\chi(p)=1} (1+2p^{-s}+2p^{-2s}+2p^{-3s}+\dots)$$

Writing $G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$, we have $\underset{\geq 0}{g(n) = 2^{w(n)}}$: $w(n) = \#\{p \mid n : \chi(p) = 1\}$.

Let σ_c be the abscissa of convergence of $\sum_{n=1}^{\infty} g(n) n^{-s}$.

By Landau's oscillation thm, $G(s)$ has a singularity at $s = \sigma_c$

We have seen that $\sigma_c \leq \frac{1}{2}$ (as it was analytic for $\sigma > \frac{1}{2}$).

For real $\sigma > \frac{1}{2}$, $G(\sigma) = \sum_{n=1}^{\infty} \frac{g(n)}{n^{\sigma}} \geq 1$.

$\therefore$ As $\sigma \to \frac{1}{2}^+$, $G(\sigma) \not\to 0 \Rightarrow !!$

<u>Corollary</u>: Dirichlet's Theorem. (as we saw).

<u>Theorem 6.12</u>: Let $(h,k)=1$. Then:

i) $$\sum_{\substack{n \leq x \\ n \equiv h\ (k)}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(k)} \log x + O_k(1).$$

ii) $$\sum_{\substack{n \leq x \\ p \equiv h (\mathrm{mod}\ k)}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O_k(1)$$

<u>Proof</u>: (ii) follows from (i) since $0 \leq \sum_{\substack{n \leq x \\ n \equiv h\ (\mathrm{mod}\ k)}} \frac{\Lambda(n)}{n} - \sum_{\substack{n \leq x \\ n \equiv h\ (k)}} \frac{\log p}{p} \leq \sum_{p,\, a \geq 2} \frac{\log p}{p^a} = O(1)$.

To see (i), by orthogonality of characters mod k (6.6 + corollaries),

$$(1) \sum_{\substack{n \leq x \\ n \equiv h\ (k)}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(k)} \sum_{\chi \in C_k} \overline{\chi}(h) \sum_{n \leq x} \frac{\Lambda(n)\chi(n)}{n}$$

Use now that $\log = 1 * \Lambda$, so $(2) \sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{\ell, m \leq x} \frac{\chi(\ell m)\Lambda(m)}{\ell m} = \sum_{m \leq x} \frac{\chi(m)\Lambda(m)}{m} \sum_{\ell \leq \frac{x}{m}} \frac{\chi(\ell)}{\ell}$,

(cont pf)

Let now $S(x) := \sum_{n\leq x} \chi(n)$ for a fixed χ.

Since $S(k)=0$ for $\chi \neq \chi_0$, (In fact, $|S(x)| \leq k$ for $\chi \neq \chi_0$, $x \geq 1$)

the LHS of (2) is $\dfrac{S(x)\log x}{x} + \int_1^x S(t)\dfrac{\log t - 1}{t^2}\,dt = O_k\left(\dfrac{\log x}{x} + \int_1^x \dfrac{\log t}{t^2}\,dt\right) = O_k(1)$

Similarly, $\displaystyle\sum_{\ell \leq y} \frac{\chi(\ell)}{\ell} = \frac{S(y)}{y} + \int_1^y \frac{S(t)}{t^2}\,dt = O_k(1)$.

Write $\displaystyle\sum_{\ell \leq y} \frac{\chi(\ell)}{\ell} = \sum_{\ell=1}^{\infty} \frac{\chi(\ell)}{\ell} - \sum_{\ell > y}\frac{\chi(\ell)}{\ell} = L(1,\chi) - \frac{S(y)}{y} - \int_y^\infty \frac{S(t)}{t^2}\,dt = L(1,\chi) + O_k\left(\frac{1}{y}\right)$

Thus RHS of (2) can be written as: $\displaystyle\sum_{m\leq x} \frac{\Lambda(m)\chi(m)}{m}\left(L(1,\chi) + O_k\left(\frac{m}{x}\right)\right) =$

$\displaystyle = L(1,\chi)\cdot \sum_{m\leq x} \frac{\Lambda(m)\chi(m)}{m} + O_k\left(\frac{1}{x}\sum_{m\leq x}\Lambda(m)\right) = L(1,\chi)\sum_{m\leq x}\frac{\Lambda(m)\chi(m)}{m} + O_k(1).$

Since $L(1,\chi)\neq 0$, conclude that for $\chi \neq \chi_0$, $\displaystyle\sum_{m\leq x} \frac{\Lambda(m)\chi(m)}{m} = O_k(1)$

When $\chi = \chi_0$, $\displaystyle\sum_{n\leq x} \frac{\Lambda(n)\chi_0(n)}{n} = \sum_{\substack{n\leq x\\(n,k)=1}} \frac{\Lambda(n)}{n} = \log x + O_k(1)$ since $\displaystyle\sum_{\substack{n\leq x\\ \underline{\underline{(n,k)>1}}}} \frac{\Lambda(n)}{n} = \sum_{\substack{p\mid k\\ a\geq 1}} \frac{\log p}{p^a} \leq$

$\displaystyle\leq \sum_{p\mid k}\sum_{a} \frac{\log p}{p^a} = \sum_{p\mid k}\frac{\log p}{p-1}$ ← only depends on k.

By (1), $\displaystyle\sum_{\substack{n\leq x\\ n\equiv h(k)}} \frac{\Lambda(n)}{n} = \frac{\log x + O_k(1)}{\varphi(k)} + O_k(1).$

Define now $\displaystyle\psi(x;q,a) := \sum_{\substack{n\leq x\\ n\equiv a\,(q)}} \Lambda(n)$, $\displaystyle\pi(x;q,a) = \sum_{\substack{p\leq x\\ p\equiv a\,(q)}} 1$

<u>Theorem 6.13</u>: (Prime N.T. for Arithmetic Progressions):

If $(a,q)=1$, then

$$\psi(x,q,a) \sim \frac{x}{\varphi(q)}$$

$$\pi(x,q,a) \sim \frac{\mathrm{li}(x)}{\varphi(q)} \sim \frac{x\log x}{\varphi(q)}$$

Primitive Characters:

Def Suppose $\chi \in C_K$, $d \mid K$ and $\chi(n) = 1$ for all $n \equiv 1 \pmod d$ with $(n,K)=1$. Then d is called an <u>induced modulus</u> for χ.

Example: 3 characters mod 15 ($\in C_{15}$):

	1	2	4	7	8	11	13	14
χ_1	1	-1	1	-1	-1	1	-1	1
χ_2	1	-1	1	1	-1	-1	1	-1
χ_3	1	$-i$	-1	i	i	-1	$-i$	1

- 5 is an induced modulus of χ_1.
- 3 is an induced modulus for χ_2.
- 15 is an induced modulus for χ_3.

Consider the character $\psi \in C_5$ given by $\psi(2) = -1$, $\psi(3) = -1$, $\psi(4) = 1$.

Note that $\chi_1(n) = \psi(n)$ when neither is 0.

So $\chi_1(n) = \psi(n) \cdot \chi_0(n)$ where χ_0 is the ~~trivial~~ principal character mod 15.

Def: If $\chi \in C_K$ has no induced modulus $< K$, we say that χ is <u>primitive</u>. Otherwise, χ is <u>imprimitive</u>.

Remarks:
- 1 is an induced modulus for $\chi \Leftrightarrow \chi = \chi_0$.
- if p is prime and $\chi \neq \chi_0$, then χ is primitive.

Theorem 6.14: Let $\chi \in C_K$ and $d \mid K$. Then TFAE:
- d is an induced modulus for χ
- whenever $a \equiv b \pmod d$ and $(a,K) = (b,K)$, $\chi(a) = \chi(b)$. (i.e. $\chi = \chi_0 \cdot F$ where F has period d)

Proof: $\Leftarrow$) clear

$\Rightarrow$) Suppose $(a,K), (b,K) = 1$, and $a \equiv b \pmod d$. Let $a' \equiv a^{-1} \pmod K$. Then $aa' \equiv 1\ (K) \Rightarrow aa' \equiv 1\ (d)$, and also $ba' \equiv 1 \pmod d$. So $\chi(aa') = \chi(ba') = 1$. //

Theorem 6.15: Let $\chi \in C_K$ and $d \mid K$. Then TFAE:
- d is an induced modulus for χ
- $\exists\, \psi \in C_d$ with $\chi = \chi_0 \cdot \psi$, where $\chi_0 \in C_K$.

Pf $\Leftarrow$) clear

$\Rightarrow$) For $(n,d) = 1$, take h_n so that $(n + h_n d, K) = 1$ (eg. $h_n = \prod_{\substack{p \mid K \\ p \nmid n}} p$).

Define now ψ by $\psi(n) := \chi(n + h_n d)$, which is well-defined by the previous theorem (define $\psi(n) = 0$ if $(n,d) > 1$).

Now, $\chi = \chi_0 \psi$, because $\chi(n + h_n d) = \chi(n)$ if $(k,n) = 1$ (χ_0 the ppal mod k).

ψ is clearly periodic mod d. Also ψ is completely-multiplicative, since χ_0 and χ are. //

Def Let $\chi \in C_k$. The smallest induced modulus of χ is called the <u>conductor</u> of χ.

<u>Theorem 6.16</u>: Every $\chi \in C_k$ can be written <u>uniquely</u> as a product $\chi = \chi_0 \cdot \psi$, where $\chi_0^{\text{ppal}} \in C_k$ and ψ is a <u>primitive</u> character modulo the conductor of χ.

Pf Let d be the conductor of χ. By the previous thm, $\chi = \chi_0 \cdot \psi$. $\psi \in C_d$, $\chi_0 \in C_k$.

If ψ were imprimitive, then $\exists\, q \mid d$, $q < d$ which is an induced modulus for ψ.

Then $\psi = \Theta \cdot \chi_0'$, $\chi_0' \in C_d$, $\Theta \in C_q$. Hence $\chi = \chi_0 \chi_0' \Theta = \chi_0 \cdot \Theta \overset{6.16}{\Rightarrow} !!$ //

To show uniqueness, spz $\chi = \chi_0 \cdot \psi = \chi_0 \cdot \psi'$. $\psi, \psi' \in C_d$ primitive.

Spz $\psi(a) \neq \psi'(a)$, where $(a,d) = 1$. Take h_a s.t. $(a + h_a d, k) = 1$.

then $\chi(a + h_a d) = \psi(a) \neq \psi'(a) = \chi(a + h_a d) \Rightarrow !!$ //

<u>Note</u>: if p prime, $\chi(p) = \chi_0(p)\,\psi(p)$.

For $\sigma > 1$, $L(s,\chi) = \prod_p (1 - \chi(p) p^{-s})^{-1} = \prod_{p \nmid k} (1 - \chi(p) p^{-s})^{-1} = \prod_{p \nmid k} (1 - \psi(p) p^{-s})^{-1} =$

$$= L(s,\psi) \cdot \prod_{p \mid k} (1 - \psi(p) p^{-s}) .$$

It could be that $\psi(p) \neq 0$ for $p \mid k$ (i.e if $p \nmid d$ as well).

So $L(s,\chi) = L(s,\psi) \prod_{\substack{p \mid k \\ p \nmid d}} (1 - \psi(p) p^{-s})$ ← entire, zeros on $\sigma = 0$.

<u>Conclusion</u>: $L(s,\chi)$ and $L(s,\psi)$ have the same zeros in the half-plane of conv. ($\sigma > 0$)

• Things about primitive L-functions:

$\chi \in C_k$, $k \geq 3$, χ primitive.

• $L(s,\chi)$ is entire. ($L(s,\chi)$ has a functional equation only for primitive characters)

Let $a := \begin{cases} 0 & \text{if } \chi(-1) = 1 \quad \leftarrow \text{even char's} \\ 1 & \text{if } \chi(-1) = -1 \quad \leftarrow \text{odd char's} \end{cases}$

Let $\xi(s,\chi) := \left(\frac{\pi}{k}\right)^{-\frac{1}{2}(s+a)} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi)$.

Then if χ is primitive, $\xi(1-s,\bar{\chi}) = c(\chi) \cdot \xi(s,\chi)$ where $|c(\chi)| = 1$.

In fact, $c(\chi) = \dfrac{i^a \cdot k^{1/2}}{\tau(\chi)}$ where $\tau(\chi) = \sum_{m\geq 1}^{k} \chi(m) e^{2\pi i m/k}$ (Gauss sum)

Zero free regions (classical type)

Theorem: There exists a constant $c > 0$ s.t. the following holds:

If χ is a complex character, then $L(s,\chi) \neq 0$ for

$$(*) \quad \left\{ \sigma \geq 1 - \frac{c}{\log(\underset{\uparrow \text{ if } \chi \in C_k}{k}(|t|+2))} \right\}.$$

If χ is a real character, then $L(s,\chi)$ has at most one zero in $(*)$.

If such a zero exists, it is real. It is called a Siegel zero.

Let now $N(T,\chi) := \#$ of zeros of $L(s,\chi)$ with $|\operatorname{Im} \rho| \leq T$.

It is known that $N(T,\chi) = \frac{T}{\pi} \log \frac{kT}{2\pi} - \frac{T}{\pi} + O(\log kT)$. $\underset{\substack{\text{if } k \text{ fixed.} \\ T \to \infty.}}{\sim} \frac{T}{\pi} \log T$

Explicit Formulas:

Let $\psi(x,\chi) = \sum_{n \leq x} \Lambda(n) \cdot \chi(n)$

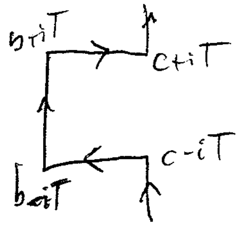
Then $\sum_{\substack{n \leq x \\ n \equiv h \,(\mathrm{mod}\, k)}} \Lambda(n) = \frac{1}{\phi(k)} \sum_{\chi \in C_k} \bar{\chi}(h) \psi(x,\chi)$.

↓

Let now ρ be a generic zero of $L(s,\chi)$ with $0 < \operatorname{Re}\rho < 1$, and assume that χ is primitive and nonprincipal.

If $x > 1$, $x \notin \mathbb{N}$ then $\psi(x,\chi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds$. $(c>1)$

Move a (truncated) contour to:



Let $b \to -\infty$
$T \to \infty$

$\lim_{T\to\infty}\left(\sum_{\rho,\,|\operatorname{Im}\rho|\le T}\frac{x^\rho}{\rho}\right)$.

trivial zeros (< real axis)

This leads to $\psi(x,\chi) = -\sum_\rho \frac{x^\rho}{\rho} + b(\chi) + \sum_{\substack{m=1\\ m\equiv a \pmod 2}}^{\infty}\frac{x^{-m}}{m} - (1-a)\log x$.

from the pole at $s=0$. (if χ is odd).

Usually, the only important term is $-\sum_\rho \frac{x^\rho}{\rho}$, as the rest is $O(\log x)$.

Let $\pi(x,k,h) = \#\{\text{primes } p \le x : p \equiv h \pmod k\}$; $\psi(x;k,h) := \sum_{\substack{n\le x\\ n\equiv h\,(k)}}\Lambda(n)$

<u>Theorem</u>: For each fixed k, there is $c = c_k$ s.t.:

$$(1)\quad \pi(x;k,h) = \frac{1}{\phi(k)}\,li(x) + O_k\left(x e^{-c_k\sqrt{\log x}}\right) \qquad (h,k)=1$$

<u>Theorem</u> (Siegel-Walfisz): Let $A > 0$. Then (1) holds uniformly for c_k = constant, and $1 \le k \le (\log x)^A$.

That is, $\forall A, \exists C$ s.t when $1 \le k \le (\log x)^A$,

$$\pi(x;k,h) = \frac{li(x)}{\phi(k)} + O_A\left(x e^{-c\cdot\sqrt{\log x}}\right)$$

<u>Theorem</u> (Gallagher, 1970): Let $f(x) \to 0$ as $x\to\infty$, decreasing.

<u>Then</u>: $\pi(x;q,a) = \frac{li(x)}{\phi(k)} + O\left(x e^{-c\sqrt{\log x}}\right)$ uniformly for $1 \le q \le x^{f(x)}$, $((a,q)=1)$ with the possible exception of those q divisible by a certain q_0.
(q_0 is a modulus where for some $\chi \in C_{q_0}$, $L(s,\chi)$ has a Siegel zero).

Another theorem:

Theorem (Bombieri, independently A. I. Vinogradov 1965):

$\forall A \geq 0$, $\exists B > 0$ s.t. if $Q = \sqrt{x}(\log x)^{-B}$, then:

"RH on average"

$$(*) \quad \sum_{q \leq Q} \max_{(a,q)=1} \max_{y \leq x} \left| \pi(y;q,a) - \frac{li(y)}{\phi(q)} \right| \lll \frac{x}{(\log x)^A}.$$

Conjecture: This is true for $Q = x^{1-\varepsilon}$ $(\forall \varepsilon > 0)$. (Elliott-Halberstam conjecture)

Theorem (Goldston, Pintz, Yildirim, 2005):

Background:

Define $\Theta := \liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log n}$. The PNT $\Rightarrow \Theta \leq 1$.

In 1940, Erdős showed $\Theta < 1$.

⋮

1988, Maier showed $\Theta < 1/4$.

The Thm GPY proves that $\Theta = 0$!. (Note that the twin prime conjecture $\Rightarrow \Theta = 0$ also).

If $(*)$ holds for $Q = x^{\frac{1}{2}+\delta}$ for some $\delta > 0$, then $p_{n+1} - p_n = O_\delta(1)$ ∞-many often!

The Extended Riemann Hypothesis (ERH) says that $L(s,\chi) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$.
This implies that $\pi(x;q,a) = \frac{li(x)}{\phi(q)} + O(\sqrt{x}\log x)$.

Examples: Let $K/\mathbb{Q}$ be a number field. We can construct:

$$\zeta_K(s) := \sum_{\mathfrak{a} \neq 0} \frac{1}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - (N\mathfrak{p})^{-s}\right)^{-1} \quad \text{(Dedekind-zeta function)}$$

- Let $E: y^2 = x^3 + ax + b$ an elliptic curve. #(solutions mod p) $= p + 1 + a_p$

$$|a_p| \leq 2\sqrt{p}$$

$$L(s,E) := \prod_p \left(\frac{1}{1 - a_p p^{-s} + p^{1-2s}} \right) \cdot \prod_{p \text{ bad}} (\dots)$$

Further Topics

- **Exponential sums:**

$$\sum_{n\in N} e^{2\pi i f(n)} \quad , \quad N \subseteq \mathbb{Z}, \quad f(n) \in \mathbb{R}.$$

Example: count solutions to an equation $F(x_1,\dots,x_n)=0$ $(*)$ where $F(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$.

Say $x_1\in N_1,\dots,x_n\in N_n$. (finite sets)

Then the number of solutions of $(*)$ with these conditions is:

$$\int_0^1 \Big(\sum_{\substack{x_1\in N_1\\ x_n\in N_n}} e^{2\pi i F(x_1,\dots,x_n)\alpha}\Big)\, d\alpha \qquad \left(\text{as } \int_0^1 e^{2\pi i k\alpha}\, d\alpha = \begin{cases} 1 & k=0\\ 0 & k\neq 0\end{cases}\right)$$

Special case: Waring's problem.

Lagrange (1770): $\forall n\in\mathbb{N}$, $n=a^2+b^2+c^2+d^2$ $(a,b,c,d\geq 0)$ (and e.g. 7 is not the sum of 3 squares).

Waring (1770): conjectured that $\forall n\in\mathbb{N}$, it was the sum of 9 non-negative cubes. (23 requires 9 cubes). (proved 1909)

Also, as 19 4th powers. (proved 1980's).

Let $g(k)=\text{minimum}\{s$ s.t every positive integer is the sum of s non-negative k^{th} powers$\}$.

Hilbert (1909): $\forall k\geq 2$ $g(k)<\infty$. (used algebraic methods).

Let $r_k(n) = \#$ of solutions of $x_1^k+\dots+x_s^k=n$, $(x_i\geq 0)$. $(x_i\leq n^{1/k})$

Hardi/Ramanujan

$$= \int_0^1 f(\alpha)^s\, e(-\alpha n)\, d\alpha, \qquad f(\alpha)=\sum_{0\leq x\leq n^{1/k}} e(\alpha x^k).$$

$$e(q)=e^{2\pi i q}$$

This allows to get, not only that $g(k)<\infty$, but also estimates (asymptotics) for $g(k)$:

Hardi-Littlewood: $|f(\alpha)|$ "large" when α is near $\frac{a}{q}$, $a\in\mathbb{Z}$, $q\in\mathbb{Z}$, q small. (major arcs)

Otherwise, $|f(\alpha)|$ is "small". (minor arcs)

Hardy and Littlewood created this, which is called "the circle method".

Another problem:

Count # solution $F(x_1, \dots, x_n) \equiv 0 \pmod{p}$.

$$N = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{x_1=0}^{p-1} \cdots \sum_{x_n=0}^{p-1} e^{2\pi i F(x_1, \dots, x_n) \frac{a}{p}}.$$

Inequalities can also be studied using exponential sums.

Riemann ζ-function:

Let $N \in \mathbb{N}$, $\sigma > 1$. $\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \int_N^\infty \frac{1}{t^s} d[t] = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{[t]}{t^s}\Big|_{N^+}^{\infty} +$

$+ s \int_N^\infty \frac{[t]}{t^{s+1}} dt$. Write $[t] = t - \{t\}$, and get

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + N^{1-s} \cancel{+} s \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{\{t\}}{t^{s+1}} dt$$

Converges for $\sigma > 0$. $\Rightarrow$ analytic cont also.

Let $\sigma \geq \frac{1}{2}$. Then $\left| \int_N^\infty \frac{\{t\}}{t^{s+1}} dt \right| \leq \int_N^\infty \frac{dt}{t^{3/2}} = \frac{2}{\sqrt{N}}$

In this case, $\left| \zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s} \right| \leq \frac{N^{1/2}}{|s-1|} + \frac{2|s|}{\sqrt{N}}$. (Take $N := |s|^2 + O(1)$ (optimal)) $= O(1)$

(if $N = |s|^2 + O(1)$)

So $\zeta(s) = \sum_{n=1}^{\lfloor |s|^2 \rfloor} \frac{1}{n^s} + O(1)$ break into intervals $\sum_{M < n \leq 2M} \frac{1}{n^s}$

Then this can be written in terms of $\sum_{M < n \leq \mathcal{M}} \frac{1}{n^{it}}$, $\sigma + it = s$ $(M < \mathcal{M} \leq 2M)$

Note that $n^{-it} = e^{2\pi i \left(\frac{t \log n}{2\pi}\right)}$ $\rightarrow$ so have an exponential sum!

These ideas led to the estimates by Vinogradov - Korobov:

$$|\zeta(1+it)| \ll \left(\log(|t|+2)\right)^{2/3} \Rightarrow \zeta(s) \neq 0 \text{ for } \sigma \geq 1 - \frac{c}{(\log|t|)^{2/3} (\log\log|t|)^{1/3}}$$

$(|t| \geq 10)$

which leads to the best known bound for the prime number theorem.

Sieve Methods

Q: Are there infinitely-many primes p with $p+2$ prime? (Twin-prime).

Q: Is every even number $n \geq 4$ the sum of two primes? (Goldbach conj.)

Q: Are there infinitely-many primes of the form x^2+1.

Known (via Sieve methods):

- ∞ly many primes p so that $p+2$ is either prime or the product of two primes. (theorem of Chen '60-'70). We say that $p+2 \in \mathbb{P}_2$ infinitely often.
- Every large even number can be written as $p+q$, where p is prime, $q \in \mathbb{P}_2$. (this is also due to Chen).
- $\mathbb{P}_2$ contains infinitely many terms of the form x^2+1 (due to Iwaniec).

Sieve methods can be used to solve other problems: ∞ly many primes of the form x^2+y^4. (due to Friedlander-Iwaniec, 1998).

- $\displaystyle\sum_{\substack{p \\ p+2 \text{ prime}}} \frac{1}{p} < \infty$. (Viggo Brun, 1915).

Basic Sieve Setup.

Have a set (finite) of integers $\mathcal{A}$.

Have a finite set of primes $\mathcal{P}$.

$$S(\mathcal{A},\mathcal{P}) := \#\{a \in \mathcal{A} : a \text{ is divisible by no prime } p \in \mathcal{P}\}.$$

Example:

$\mathcal{A} := \{1, 2, \dots, N\}$, $\mathcal{P} := \{\text{primes } p \leq \sqrt{N}\}$.

Then $S(\mathcal{A},\mathcal{P}) = \pi(N) + \underbrace{1}_{\text{the number 1 is unsieved!}} - \pi(\sqrt{N}) = \pi(N) + O(\sqrt{N})$.

Twin primes:

$\mathcal{A} := \{ n(n+2) : 1 \leq n \leq N \}$.

$\mathcal{P} := \{ \text{primes} \leq \sqrt{N+2} \}$.

Then $S(\mathcal{A},\mathcal{P}) = \# \{ \sqrt{N+2} < n \leq N : n, n+2 \text{ are both prime} \}$. $\left(\text{is } S(\mathcal{A},\mathcal{P}) \underset{N\to\infty}{\to} \infty \text{ ?} \right)$

Goldbach: (given N, to be represented as $N = p_1 + p_2$).

$\mathcal{A} := \{ n\cdot(N-n) : 1 \leq n \leq N-1 \}$.

$\mathcal{P} := \{ \text{primes} \leq \sqrt{N} \}$.

$S(\mathcal{A},\mathcal{P}) = \# \{ \sqrt{N} < n \leq N-1 : n \;\&\; N-n \text{ are both prime} \}$. $\left(\text{is it} \geq 1 \text{ ?} \right)$

What do we expect?

Define $\nu(p) := \dfrac{\# \{ a \in \mathcal{A} : p \mid a \}}{\# \mathcal{A}}$

if the events were independent.

We expect then that $S(\mathcal{A},\mathcal{P}) \approx \#\mathcal{A} \cdot \prod_{p\in\mathcal{P}} (1-\nu(p))$

If we applied this "heuristic" to $\mathcal{A} = \{1,\dots,N\}$, $\mathcal{P} = \{\text{primes} \leq \sqrt{N}\}$.

Then $S(\mathcal{A},\mathcal{P}) \approx N \cdot \prod_{p\leq\sqrt{N}} \left(1 - \frac{1}{p}\right) \underset{\text{asymptotic}}{\sim} N \cdot \dfrac{e^{-\gamma}}{\log \sqrt{N}} = 2e^{-\gamma} \dfrac{N}{\log N}$ ← bad.

!

In general, we get from this upper bounds:

$S(\mathcal{A},\mathcal{P}) \ll \#\mathcal{A} \prod_{p\in\mathcal{P}} (1-\nu(p))$ can be proved.

If we shrink the set of primes, we can estimate things... but there are other problems:

$\mathcal{A} := \{ n(n+2) : 1 \leq n \leq N \}$

$\mathcal{P} = \{ \text{primes} \leq \sqrt[3]{N} \}$.

Then $S(\mathcal{A},\mathcal{P})$ will include numbers n where $n, n+2$ are both in $\mathcal{P}_2$!

That's what allows one to prove that type of theorems we've seen.

E.O.C.

when α is not differentiable or even when α is discontinuous. In fact, it is in dealing with *discontinuous* α that the importance of the Stieltjes integral becomes apparent. By a suitable choice of a discontinuous α, any finite or infinite sum can be expressed as a Stieltjes integral, and summation and ordinary Riemann integration then become special cases of this more general process. Problems in physics which involve mass distributions that are partly discrete and partly continuous can also be treated by using Stieltjes integrals. In the mathematical theory of probability this integral is a very useful tool that makes possible the simultaneous treatment of continuous and discrete random variables.

9–2 Notations. For brevity, we shall make certain stipulations concerning notations and terminology to be used in this chapter. We shall be working with a finite interval $[a, b]$ and, unless otherwise stated, all functions denoted by f, g, α, or β will be assumed to be real-valued functions defined and *bounded* on $[a, b]$. Complex-valued functions will be dealt with in the latter part of the chapter. Extensions to unbounded functions and infinite intervals will be discussed in Chapter 14.

As in Chapter 8, a partition P of $[a, b]$ is a finite set of points, say $P = \{x_0, x_1, \ldots, x_n\}$, such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The norm of P is the largest of the n numbers $\Delta x_k = x_k - x_{k-1}$, and is denoted by $|P|$. The partition P' of $[a, b]$ is finer than P (or a refinement of P) if $P \subset P'$. Thus, $P \subset P'$ implies $|P'| \leq |P|$, but the converse does not hold. The symbol $\Delta\alpha_k$ denotes the difference $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$, so that $\sum_{k=1}^{n} \Delta\alpha_k = \alpha(b) - \alpha(a)$. Finally, the set of all possible partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$.

9–3 The definition of the Riemann-Stieltjes integral.

9–1 DEFINITION. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the subinterval $[x_{k-1}, x_k]$. A sum of the form*

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k$$

is called a Riemann-Stieltjes sum of f with respect to α. We say f is Riemann-integrable with respect to α on $[a, b]$, and we write "$f \in R(\alpha)$ on $[a, b]$" if there exists a number A having the following property: For every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that for every partition P finer than P_ϵ and for every choice of the points t_k in $[x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \epsilon$.

When such a number A exists, it is uniquely determined and is denoted by $\int_a^b f\, d\alpha$ or by $\int_a^b f(x)\, d\alpha(x)$. We also say that the Riemann-Stieltjes integral $\int_a^b f\, d\alpha$ *exists*. The functions f and α are referred to as the *integrand* and the *integrator*, respectively. In the special case when $\alpha(x) = x$, we write $S(P, f)$ instead of $S(P, f, \alpha)$, and $f \in R$ instead of $f \in R(\alpha)$. The integral is then called a Riemann integral and is denoted by $\int_a^b f\, dx$ or by $\int_a^b f(x)\, dx$. The numerical value of $\int_a^b f(x)\, d\alpha(x)$ depends only on f, α, a, and b, and does not depend on the symbol x. The letter x is a "dummy variable" and may be replaced by any other convenient symbol.

NOTE. This is one of several accepted definitions of the Riemann-Stieltjes integral. An alternative (but not equivalent) definition is stated in Exercise 9–3.

9–4 Linearity properties. It is an easy matter to prove that the integral operates in a linear fashion on both the integrand and the integrator. This is the content of the next two theorems.

9–2 THEOREM. *If $f \in R(\alpha)$ and if $g \in R(\alpha)$ on $[a, b]$, then $c_1 f + c_2 g \in R(\alpha)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

Proof. Let $h = c_1 f + c_2 g$. Given a partition P of $[a, b]$, we can write

$$S(P, h, \alpha) = \sum_{k=1}^{n} h(t_k)\, \Delta\alpha_k = c_1 \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k + c_2 \sum_{k=1}^{n} g(t_k)\, \Delta\alpha_k$$
$$= c_1 S(P, f, \alpha) + c_2 S(P, g, \alpha).$$

Given $\epsilon > 0$, choose P'_ϵ so that $P'_\epsilon \subset P$ implies $|S(P, f, \alpha) - \int_a^b f\, d\alpha| < \epsilon$, and choose P''_ϵ so that $P''_\epsilon \subset P$ implies $|S(P, g, \alpha) - \int_a^b g\, d\alpha| < \epsilon$. If we take $P_\epsilon = P'_\epsilon \cup P''_\epsilon$, then, for P finer than P_ϵ, we have

$$\left| S(P, h, \alpha) - c_1 \int_a^b f\, d\alpha - c_2 \int_a^b g\, d\alpha \right| \leq |c_1|\epsilon + |c_2|\epsilon,$$

and this proves the theorem.

9–3 THEOREM. *If $f \in R(\alpha)$ and $f \in R(\beta)$ on $[a, b]$, then $f \in R(c_1\alpha + c_2\beta)$ on $[a, b]$ (for any two constants c_1 and c_2) and we have*

$$\int_a^b f\, d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta.$$

The proof is similar to that of Theorem 9-2 and is left as an exercise.

A result somewhat analogous to the previous two theorems tells us that the integral is also "additive" with respect to the interval of integration.

9-4 THEOREM. *Assume that $c \in (a, b)$. If two of the three integrals in (1) exist, then the third also exists and we have*

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha. \tag{1}$$

Proof. If P is a partition of $[a, b]$ such that $c \in P$, let $P' = P \cap [a, c]$ and $P'' = P \cap [c, b]$ denote the corresponding partitions of $[a, c]$ and $[c, b]$, respectively. The Riemann-Stieltjes sums for these partitions are connected by the equation

$$S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha).$$

Assume that $\int_a^c f\,d\alpha$ and $\int_c^b f\,d\alpha$ exist. Then, given $\epsilon > 0$, there is a partition P'_ϵ of $[a, c]$ such that

$$\left|S(P', f, \alpha) - \int_a^c f\,d\alpha\right| < \epsilon/2 \qquad \text{whenever } P' \text{ is finer than } P'_\epsilon,$$

and a partition P''_ϵ of $[c, b]$ such that

$$\left|S(P'', f, \alpha) - \int_c^b f\,d\alpha\right| < \epsilon/2 \qquad \text{whenever } P'' \text{ is finer than } P''_\epsilon.$$

Then $P_\epsilon = P'_\epsilon \cup P''_\epsilon$ is a partition of $[a, b]$ such that P finer than P_ϵ implies $P'_\epsilon \subset P'$ and $P''_\epsilon \subset P''$. Hence, if P is finer than P_ϵ, we can combine the foregoing results to obtain the inequality

$$\left|S(P, f, \alpha) - \int_a^c f\,d\alpha - \int_c^b f\,d\alpha\right| < \epsilon.$$

This proves that $\int_a^b f\,d\alpha$ exists and equals $\int_a^c f\,d\alpha + \int_c^b f\,d\alpha$. The reader can easily verify that a similar argument proves the theorem in the remaining cases.

Using mathematical induction, we can prove a similar result for a decomposition of $[a, b]$ into a finite number of subintervals.

NOTE. The preceding type of argument cannot be used to prove that the integral $\int_a^c f\,d\alpha$ exists whenever $\int_a^b f\,d\alpha$ exists. The conclusion is correct, however. For integrators α of bounded variation, this fact will later be proved in Theorem 9-21.

9-5 DEFINITION. *If $a < b$, we define $\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha$ whenever $\int_a^b f\,d\alpha$ exists. We also define $\int_a^a f\,d\alpha = 0$.*

The equation in Theorem 9-4 can now be written as follows:

$$\int_a^b f\,d\alpha + \int_b^c f\,d\alpha + \int_c^a f\,d\alpha = 0.$$

9-5 Integration by parts. A remarkable connection exists between the integrand and the integrator in a Riemann-Stieltjes integral. The existence of $\int_a^b f\,d\alpha$ implies the existence of $\int_a^b \alpha\,df$, and the converse is also true. Moreover, a very simple relation holds between the two integrals.

9-6 THEOREM. *If $f \in R(\alpha)$ on $[a, b]$, then $\alpha \in R(f)$ on $[a, b]$ and we have*

$$\int_a^b f(x)\,d\alpha(x) + \int_a^b \alpha(x)\,df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

NOTE. This equation, which provides a kind of reciprocity law for the integral, is known as the *formula for integration by parts.*

Proof. Let $\epsilon > 0$ be given. Since $\int_a^b f\,d\alpha$ exists, there is a partition P_ϵ of $[a, b]$ such that for every P' finer than P_ϵ, we have

$$\left|S(P', f, \alpha) - \int_a^b f\,d\alpha\right| < \epsilon. \tag{2}$$

Consider an arbitrary Riemann-Stieltjes sum for the integral $\int_a^b \alpha\,df$, say

$$S(P, \alpha, f) = \sum_{k=1}^n \alpha(t_k)\,\Delta f_k = \sum_{k=1}^n \alpha(t_k)f(x_k) - \sum_{k=1}^n \alpha(t_k)f(x_{k-1}),$$

where P is finer than P_ϵ. Writing $A = f(b)\alpha(b) - f(a)\alpha(a)$, we have the identity

$$A = \sum_{k=1}^n f(x_k)\alpha(x_k) - \sum_{k=1}^n f(x_{k-1})\alpha(x_{k-1}).$$

Subtracting the last two displayed equations, we find

$$A - S(P, \alpha, f) = \sum_{k=1}^{n} f(x_k)[\alpha(x_k) - \alpha(t_k)] + \sum_{k=1}^{n} f(x_{k-1})[\alpha(t_k) - \alpha(x_{k-1})].$$

The two sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where P' is that partition of $[a, b]$ obtained by taking the points x_k and t_k together. Then P' is finer than P and hence finer than P_ϵ. Therefore the inequality (2) is valid and this means that we have

$$\left|A - S(P, \alpha, f) - \int_a^b f\, d\alpha\right| < \epsilon$$

whenever P is finer than P_ϵ. But this is exactly the statement that $\int_a^b \alpha\, df$ exists and equals $A - \int_a^b f\, d\alpha$.

9–6 Change of variable in a Riemann-Stieltjes integral.

9–7 THEOREM. *Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having endpoints c and d. Assume that $a = g(c)$, $b = g(d)$. Let h and β be the composite functions defined as follows:*

$$h(x) = f[g(x)], \qquad \beta(x) = \alpha[g(x)], \qquad \text{if } x \in S.$$

Then $h \in R(\beta)$ on S and we have

$$\int_a^b f\, d\alpha = \int_c^d h\, d\beta.$$

Proof. For definiteness, assume that g is strictly increasing on S. (This implies $c < d$.) Then g is one-to-one and has a strictly increasing, continuous inverse g^{-1} defined on $[a, b]$. Therefore, for every partition $P = \{y_0, \ldots, y_n\}$ of $[c, d]$, there corresponds one and only one partition $P' = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $x_k = g(y_k)$. In fact, we can write $P' = g(P)$ and $P = g^{-1}(P')$. Furthermore, a refinement of P produces a corresponding refinement of P', and the converse also holds.

If $\epsilon > 0$ is given, there is a partition P'_ϵ of $[a, b]$ such that P' finer than P'_ϵ implies $|S(P', f, \alpha) - \int_a^b f\, d\alpha| < \epsilon$. Let $P_\epsilon = g^{-1}(P'_\epsilon)$ be the corresponding partition of $[c, d]$, and let $P = \{y_0, \ldots, y_n\}$ be a partition of $[c, d]$ finer than P_ϵ. Form a Riemann-Stieltjes sum

$$S(P, h, \beta) = \sum_{k=1}^{n} h(u_k)\, \Delta\beta_k,$$

where $u_k \in [y_{k-1}, y_k]$ and $\Delta\beta_k = \beta(y_k) - \beta(y_{k-1})$. If we put $t_k = g(u_k)$ and $x_k = g(y_k)$, then $P' = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$ finer than P'_ϵ. Moreover, we then have

$$S(P, h, \beta) = \sum_{k=1}^{n} f[g(u_k)]\{\alpha[g(y_k)] - \alpha[g(y_{k-1})]\}$$
$$= \sum_{k=1}^{n} f(t_k)\{\alpha(x_k) - \alpha(x_{k-1})\} = S(P', f, \alpha),$$

since $t_k \in [x_{k-1}, x_k]$. Therefore, $|S(P, h, \beta) - \int_a^b f\, d\alpha| < \epsilon$ and the theorem is proved.

NOTE. This theorem applies, in particular, to Riemann integrals, that is, when $\alpha(x) = x$. Another theorem of this type, in which g is not required to be monotonic, will later be proved for Riemann integrals. (See Theorem 9–33.)

9–7 Reduction to a Riemann integral.

The next theorem tells us that we are permitted to replace the symbol $d\alpha(x)$ by $\alpha'(x)\, dx$ in the integral $\int_a^b f(x)\, d\alpha(x)$ whenever α has a continuous derivative α'.

9–8 THEOREM. *Let $f \in R(\alpha)$ on $[a, b]$ and assume that α has a continuous derivative α' on $[a, b]$. Then the Riemann integral $\int_a^b f(x)\alpha'(x)\, dx$ exists and we have*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx.$$

Proof. Let $g(x) = f(x)\alpha'(x)$ and consider a Riemann sum

$$S(P, g) = \sum_{k=1}^{n} g(t_k)\, \Delta x_k = \sum_{k=1}^{n} f(t_k)\alpha'(t_k)\, \Delta x_k.$$

The same partition P and the same choice of the t_k can be used to form the Riemann-Stieltjes sum

$$S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)\, \Delta\alpha_k.$$

Applying the Mean Value Theorem, we can write

$$\Delta\alpha_k = \alpha'(v_k)\,\Delta x_k, \qquad \text{where} \qquad v_k \in (x_{k-1}, x_k),$$

and hence

$$S(P, f, \alpha) - S(P, g) = \sum_{k=1}^{n} f(t_k)[\alpha'(v_k) - \alpha'(t_k)]\,\Delta x_k.$$

Since f is bounded, we can write $|f(x)| \leq M$ for all x in $[a, b]$, where $M > 0$. Continuity of α' on $[a, b]$ implies uniform continuity on $[a, b]$. Hence, if $\epsilon > 0$ is given, there exists a $\delta > 0$ (depending only on ϵ) such that

$$0 \leq |x - y| < \delta \qquad \text{implies} \qquad |\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{2M(b-a)}.$$

If we take a partition P'_ϵ with norm $|P'_\epsilon| < \delta$, then for any finer partition P we will have $|\alpha'(v_k) - \alpha'(t_k)| < \epsilon/[2M(b - a)]$ in the preceding equation. For such P we therefore have

$$|S(P, f, \alpha) - S(P, g)| < \frac{\epsilon}{2}.$$

On the other hand, since $f \in R(\alpha)$ on $[a, b]$, there exists a partition P''_ϵ such that P finer than P''_ϵ implies

$$\left| S(P, f, \alpha) - \int_a^b f\,d\alpha \right| < \frac{\epsilon}{2}.$$

Combining the last two inequalities, we see that when P is finer than $P_\epsilon = P'_\epsilon \cup P''_\epsilon$, we will have $|S(P, g) - \int_a^b f\,d\alpha| < \epsilon$, and this proves the theorem.

9–8 Step functions as integrators. If α is constant throughout $[a, b]$, the integral $\int_a^b f\,d\alpha$ exists and has the value 0, since each sum $S(P, f, \alpha) = 0$. However, if α is constant except for a jump discontinuity at one point, the integral $\int_a^b f\,d\alpha$ need not exist and, if it does exist, its value need not be zero. The situation is described more fully in the following theorem:

9–9 THEOREM. *Given $a < c < b$. Define α on $[a, b]$ as follows: The values $\alpha(a)$, $\alpha(c)$, $\alpha(b)$ are arbitrary; $\alpha(x) = \alpha(a)$ if $a \leq x < c$, and $\alpha(x) = \alpha(b)$ if $c < x \leq b$. Let f be defined on $[a, b]$ in such a way that at least one of the functions f or α is continuous from the left at c and at least one is continuous from the right at c. Then $f \in R(\alpha)$ on $[a, b]$ and we have*

$$\int_a^b f\,d\alpha = f(c)[\alpha(c+) - \alpha(c-)].$$

NOTE. The result also holds if $c = a$, provided that we write $\alpha(c)$ for $\alpha(c-)$, and it holds for $c = b$ if we write $\alpha(c)$ for $\alpha(c+)$. We will prove later (Theorem 9–28) that the integral does not exist if both f and α are discontinuous from the right or from the left at c.

Proof. If $c \in P$, every term in the sum $S(P, f, \alpha)$ is zero except the two terms arising from the subinterval separated by c, say

$$S(P, f, \alpha) = f(t_{k-1})[\alpha(c) - \alpha(c-)] + f(t_k)[\alpha(c+) - \alpha(c)],$$

where $t_{k-1} \leq c \leq t_k$. This equation can also be written as follows:

$$\Delta = [f(t_{k-1}) - f(c)][\alpha(c) - \alpha(c-)] + [f(t_k) - f(c)][\alpha(c+) - \alpha(c)],$$

where $\Delta = S(P, f, \alpha) - f(c)[\alpha(c+) - \alpha(c-)]$. Hence we have

$$|\Delta| \leq |f(t_{k-1}) - f(c)||\alpha(c) - \alpha(c-)| + |f(t_k) - f(c)||\alpha(c+) - \alpha(c)|.$$

If f is continuous at c, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|P| < \delta$ implies

$$|f(t_{k-1}) - f(c)| < \epsilon \qquad \text{and} \qquad |f(t_k) - f(c)| < \epsilon.$$

In this case, we obtain the inequality

$$|\Delta| \leq \epsilon|\alpha(c) - \alpha(c-)| + \epsilon|\alpha(c+) - \alpha(c)|.$$

But this inequality holds whether or not f is continuous at c. For example, if f is discontinuous both from the right and from the left at c, then $\alpha(c) = \alpha(c-)$ and $\alpha(c) = \alpha(c+)$ and we get $\Delta = 0$. On the other hand, if f is continuous from the left and discontinuous from the right at c, we must have $\alpha(c) = \alpha(c+)$ and we get $|\Delta| \leq \epsilon|\alpha(c) - \alpha(c-)|$. Similarly, if f is continuous from the right and discontinuous from the left at c, we have $\alpha(c) = \alpha(c-)$ and $|\Delta| \leq \epsilon|\alpha(c+) - \alpha(c)|$. Hence the last displayed inequality holds in every case. This proves the theorem.

Theorem 9–9 tells us that the value of a Riemann-Stieltjes integral can be altered by changing the value of f at a single point and it is easy to see that the *existence* of the integral can also be affected by such a change.

Consider the following example:

$$\alpha(x) = 0 \text{ if } x \neq 0, \quad \alpha(0) = -1,$$
$$f(x) = 1, \qquad \text{if } -1 \leq x \leq +1.$$

In this case Theorem 9-9 implies $\int_{-1}^{1} f\,d\alpha = 0$. But if we re-define f so that $f(0) = 2$ and $f(x) = 1$ if $x \neq 0$, we can easily see that $\int_{-1}^{1} f\,d\alpha$ will not exist. In fact, when P is a partition which includes 0 as a point of subdivision, we find

$$S(P, f, \alpha) = f(t_k)[\alpha(x_k) - \alpha(0)] + f(t_{k-1})[\alpha(0) - \alpha(x_{k-2})]$$
$$= f(t_k) - f(t_{k-1}),$$

where $x_{k-2} \leq t_{k-1} \leq 0 \leq t_k \leq x_k$. The value of this sum is 0, 1, or -1, depending on the choice of t_k and t_{k-1}. Hence, $\int_{-1}^{1} f\,d\alpha$ does not exist in this case. However, in a Riemann integral $\int_a^b f(x)\,dx$, the values of f *can* be changed at a finite number of points without affecting either the existence or the value of the integral. To prove this, it suffices to consider the case where $f(x) = 0$ for all x in $[a, b]$ except for one point, say $x = c$. But for such a function it is obvious that $|S(P, f)| \leq |f(c)||P|$. Since $|P|$ can be made arbitrarily small, it follows that $\int_a^b f(x)\,dx = 0$.

The integrator α in Theorem 9-9 is a special case of an important class of functions known as *step functions*. These are functions which are constant throughout an interval except for a finite number of jump discontinuities.

9-10 DEFINITION (*Step function*). *Let α be defined on $[a, b]$ in such a way that α is discontinuous at a finite number of points c_k, where*

$$a \leq c_1 < c_2 < \cdots < c_n \leq b.$$

If α is constant in each open subinterval (c_{k-1}, c_k), then α is called a step function and the number $\alpha(c_k+) - \alpha(c_k-)$ is called the jump at c_k. If $c_1 = a$, the jump at c_1 is $\alpha(c_1+) - \alpha(c_1)$, and if $c_n = b$, the jump at c_n is $\alpha(c_n) - \alpha(c_n-)$.

Step functions provide the connecting link between Riemann-Stieltjes integrals and finite sums:

9-11 THEOREM (*Reduction of a Riemann-Stieltjes integral to a finite sum*). *Let α be a step function defined on $[a, b]$ with jump α_k at x_k, where $a \leq x_1 < x_2 < \cdots < x_n \leq b$. Let f be defined on $[a, b]$ in such a*

way that not both f and α are discontinuous from the right or from the left at each x_k. Then $\int_a^b f\,d\alpha$ exists and we have

$$\int_a^b f(x)\,d\alpha(x) = \sum_{k=1}^{n} f(x_k)\alpha_k.$$

Proof. By Theorem 9-4, $\int_a^b f\,d\alpha$ can be written as a sum of integrals of the type considered in Theorem 9-9.

One of the simplest step functions is the *greatest-integer function*. Its value at x is the greatest integer which is less than or equal to x and is denoted by $[x]$. Thus, $[x]$ is the unique integer satisfying the inequalities $[x] \leq x < [x] + 1$.

9-12 THEOREM. *Every finite sum can be written as a Riemann-Stieltjes integral. In fact, given a sum $\sum_{k=1}^{n} a_k$, define f on $[0, n]$ as follows:*

$$f(x) = a_k \text{ if } k - 1 < x \leq k \qquad (k = 1, 2, \ldots, n), \; f(0) = 0.$$

Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)\,d[x],$$

where $[x]$ is the greatest integer $\leq x$.

Proof. The greatest-integer function is a step function, continuous from the right and having jump 1 at each integer. The function f is continuous from the left at $1, 2, \ldots, n$. Now apply Theorem 9-11.

We shall illustrate the use of Riemann-Stieltjes integrals by deriving a remarkable formula known as *Euler's summation formula*, which relates the integral of a function over an interval $[a, b]$ with the sum of the function values at the *integers* in $[a, b]$. It can sometimes be used to approximate integrals by sums or, conversely, to estimate the values of certain sums by means of integrals.

9-13 THEOREM (*Euler's summation formula*). *If f has a continuous derivative f' on $[a, b]$, then we have*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)((x))\,dx + f(a)((a)) - f(b)((b)),$$

where $((x)) = x - [x]$. *When* a *and* b *are integers, this becomes*

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)dx + \frac{f(a)+f(b)}{2}.$$

NOTE. $\sum_{a<n\le b}$ means the sum from $n = [a] + 1$ to $n = [b]$.

Proof. Applying Theorem 9–6 (integration by parts), we can write

$$\int_a^b f(x)\,d(x - [x]) + \int_a^b (x - [x])\,df(x)$$
$$= f(b)(b - [b]) - f(a)(a - [a]).$$

Since the greatest-integer function has unit jumps in $[a, b]$ at the integers $x = [a] + 1, [a] + 2, \ldots, [b]$, we can write

$$\int_a^b f(x)\,d[x] = \sum_{a<n\le b} f(n).$$

If we combine this with the previous equation, the theorem follows at once.

9–9 Monotonically increasing integrators. Upper and lower integrals. The further theory of Riemann-Stieltjes integration will now be developed for monotonically increasing integrators, and we shall see later (in Theorem 9–20) that for many purposes this is just as general as studying the theory for integrators which are of bounded variation.

When α is increasing, the differences $\Delta\alpha_k$ which appear in the Riemann-Stieltjes sums are all non-negative. This simple fact plays a vital role in the development of the theory. For brevity, we shall use the abbreviation "$\alpha \nearrow$ on $[a, b]$" to mean that "α is increasing on $[a, b]$."

It was stated earlier that the problem of finding the area under the graph of a function f is approached by considering the Riemann sums $\sum f(t_k)\,\Delta x_k$ as approximations to the area by means of rectangles. Such sums also arise quite naturally in certain physical problems requiring the use of integration for their solution. Another approach to these problems is by means of *upper* and *lower* Riemann sums. For example, in the case of areas, we can consider approximations from "above" and from "below" by means of the sums $\sum M_k\,\Delta x_k$ and $\sum m_k\,\Delta x_k$, where M_k and m_k denote, respectively, the sup and inf of the function values in the kth subinterval. Our geometric intuition tells us that the upper sums are at least as big as the area we seek, whereas the lower sums cannot exceed this area. (See Fig. 9–1.) Therefore it seems natural to ask: What is the smallest possible

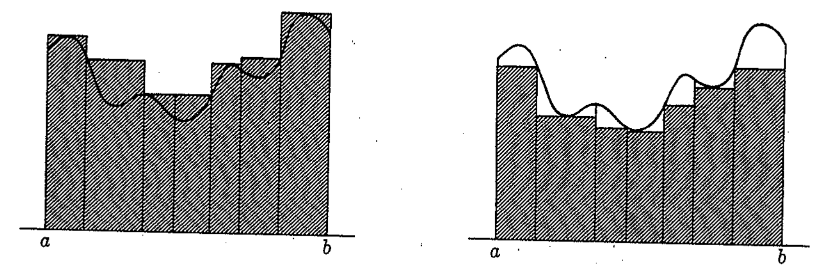


FIG. 9–1. Upper and lower Riemann sums as areas.

value of the upper sums? This leads us to consider the inf of all upper sums, a number called the *upper integral* of f. The *lower integral* is similarly defined to be the sup of all lower sums. For "reasonable" functions (for example, continuous functions) both these integrals will be equal to $\int_a^b f(x)\,dx$. However, in general, these integrals will be different and hence it becomes an important problem to find conditions on the function which will ensure that the upper and lower integrals will be the same. We shall now discuss this type of problem for Riemann-Stieltjes integrals.

9–14 DEFINITION. *Let* P *be a partition of* $[a, b]$ *and let*

$$M_k(f) = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\},$$
$$m_k(f) = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

The numbers

$$U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f)\,\Delta\alpha_k \quad \textit{and} \quad L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f)\,\Delta\alpha_k$$

are called, respectively, the upper and lower Stieltjes sums of f *with respect to* α *for the partition* P.

NOTE. We always have $m_k(f) \le M_k(f)$. *If* $\alpha \nearrow$ on $[a, b]$, then $\Delta\alpha_k \ge 0$ and we can also write $m_k(f)\,\Delta\alpha_k \le M_k(f)\,\Delta\alpha_k$, from which it follows that the lower sums do not exceed the upper sums. Furthermore, if $t_k \in [x_{k-1}, x_k]$, then $m_k(f) \le f(t_k) \le M_k(f)$. Therefore, when $\alpha \nearrow$, we have the inequalities

$$L(P, f, \alpha) \le S(P, f, \alpha) \le U(P, f, \alpha)$$

relating the upper and lower sums to the Riemann-Stieltjes sums. These inequalities, which are frequently used in the material that follows, do not necessarily hold when α is not an increasing function.