Stability of Calderón's inverse problem in 2D

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ICMAT INSTITUTO DE CIENCIAS MATEMÁTICAS

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Introduction

Uniformly strongly elliptic boundary value problems

Let $K \geqslant 1$, $\Omega \subset \mathbb{C}$ bounded domain. We say $\gamma \in \mathcal{G}(K, \Omega)$ when

- Compactly supported: $\operatorname{supp}(\gamma - 1) \subset \overline{\Omega}$.
- Strongly elliptic: $\|\gamma\|_\infty \leqslant K$, $\|\gamma^{-1}\|_\infty \leqslant K$.
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DtN map: $\quad \Lambda_\gamma : f \mapsto (\gamma \partial_\nu u_{\gamma,f})|_{\partial\Omega}.$

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is continuous for the distance $\|\gamma_1 - \gamma_2\|_\infty$.Given boundary measurements can we recover the conductivity? That is, find the inverse map

$$\begin{array}{rccc} \Lambda^{-1} : & \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right) & \to & \mathcal{G}(K,\Omega), \\ & \Lambda_\gamma & \mapsto & \gamma. \end{array}$$

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Calderón's CIP is severely "ill-posed".

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- Yields a characterization for conductivities supported away from the boundary.
- Settles Alessandrini's 2007 conjecture.

Moduli of continuity

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$$\left|\langle(\Lambda_{\gamma_0} - \Lambda_{\gamma_j})\varphi, \varphi\rangle\right| = \left|\int_{\Omega} (\gamma_0 - \gamma_j)\nabla u_0 \cdot \nabla u_j\right|$$

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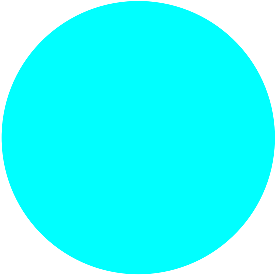
Stability counterexamples



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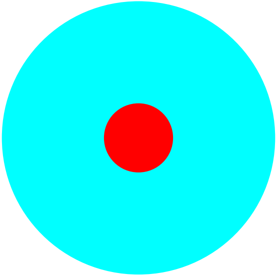

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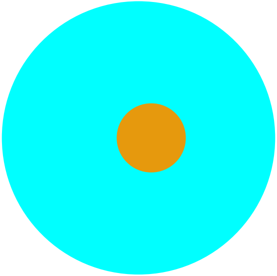

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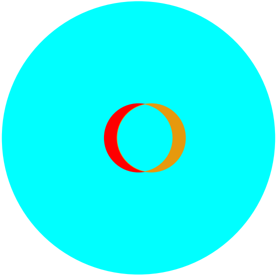

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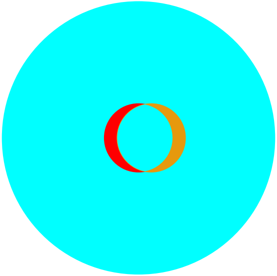

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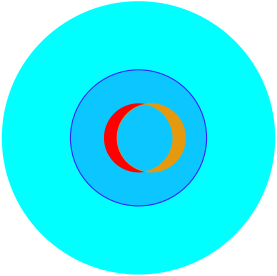

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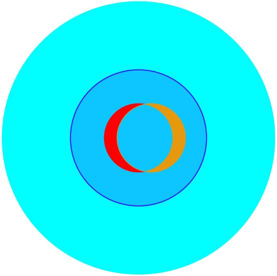

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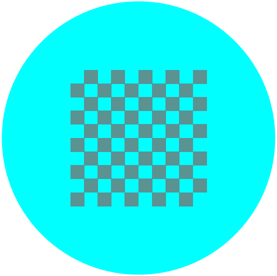

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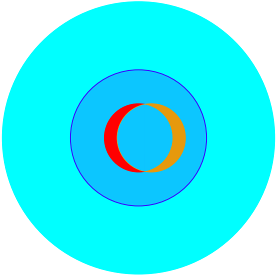

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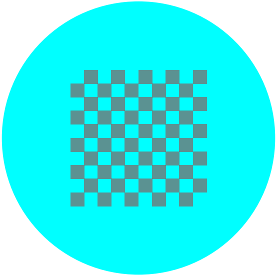

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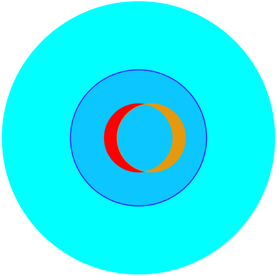

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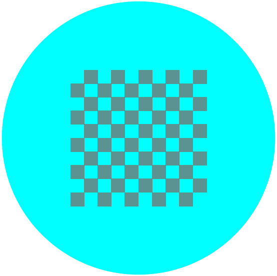

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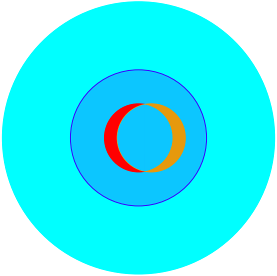

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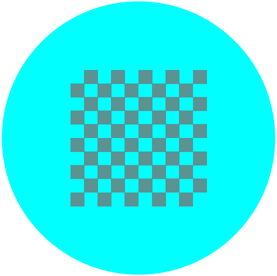

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L^p stability fails in general! Thus, we seek a priori conditions.

Compactness issues

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Continuity forward map + Uniqueness ([AP]) + compactness imply continuity of inverse. But no control on its modulus of continuity.

Compactness issues

Theorem (Mandache'01)

$\Lambda(\mathcal{G}(K, r_0\mathbb{D}))$ is a pre-compact subset of $\mathcal{L}(H^{1/2}(\partial\mathbb{D}), H^{-1/2}(\partial\mathbb{D}))$.

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Continuity forward map + Uniqueness ([AP]) + compactness imply continuity of inverse. But no control on its modulus of continuity.

Theorem

Let $K \geqslant 1$, let $r_0 < 1$ and let $\mathcal{F} \subset \mathcal{G}(K, r_0\mathbb{D})$. The family $\mathcal{F}$ is L^2-stable for $\mathbb{D}$ if and only if it is pre-compact.

Alessandrini conjecture

Let $\tau_y f(x) = f(x-y)$. Integral modulus of continuity of f:

$$\omega_p f(t) := \sup_{|y|\leqslant t} \|f - \tau_y f\|_{L^p} \qquad \text{for } 0 \leqslant t \leqslant \infty,$$

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Problem: Quantify continuity of inverse mapping for any ω.

Our result

Theorem

Let $K \geqslant 1$, let $0 < p < \infty$, let Ω be a bounded domain and let ω be a modulus of continuity. Then the family $\mathcal{G}(K, \Omega, p, \omega)$ is L^2-stable for Ω.

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$$\|\gamma_1 - \gamma_2\|_2 \leqslant C\ \eta\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\mathcal{L}(\partial\Omega)}\right)^{\frac{1}{2}}$$

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$$\eta(\rho) \lesssim_{K,p} (Id + \omega) \left(C_{K,p}\, \omega \left(\frac{C_K}{|\log(\rho)|^{\frac{1}{K}}} \right)^{b_{K,p}} + \frac{C_K}{|\log(\rho)|^{\alpha_K}} \right).$$

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We have gotten every bounded domain and every modulus of continuity. No "compactly supported" condition!! Every conductivity has an integral modulus of continuity.

Tools

Complex Geometric Optics Solution

The DtN map is a matrix on an appropriate base:
spherical harmonics.

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$$\begin{cases} \nabla \cdot (\gamma \nabla u_\gamma(\cdot, k)) \equiv 0, \\ u_\gamma(z, k) = e^{ikz} \left(1 + R(z, k)\right), \text{ with } R(\cdot, k) \in W^{1,p} \end{cases}$$

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Interesting behavior in k: for every z

$$\frac{\partial_{\overline{k}} u_\gamma(z,k)}{-i\overline{u_\gamma(z,k)}} = ct(k) =: \tau_\gamma(k).$$

(scattering transform).

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$R_\gamma(\cdot, k)$

τ_γ

$\log u_\gamma$

u_γ

γ

Quasiconformal mappings

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Conformal mappings

Preserves angles

"Circles to circles"

Cauchy-Riemann:

$\frac{1}{2}\left(\partial_x f + i\partial_y f\right) = 0$

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Quasiconformal mappings
Angle distortion bounded.
"Circles to ellipses".
$|\bar{\partial} f| \leqslant k |\partial f|$

Hodge-* conjugation

Dictionary of divergence equation and Beltrami equation:

$$\Lambda_\gamma$$

$$R_\gamma(\cdot, k)$$

$$\tau_\gamma$$

$$\log u_\gamma$$

$$u_\gamma$$

$$\gamma$$

Hodge-* conjugation

Dictionary of divergence equation and Beltrami equation:
Let $\mu := \frac{1-\gamma}{1+\gamma}$. Let $f_\mu := \mathrm{Re}\, u_\gamma + i\, \mathrm{Im} u_{\gamma^{-1}}$. Then

$$\begin{cases} \bar\partial f_\mu = \mu \overline{\partial f_\mu} \\ f_\mu(z,k) = e^{ikz}\left(1 + M_\mu\right), \text{ with } M_\mu(\cdot,k) \in W^{1,p}(\mathbb{C}) \end{cases}$$

$\Delta\Lambda_\gamma$

$\Delta M_\mu(\cdot, k)$

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We have Lipschitz continuity on the mapping

$$\begin{array}{ccccc} \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right) & \to & W^{1,p}(\mathbb{D}^c) & \to & \mathbb{C}, \\ \Lambda_\gamma & \mapsto & M_\mu(\cdot,k) & \mapsto & \tau_\mu(k). \end{array}$$

with $|\tau_1(k) - \tau_2(k)| \lesssim e^{C|k|}\rho$ ([BFR'07])

$\rho := \|\Delta\Lambda_\gamma\|_{\mathcal{L}}$

$\|\Delta M_\mu(\cdot,k)\|_{W\mathbb{D}^c}$

$|\Delta\tau_\mu(k)| \lesssim \rho e^{C|k|}$

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Subexponential behavior in k

The logarithm $\varphi_\mu := \frac{\log(f_\mu)}{ik}$ is a quasiconformal principal mapping of $\mathbb{C}$.

$$\bar\partial \varphi_\mu(\cdot, k) = -\frac{\bar k}{k} \mu(\cdot)\, e_{-k}(\varphi_\mu(\cdot, k))\, \overline{\partial \varphi_\mu(\cdot, k)}.$$

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We show that $\|\varphi_\mu(\cdot, k) - Id\|_{L^\infty} \leqslant \upsilon(|k|^{-1})$.

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We show that $\|\varphi_\mu(\cdot, k) - Id\|_{L^\infty} \leqslant \upsilon(|k|^{-1})$.
Tools: interaction of modulus of continuity with translation invariant operators and Fourier transform, control of the Neumann series in k, interaction of the modulus of continuity when composing with qc-maps,...

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Cauchy problem

Next we need to solve the Cauchy problem

$$\partial_{\overline{k}} u_\gamma(z,k) = -i\tau_\mu(k)\overline{u_\gamma(z,k)}.$$

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There is not enough decay of τ_μ to solve it by standard means (fixed point, Cauchy transform,...). Instead, we get uniqueness and (bold) stability by using both variables at the same time:

$$\|u_1 - u_2\|_\infty \leqslant \iota(\|\Lambda_1 - \Lambda_2\|_{\mathcal{L}}).$$

Tools: Browder degree, argument principle, CZ estimates.

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Moltes gràcies!!

谢谢！

Quasiconformal mappings and moduli

Lemma

Let ϕ be K-qc, and let $\mu \in L^\infty_c$. Consider $0 < p \leqslant \infty$ and $\frac{1}{q} > \frac{K}{p}$. For t small enough

$$\omega_q(\mu \circ \phi)(t) \leqslant C_{K,q,p}\, \omega_p \mu(C_K t^{\frac{1}{K}}).$$

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Theorem

Let $\mu \in L^\infty_c$ with $\|\mu\|_{L^\infty} \leqslant \kappa < 1$ and support in $\mathbb{D}$. Let f be a quasiregular solution to

$$\bar\partial f = \mu \overline{\partial f}.$$

Let $1 < p < p_\kappa$ satisfy that $\kappa \|\mathcal{B}\|_{L^p \to L^p} < 1$, let $r \in [p, p_\kappa)$ and let q be defined by $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, we have that

$$\omega_p(\bar\partial f)(t) \lesssim_{\kappa,r,p} \|f\|_{L^r(2\mathbb{D})} \omega_q \mu(t) + \|f\|_{W^{1+p}(2\mathbb{D})} |t|^{1-\frac{2}{p}}.$$