Regularity of planar quasiconformal mappings

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Introduction
Measuring smoothness and integrability in $\mathbb{R}^d$

Lebesgue spaces $\rightarrow$ integrability.

\[ \|f\|_{L^p} = \left( \int |f|^p \right)^{1/p}, \]
\[ \|f\|_{L^\infty} = \text{ess sup}|f| \]
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Differentiability classes $\rightarrow$ smoothness.

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Sobolev spaces $\rightarrow$ both together.

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- $\| f \|_{C^s} = \| f \|_{L^\infty} + \cdots + \sup \frac{\| \nabla^{|s|} f(x) - \nabla^{|s|} f(y) \|}{|x-y|^{\{s\}}}$
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- $\|f\|_{W^{s,p}}, \|f\|_{B^{s}_{p,q}}, \|f\|_{F^{s}_{p,q}}$
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Lebesgue spaces → integrability.
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By means of Sobolev embeddings, we have either continuity or extra integrability.
Quasiconformal mappings
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Conformal mappings
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Cauchy-Riemann:
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Quasiconformal mappings
Angle distortion bounded.
“Circles to ellipses”.
\[ |\bar{\partial} f| \leq k |\partial f| \]
The Beurling transform

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

$$Bf(z) = \frac{1}{-\pi} \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} \, dm(w).$$
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It is essential to quasiconformal mappings because

$$\mathcal{B}(\bar{\partial}f) = \bar{\partial}f \quad \forall f \in W^{1,p}.$$
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It is essential to quasiconformal mappings because

$$\mathcal{B}(\overline{\partial}f) = \partial f \quad \forall f \in W^{1,p}.$$

Recall that $\mathcal{B} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$.

Also $\mathcal{B} : W^{s,p}(\mathbb{C}) \to W^{s,p}(\mathbb{C})$ is bounded for $1 < p < \infty$ and $s > 0$. 
QC mappings of the whole plane
Let $\mu \in L^\infty_c(\mathbb{C})$ with $\kappa := \|\mu\|_\infty < 1$. 

$p = \infty$ \hspace{1cm} $p = 1$ 

$\mu$ $\frac{1}{p}$
The Beltrami equation

Let $\mu \in L^\infty_c(\mathbb{C})$ with $\kappa := \|\mu\|_\infty < 1$. The Beltrami equation

$$\bar{\partial}f(z) = \mu(z) \partial f(z)$$

has a unique solution $f \in W^{1,2}_{loc}$ such that $f(z) = z + O(1/z)$ as $z \to \infty$. 

![Diagram showing the Beltrami equation and its solution]
Let $\mu \in L_{C}^{\infty}(\mathbb{C})$ with $\kappa := \|\mu\|_{\infty} < 1$. The Beltrami equation

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has a unique solution $f \in W_{loc}^{1,2}$ such that $f(z) = z + O(1/z)$ as $z \to \infty$.

Consider

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h := \mu + \mu B(\mu) + \mu B(\mu B(\mu)) + \cdots
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since $\|\mu \cdot \mathcal{B}\|_{(2,2)} \leq \kappa \|\mathcal{B}\|_{(2,2)} = \kappa < 1$. 
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Then, $h \in L^2$.
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Then, \( h \in L^2 \) and \( f = \frac{1}{\pi z} \ast h + z \).
The Beltrami equation

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Then, $h \in L^2$ and $f = \frac{1}{\pi z} \ast h + z$. This remains true if $\|B\|_{(p,p)} < 1/\kappa$. 
Let $\mu \in L_{c}^{\infty}(\mathbb{C})$ with $\kappa := \|\mu\|_{\infty} < 1$.
Results without boundaries

Let $\mu \in L^\infty_c(\mathbb{C})$ with $\kappa := \|\mu\|_\infty < 1$.

- $h \in L^p$ for $\frac{1}{p_\kappa} < \frac{1}{p}$ [A92, AIS01].
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Recent progress

Theorem (P.)

Let $0 < s < 2$, $1 < p < \infty$, let $\mu \in W^{s,p} \cap L^\infty$, with $\mu \leq \kappa \chi_D$ and let $f$ be the principal solution to the Beltrami equation $\bar{\partial} f = \mu \partial f$.

If $s = \frac{2}{p}$, then

$$\bar{\partial} f \in W^{s,q} \quad \text{for every } \frac{1}{q} > \frac{1}{p}.$$

If $s < \frac{2}{p}$ and $\frac{1}{p} < \frac{1}{p'} - \frac{1}{p_\kappa} = \frac{1-\kappa}{1+\kappa}$, then

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See [Clop, Faraco, Ruiz] for previous weaker results and Baisón’s thesis for a stronger result in the critical setting with $s > 1/2$. 
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See [Clop, Faraco, Ruiz] for previous weaker results and Baisón’s thesis for a stronger result in the critical setting with $s > 1/2$.

It remains unclear if the condition $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$ can be replaced by $\frac{1}{p} < \frac{1}{p'_\kappa}$, which is more natural and is achieved for $s = 1$. 
Idea

\[ \mu \in L^\infty \cap W^{s,p} \]
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\[ \mu \in L^\infty \cap \mathcal{W}^{s,p} \]

\[ h := (I - \mu \mathcal{B})^{-1}(\mu) \]
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\[ \mu \in L^\infty \cap W^{s,p} \]
\[ h := (1 - \mu B)^{-1}(\mu) \]
\[ h = \mu Bh + \mu. \]
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\[ \partial h = \partial(\mu Bh) + \partial \mu \]

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**Idea**

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\[ D^s h = \mu D^s Bh - [D^s, \mu](Bh) + D^s(\mu) \]

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**Idea**

$$\mu \in L^\infty \cap W^{s,p}$$

$$h := (I - \mu B)^{-1}(\mu)$$

$$h = \mu \mathcal{B} h + \mu.$$ 

$s < 1$

$$D^s h = D^s(\mu \mathcal{B} h) + D^s \mu$$

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\[ \partial h = \mu \partial B h + \partial \mu B h + \partial \mu \]
\[ \partial h = \mu B \partial h + \partial \mu B h + \partial \mu \]
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\[ s < 1 \]
\[ D^s h = D^s (\mu B h) + D^s \mu \]
\[ D^s h = \mu D^s B h - [D^s, \mu] (B h) + D^s \mu \]
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\[ (1 - \mu B) D^s h = -[D^s, \mu] (B h) + D^s \mu \]
Idea

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\begin{align*}
\mu & \in L^{\infty} \cap W^{s,p} \\
h & := (1 - \mu B)^{-1}(\mu) \\
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\end{align*}
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D^s h &= (I - \mu B)^{-1}(D^s \mu - [D^s, \mu](B h))
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Idea

\[ \mu \in L^\infty \cap W^{s,p} \]
\[ h := (I - \mu B)^{-1}(\mu) \]
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\[ \partial \mu Bh \in L^q, \quad q \in \left( p_{\kappa}', p_{\kappa} \right): \text{Hölder} \]

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\[ h := (I - \mu B)^{-1}(\mu) \]
\[ h = \mu B h + \mu. \]
\[ \partial \mu B h \in L^q, \; q \in (p'_\kappa, p_\kappa): \text{Hölder} \]
\[ [D^s, \mu] B h \in L^q: \text{Kato-Ponce} \]

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In [CFMOZ], the subcritical case and critical cases are improved using \( \log(\partial f) \) to avoid the restriction \( \frac{1}{p} < \frac{1}{p_{p}} - \frac{1}{p_{p}} \) when \( s = 1 \).
In [CFMOZ], the subcritical case and critical cases are improved using \( \log(\partial f) \) to avoid the restriction \( \frac{1}{p} < \frac{1}{p_{rc}} - \frac{1}{p_{rc}} \) when \( s = 1 \).

This technique cannot be used for fractional derivatives. Can we bypass it?
In [CFMOZ], the subcritical case and critical cases are improved using $\log(\partial f)$ to avoid the restriction $\frac{1}{p} < \frac{1}{p_{r_c}'} - \frac{1}{p_{r_c}}$ when $s = 1$. This technique cannot be used for fractional derivatives. Can we bypass it?

In the critical setting with fractional derivatives, Baisón et al. could do it combining the use of the logarithm with certain potentials to give some better results, namely $\log(\partial f) \in W^{s,p}$, but they were forced to work only with $s > 1/2$. Is this restriction natural? Can this procedure be adapted to the subcritical setting?
QC mappings on domains
Consider a Riemann mapping from $\mathbb{D}$ to the Koch Snowflake. Since it is conformal, $\bar{\partial}\varphi = 0$. Thus, $\mu = 0$ and $\mu \in W^{s,p}$ for every $s, p$. 

What about quasiconformal mappings on domains?
Consider a Riemann mapping from $\mathbb{D}$ to the Koch Snowflake. Since it is conformal, $\bar{\partial}\varphi = 0$. Thus, $\mu = 0$ and $\mu \in W^{s,p}$ for every $s, p$. However, $\varphi'$ does not extend to $\partial\mathbb{D}$. Thus, $\varphi \notin C^1(\overline{\mathbb{D}})$ and, as a consequence, $\partial\varphi$ is not in any supercritical Sobolev space.
What about quasiconformal mappings on domains?

Consider a Riemann mapping from $\mathbb{D}$ to the Koch Snowflake. Since it is conformal, $\partial \varphi = 0$. Thus, $\mu = 0$ and $\mu \in W^{s,p}$ for every $s, p$. However, $\varphi'$ does not extend to $\partial \mathbb{D}$. Thus, $\varphi \notin C^1(\overline{\mathbb{D}})$ and, as a consequence, $\partial \varphi$ is not in any supercritical Sobolev space. The moral is that in order to study the regularity of $\mu$-quasiconformal mappings between domains we must take into account both the regularity of the boundary and the regularity of $\mu$. 
Let $g : \Omega_1 \to \Omega_3$ to be $\mu$-QC, with $\mu \in W^{s,p}(\Omega_1)$ and $\partial \Omega_1$, $\partial \Omega_3$ regular enough. Can we say that $\partial g \in W^{s,p}(\Omega_1)$?
By Stoilow factorization, \( g = h \circ f \) where \( f : \mathbb{C} \rightarrow \mathbb{C} \) is the \( \mu \)-principal mapping and \( h : \Omega_2 \rightarrow \Omega_3 \) is conformal.
We can find Riemann mappings (conformal) if the domains are simply connected.
We study supercritical case.

Theorem (P)

Let $\Omega \subset \mathbb{C}$ be a bdd domain, with normal vector $N \in B_{p,p}^{s-1/p}(\partial \Omega)$, $s \in \mathbb{N}$ and $p > 2$. 
The principal mapping

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Let $\Omega \subset \mathbb{C}$ be a bdd domain, with normal vector $N \in B_{p,p}^{s-1/p}(\partial \Omega)$, $s \in \mathbb{N}$ and $p > 2$. Let $\mu \in W^{s,p}(\Omega) \cap L^\infty$ with $k := \|\mu\|_\infty < 1$ with $\text{supp} \mu \subset \overline{\Omega}$. Then $I_\Omega - \mu B_\Omega$ is invertible in $W^{s,p}(\Omega)$. 

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**Introduction**

**QC mappings of the whole plane**

**QC mappings on domains**

**The end**
The principal mapping

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Let $\Omega \subset \mathbb{C}$ be a bdd domain, with normal vector $N \in B_{p,p}^{s-1/p}(\partial \Omega)$, $s \in \mathbb{N}$ and $p > 2$. Let $\mu \in W^{s,p}(\Omega) \cap \mathbb{L}^\infty$ with $k := \|\mu\|_\infty < 1$ with $\text{supp} \mu \subset \overline{\Omega}$. Then the principal solution $f \in W^{s+1,p}(\Omega)$ and it is bi-Lipschitz.
General case

Conjecture (Theorem in progress with K. Astala)

Let $s \in \mathbb{N}$ and $p > 2$. If $\Omega$ is a simply connected $B_{p,p}^{s+1-\frac{1}{p}}$-domain, then any Riemann mapping $\varphi : \mathbb{D} \to \Omega$ satisfies that $\varphi \in W^{s+1,p}(\mathbb{D})$ and it is bi-Lipschitz.
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Conjectured corollary

Let $s \in \mathbb{N}$ and $p > 2$, let $\Omega_1$ and $\Omega_3$ be simply connected $B_{p,p}^{s+1-\frac{1}{p}}$-domains and let $g : \Omega_1 \to \Omega_3$ be a $\mu$-quasiconformal mapping with $\mu \in W^{s,p}(\Omega_1)$. Then $g \in W^{s+1,p}(\Omega_1)$. 
Idea of the proof of the corollary

\[ f \in W^{s+1,p}(\Omega) \] and it is bi-Lipschitz by the Theorem.
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\[ f \in W^{s+1,p}(\Omega) \] and it is bi-Lipschitz by the Theorem.
\[ \varphi_1 \in W^{s+1,p}(\mathbb{D}) \] and it is bi-Lipschitz by the conjecture.
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\( f \in W^{s+1,p}(\Omega) \) and it is bi-Lipschitz by the Theorem.
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By the trace condition, \( f \circ \varphi_1 \) is a \( B^{s+1-\frac{1}{p}}_{p,p} \) parameterization of \( \partial \Omega_2 \).
Idea of the proof of the corollary

\( f \in W^{s+1,p}(\Omega) \) and it is bi-Lipschitz by the Theorem. \( \varphi_1 \in W^{s+1,p}(\mathbb{D}) \) and it is bi-Lipschitz by the conjecture. By the trace condition, \( f \circ \varphi_1 \) is a \( B_{p,p}^{s+1-\frac{1}{p}} \) parameterization of \( \partial \Omega_2 \). By the conjecture, \( h \circ \varphi_2 \) and \( \varphi_2 \) are in \( W^{s+1,p}(\mathbb{D}) \).
Idea of the proof of the corollary

\[ f \in W^{s+1,p}(\Omega) \text{ and it is bi-Lipschitz by the Theorem.} \]
\[ \varphi_1 \in W^{s+1,p}(\mathbb{D}) \text{ and it is bi-Lipschitz by the conjecture.} \]
By the trace condition, \( f \circ \varphi_1 \) is a \( B_{p,p}^{s+1-\frac{1}{p}} \) parameterization of \( \partial \Omega_2 \).
By the conjecture, \( h \circ \varphi_2 \) and \( \varphi_2 \) are in \( W^{s+1,p}(\mathbb{D}) \).
Then, \( g = (h \circ \varphi_2) \circ (\varphi_2^{-1}) \circ f \).
In the complex plane, if \( N \in B_{p,p}^{s-1/p}(\partial \Omega) \) and \( p > 2 \), then \( \mu \in W^{s,p}(\Omega) \implies f, g \in W^{s+1,p}(\Omega) \).
Conclusions

- In the complex plane, if $N \in B_{p,p}^{s-1/p}(\partial \Omega)$ and $p > 2$, then $\mu \in W^{s,p}(\Omega) \implies f, g \in W^{s+1,p}(\Omega)$.

- Expected further results:
  - The results hold apparently for $0 < s < 1$, $sp > 2$ (work in progress with Eero Saksman) and for Hölder spaces.
  - Subcritical situation: is there any condition on $\partial \Omega$ which can lead to analogous results?
Moltes gràcies!!
Muchas gracias!!