Measuring Triebel-Lizorkin fractional smoothness on domains in terms of first-order differences

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BAC, September 5th, 2017

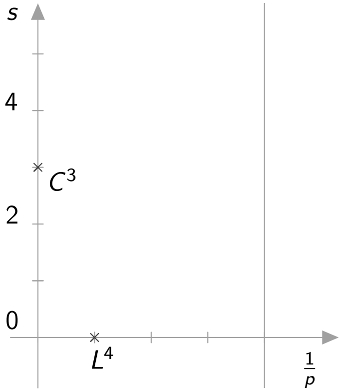
Introduction

Measuring smoothness and integrability in $\mathbb{R}^d$

Lebesgue spaces → integrability.

- $\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}$,
  $\|f\|_{L^\infty} = \operatorname{ess\,sup}|f|$

Measuring smoothness and integrability in $\mathbb{R}^d$



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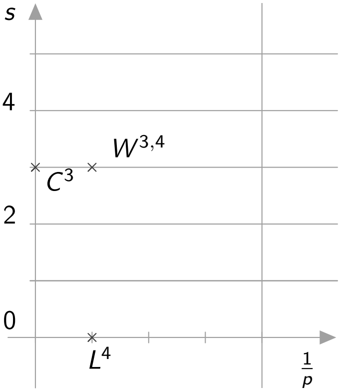

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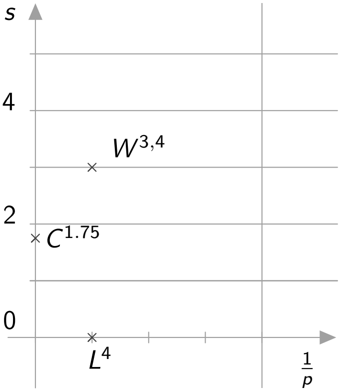

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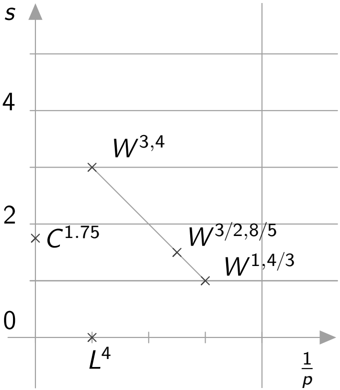

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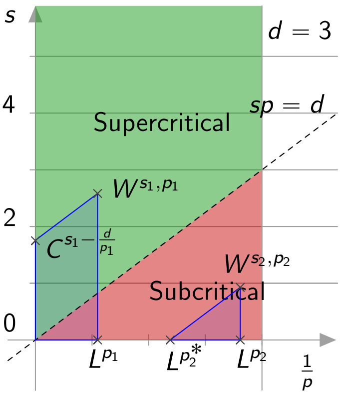

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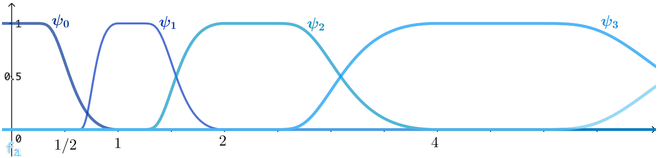
By means of Sobolev embeddings, we have either continuity or extra integrability.

Non-homogeneous Triebel-Lizorkin spaces



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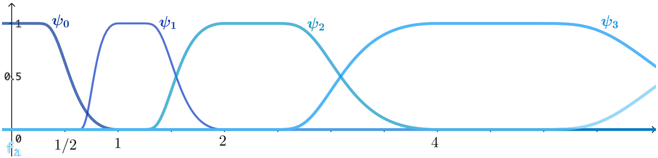

A tempered distribution f is said to belong to the Triebel-Lizorkin space $F^s_{p,q}$ if

$$\|f\|_{F^s_{p,q}} = \left\| \left\| \left\{ 2^{sj} \left(\psi_j \hat{f} \right)^{\vee} \right\}_{j=0}^{\infty} \right\|_{\ell^q} \right\|_{L^p} < \infty.$$

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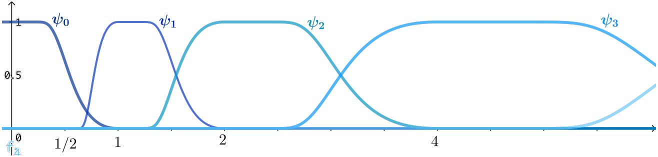

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This family includes Sobolev ($s \in \mathbb{N}$, $q = 2$), Bessel potential ($q = 2$) and trace spaces (diagonal Besov, $p = q$).

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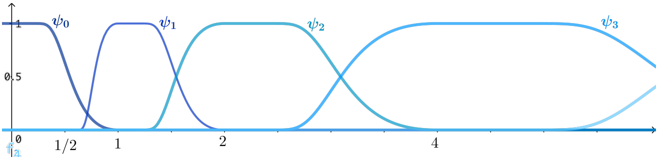

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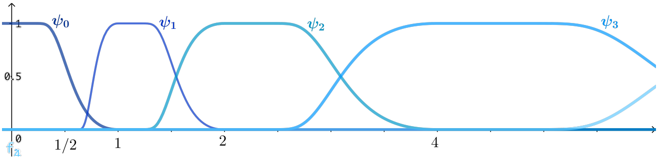

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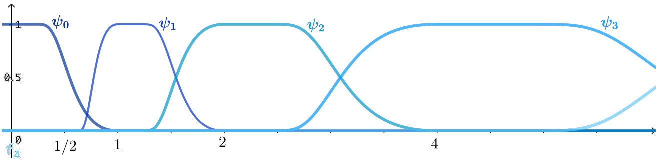

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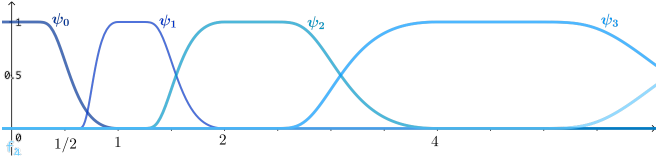

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Then

$$\|f\|_{F^s_{p,q}} \approx \|f\|_{W^{k,p}} + \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|\nabla^k f(x) - \nabla^k f(y)|^q}{|x-y|^{\sigma q + d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Triebel-Lizorkin spaces on domains

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Question 2: for which domains Ω the norm $\|f\|_{F^s_{p,q}(\Omega)}$ is equivalent to

$$\|f\|_{\widetilde{A}^s_{p,q}(\Omega)} := \|f\|_{W^{k,p}(\Omega)} + \left(\int_\Omega \left(\int_{\frac{1}{2}d_\Omega(x)} \frac{|\nabla^k f(x) - \nabla^k f(y)|^q}{|x-y|^{\sigma q + d}}\, dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}},$$

where $d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$?

Tools

Uniform domains

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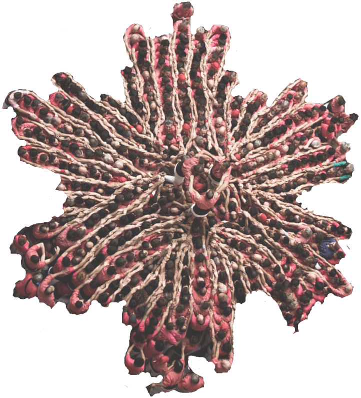

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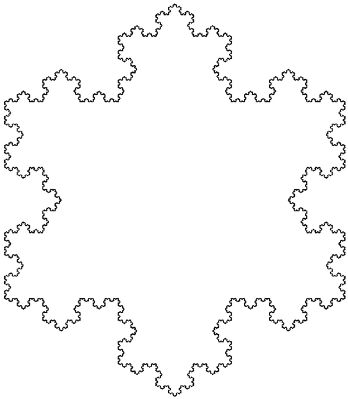

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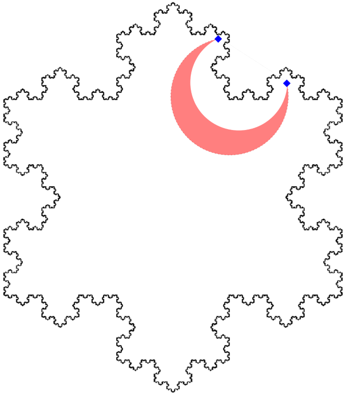

Uniform domain:

Cigars joining points x and y:

- $\mathrm{dist}(x, y) \approx \ell(\gamma)$
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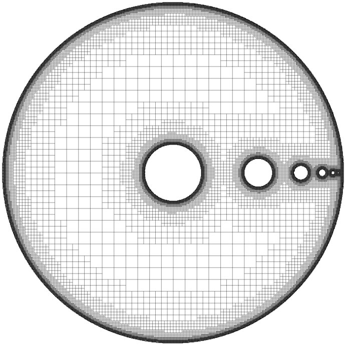

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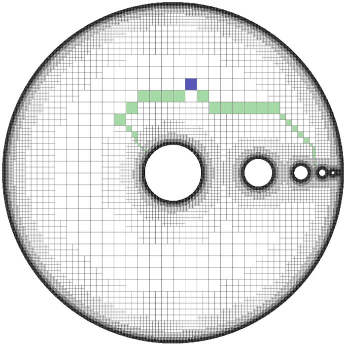

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'Cigar' paths joining Q and S

- $D(Q, S) \approx \ell([Q, S]) := \sum_P \ell(P)$
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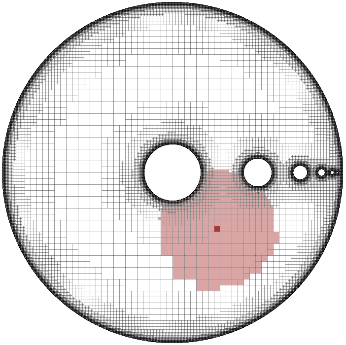

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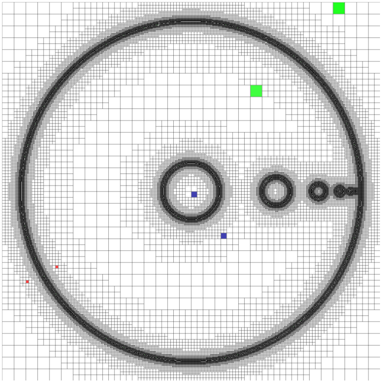

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Symmetrized cubes for the complement:
If $Q \in \mathcal{W}(\bar{\Omega}^c)$, then $\exists Q^* \in \mathcal{W}(\Omega)$ with $\ell(Q^*) = \ell(Q) \approx D(Q, Q^*)$

Meyers' approximating polynomials

Let $Q \subset \mathbb{R}^d$. Given $f \in W^{k,1}(Q)$, $\exists! \mathbf{P}_Q^k f \in \mathcal{P}^k$ such that

$$\int_Q D^\beta \mathbf{P}_Q^k f \, dm = \int_Q D^\beta f \, dm \tag{1}$$

for every multiindex $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant k$.

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Note that by Poincaré-Wirtinger,

$$\left\| f - \mathbf{P}_Q^k f \right\|_p \lesssim \ell(Q) \left\| \nabla (f - \mathbf{P}_Q^k f) \right\|_p \lesssim \cdots \lesssim \ell(Q)^k \left\| \nabla^k (f - \mathbf{P}_Q^k f) \right\|_p$$

Peter Jones extension operator

Extension is defined as

$$\Lambda_k f(x) = f(x)\chi_\Omega(x) + \sum_{Q\in\mathcal{W}_3} \psi_Q(x)\mathbf{P}^k_{Q*} f(x) \text{ for any } f \in W^{k,1}_{loc}(\Omega).$$

where $\mathcal{W}_3 = \{Q \in \mathcal{W}(\bar{\Omega}^c) : \ell(Q) < c\}$.

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Theorem (Peter Jones'81)

The operator Λ_k is an extension operator for $W^{k+1,p}(\Omega)$ for $1 \leqslant p \leqslant \infty$, that is, a right inverse for the restriction operator $f \mapsto \chi_\Omega f$ which is bounded from $W^{k+1,p}(\Omega)$ to $W^{k+1,p}$.

Extension for TL spaces

Extension for Triebel-Lizorkin spaces

Since $A^{k+\sigma}_{p,q}(\mathbb{R}^d) = F^{k+\sigma}_{p,q}(\mathbb{R}^d)$, we obtain that

$$\|f\|_{A^{k+\sigma}_{p,q}(\Omega)} \leqslant \inf_g \|g\|_{A^{k+\sigma}_{p,q}(\mathbb{R}^d)} \approx \inf_g \|g\|_{F^{k+\sigma}_{p,q}(\mathbb{R}^d)} = \|f\|_{F^{k+\sigma}_{p,q}(\Omega)},$$

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and if there is an extension operator E,

$$\|f\|_{F^{k+\sigma}_{p,q}(\Omega)} \leqslant \|Ef\|_{F^{k+\sigma}_{p,q}(\mathbb{R}^d)} \approx \|Ef\|_{A^{k+\sigma}_{p,q}(\mathbb{R}^d)} \leqslant C\|f\|_{A^{k+\sigma}_{p,q}(\Omega)},$$

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Theorem (P-Saksman'17, P'19)

The operator Λ_k is an extension operator mapping $A^{k+\sigma}_{p,q}(\Omega)$ to $A^{k+\sigma}_{p,q}(\mathbb{R}^d)$ for $k \in \mathbb{N}$, $0 < \sigma < 1$, $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ with $\sigma > \frac{d}{p} - \frac{d}{q}$.

Idea of the proof: $k = 0$

$$\Lambda_0 f(x) = f(x)\chi_\Omega(x) + \sum_{Q \in \mathcal{W}_3} \psi_Q(x) f_{Q*} \text{ for any } f \in L^1_{loc}(\Omega)$$

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$$\textcircled{a} := \int_\Omega \left(\int_{\Omega^c} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \leqslant \|f\|_{A^s_{p,q}(\Omega)}.$$

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$$\textcircled{a} = \int_\Omega \left(\int_{\Omega^c} \frac{|f(x) - \sum_{S\in\mathcal{W}_3} \psi_S(y) f_{S*}|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \leqslant \|f\|_{A^s_{p,q}(\Omega)}.$$

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$$\Lambda_0 f(x) = f(x)\chi_\Omega(x) + \sum_{Q\in\mathcal{W}_3} \psi_Q(x) f_{Q*} \text{ for any } f \in L^1_{loc}(\Omega)$$

- $\|\Lambda_0 f\|^p_{L^p(\Omega^c)} \lesssim \|f\|^p_{L^p(\Omega)}$ follows by Jensen.
- Regarding the seminorm, we want, to control three terms, the first one being

$$\text{ⓐ} = \int_\Omega \left(\int_{\Omega^c} \frac{|f(x) - \sum_{S\in\mathcal{W}_3} \psi_S(y) f_{S*}|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \lesssim \|f\|_{A^s_{p,q}(\Omega)}.$$

Adding and subtracting $\sum_{S\in\mathcal{W}_3} \psi_S(y) f(x)$,

$$\text{ⓐ} \lesssim \sum_{Q\in\mathcal{W}_1} \int_Q \left(\sum_{S\in\mathcal{W}_3} \frac{|f(x) - f_{S*}|^q}{D(Q,S)^{sq+d}} \int_{\frac{11}{10}S} \psi_S(y)\, dy \right)^{\frac{p}{q}} dx$$

$$+ \sum_{Q\in\mathcal{W}_1} \int_Q \left(\sum_{S\in\mathcal{W}_3 : \ell(S)=c} \int_{\frac{11}{10}S} \frac{\left|\left(1 - \sum_{P\in\mathcal{W}_3} \psi_P(y)\right) f(x)\right|^q}{D(Q,S)^{sq+d}} \, dy \right)^{\frac{p}{q}} dx.$$

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- Indeed $\|\Lambda_0(D^\alpha f)\|_{\dot{A}^\sigma_{p,q}} \leqslant \|D^\alpha f\|_{A^\sigma_{p,q}}$.

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- Indeed $\|\Lambda_0(D^\alpha f)\|_{\dot{A}^\sigma_{p,q}} \leqslant \|D^\alpha f\|_{A^\sigma_{p,q}}$.
- To bound for $|\beta| < \alpha$

$$\left\| \sum_{Q\in\mathcal{W}_3} D^{\alpha-\beta}\psi_Q D^\beta P^k_{Q*} f \right\|^p_{\dot{A}^\sigma_{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{A^s_{p,q}(\Omega)},$$

use Peter Jones techniques and the properties of the partition of the unity wisely. Proofs are more tricky than in $k = 0$.

Reduction of the domain

Reduction of the domain of integration

Theorem

Let $\Omega \subset \mathbb{R}^d$ *be a bounded uniform domain,* $0 < s < 1$*. Then*

$$\|f\|_{F^s_{p,q}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_\Omega \left(\int_{B\left(x, \frac{1}{2}\delta(x)\right)} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.$$

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Seeger proof uses oscillation based norms, build on work of Jones, Christ, Kalyabin,... We use Whitney cubes and their properties in uniform domains.

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$$\sum_{Q,S} \int_Q \int_S \frac{|f(x) - f_Q|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{Hölder}}{\leqslant} \sum_{Q \in \mathcal{W}} \frac{\int_{Q \times Q} |f(x) - f(\xi)| G(x) \, dx}{\ell(Q)^{s+d}}$$

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By duality it is enough to control for $g \in L^{p'}(L^{q'}(\Omega))$ with norm one

$$\sum_{Q,S} \int_Q \int_S \frac{|f(x) - f_Q|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{Hölder}}{\leqslant} \|f\|_{\tilde{A}^s_{p,q}(\Omega)}.$$

$$\sum_{Q,S} \int_Q \int_S \frac{|f_Q - f_{Q_S}|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{Hölder}}{\lesssim} \|f\|_{\tilde{A}^s_{p,q}(\Omega)}.$$

$$\sum_{Q,S} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{abs}}{\lesssim} \sum_R \int_R \int_{\mathbf{Sh}(R)} \int_{\mathbf{Sh}(R)} \frac{|f(\xi) - f(y)|}{\ell(R)^{s+d+\frac{d}{q}}} g(x,y)$$

Where $G(x) := \|g(x,\cdot)\|_{L^{q'}(\Omega)}$.

Idea of the proof I

First we check that for $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $s > \frac{d}{p} - \frac{d}{q}$,

$$\|f\|^p_{\dot{A}^s_{p,q}(\Omega)} \lesssim \sum_{Q \in \mathcal{W}} \int_Q \left(\int_{Sh(Q)} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx.$$

By duality it is enough to control for $g \in L^{p'}(L^{q'}(\Omega))$ with norm one

$$\sum_{Q,S} \int_Q \int_S \frac{|f(x) - f_Q|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{Hölder}}{\leqslant} \|f\|_{\tilde{A}^s_{p,q}(\Omega)}.$$

$$\sum_{Q,S} \int_Q \int_S \frac{|f_Q - f_{Q_S}|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{Hölder}}{\lesssim} \|f\|_{\tilde{A}^s_{p,q}(\Omega)}.$$

$$\sum_{Q,S} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \overset{\text{Höl.+def}}{\lesssim} \sum_R \int_R \left(\int_{\mathbf{Sh}(R)} \frac{|f(\xi) - f(y)|^q}{\ell(R)^{sq+d}} \, dy \right)^{\frac{1}{q}} MG(\xi) \, d\xi$$

Where $G(x) := \|g(x,\cdot)\|_{L^{q'}(\Omega)}$.

Idea of the proof II

To improve the last bound

$$\left(\sum_{Q,S}\int_Q\int_S \frac{|f_{Q_S}-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}}g(x,y)\right)^p \overset{\text{Höl}}{\lesssim} \sum_R \left(\sum_{S\in\mathbf{SH}(R)}\int_S |f_R - f(y)|^q\right)^{\frac{p}{q}} \frac{\ell(R)^d}{\ell(R)^{sp+\frac{dp}{q}}}.$$

Idea of the proof II

To improve the last bound

$$\left(\sum_{Q,S}\int_Q\int_S \frac{|f_{Q_S}-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\right)^p \overset{\text{chain}}{\lesssim} \sum_R \frac{\left(\sum_{S\in \mathbf{SH}(R)} \left|\sum_{P\in[S,R)}(f_P - f_{\mathcal{N}_P})\right|^q \ell(S)^d\right)^{\frac{p}{q}}}{\ell(R)^{sp+d\frac{p}{q}-d}}$$

Idea of the proof II

To improve the last bound

$$\left(\sum_{Q,S}\int_Q\int_S \frac{|f_{Q_S}-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}}g(x,y)\right)^p \overset{\text{chain}}{\lesssim} \sum_R \frac{\left(\sum_{S\in\mathbf{SH}(R)}\left|\sum_{P\in[S,R)}(f_P-f_{\mathcal{N}_P})\right|^q \ell(S)^d\right)^{\frac{p}{q}}}{\ell(R)^{sp+d\frac{p}{q}-d}}$$

$$\overset{\text{Höl.}}{\lesssim} \sum_R \left(\sum_{\substack{S\in\mathbf{SH}(R)\\ P\in[S,R)}} \frac{|f_P-f_{\mathcal{N}(P)}|^q}{\ell(P)^\varepsilon}\left(\sum_{P\in[S,R)}\ell(P)^{\frac{\varepsilon q'}{q}}\right)^{\frac{q}{q'}}\ell(S)^d\right)^{\frac{p}{q}}\ell(R)^{d-sp-d\frac{p}{q}}$$

Idea of the proof II

To improve the last bound

$$\left(\sum_{Q,S}\int_Q\int_S \frac{|f_{Q_S}-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}}g(x,y)\right)^p \overset{\text{chain}}{\lesssim} \sum_R \frac{\left(\sum_{S\in\mathbf{SH}(R)}\left|\sum_{P\in[S,R)}(f_P-f_{\mathcal{N}_P})\right|^q \ell(S)^d\right)^{\frac{p}{q}}}{\ell(R)^{sp+d\frac{p}{q}-d}}$$

$$\overset{\text{Höl.}}{\lesssim} \sum_R \left(\sum_{\substack{S\in\mathbf{SH}(R)\\ P\in[S,R)}} \frac{|f_P-f_{\mathcal{N}(P)}|^q}{\ell(P)^{\varepsilon}}\left(\sum_{P\in[S,R)}\ell(P)^{\frac{\varepsilon q'}{q}}\right)^{\frac{q}{q'}}\ell(S)^d\right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}}$$

$$\overset{\text{unif.}}{\lesssim} \sum_R \left(\sum_{P\in\mathbf{SH}^2(R)} \frac{\left(⨍_{5P}|f(\xi)-f_P|\,d\xi\right)^q \ell(P)^d}{\ell(P)^{\varepsilon}}\right)^{\frac{p}{q}} \ell(R)^{d+\frac{\varepsilon p}{q}-sp-\frac{dp}{q}}$$

Idea of the proof II

To improve the last bound (in case $1 \leqslant q \leqslant p < \infty$):

$$\left(\sum_{Q,S} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\right)^p \overset{\text{chain}}{\lesssim} \sum_R \frac{\left(\sum_{S \in \mathbf{SH}(R)} \left|\sum_{P \in [S,R)} (f_P - f_{\mathcal{N}_P})\right|^q \ell(S)^d\right)^{\frac{p}{q}}}{\ell(R)^{sp+d\frac{p}{q}-d}}$$

$$\overset{\text{Höl.}}{\lesssim} \sum_R \left(\sum_{\substack{S \in \mathbf{SH}(R) \\ P \in [S,R)}} \frac{|f_P - f_{\mathcal{N}(P)}|^q}{\ell(P)^\varepsilon} \left(\sum_{P \in [S,R)} \ell(P)^{\frac{\varepsilon q'}{q}} \right)^{\frac{q}{q'}} \ell(S)^d \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}}$$

$$\overset{\text{unif.}}{\lesssim} \sum_R \left(\sum_{P \in \mathbf{SH}^2(R)} \frac{\left(⨍_{5P} |f(\xi) - f_P|\, d\xi\right)^q \ell(P)^d}{\ell(P)^\varepsilon} \right)^{\frac{p}{q}} \ell(R)^{d+\frac{\varepsilon p}{q}-sp-\frac{dp}{q}}$$

$$\overset{q \leqslant p}{\lesssim} \sum_R \sum_{P \in \mathbf{SH}^2(R)} \frac{\left(⨍_{5P} |f(\xi) - f_P|\, d\xi\right)^p \ell(P)^d}{\ell(P)^{\frac{\varepsilon p}{q}}} \ell(R)^{d\left(1-\frac{q}{p}\right)\frac{p}{q}} \ell(R)^{d-sp-\frac{dp}{q}+\frac{\varepsilon p}{q}}.$$

Idea of the proof II

To improve the last bound (in case $1 \leqslant q \leqslant p < \infty$):

$$\left(\sum_{Q,S}\int_Q\int_S \frac{|f_{Q_S}-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}}g(x,y)\right)^p \overset{\text{chain}}{\lesssim} \sum_R \frac{\left(\sum_{S\in\mathbf{SH}(R)}\left|\sum_{P\in[S,R)}(f_P-f_{\mathcal{N}_P})\right|^q \ell(S)^d\right)^{\frac{p}{q}}}{\ell(R)^{sp+d\frac{p}{q}-d}}$$

$$\overset{\text{Höl.}}{\lesssim} \sum_R \left(\sum_{\substack{S\in\mathbf{SH}(R)\\P\in[S,R)}} \frac{|f_P-f_{\mathcal{N}(P)}|^q}{\ell(P)^\varepsilon}\left(\sum_{P\in[S,R)}\ell(P)^{\frac{\varepsilon q'}{q}}\right)^{\frac{q}{q'}}\ell(S)^d\right)^{\frac{p}{q}}\ell(R)^{d-sp-d\frac{p}{q}}$$

$$\overset{\text{unif.}}{\lesssim} \sum_R \left(\sum_{P\in\mathbf{SH}^2(R)} \frac{\left(\fint_{5P}|f(\xi)-f_P|\,d\xi\right)^q\ell(P)^d}{\ell(P)^\varepsilon}\right)^{\frac{p}{q}}\ell(R)^{d+\frac{\varepsilon p}{q}-sp-\frac{dp}{q}}$$

$$\overset{q\leqslant p}{\lesssim} \sum_R \sum_{P\in\mathbf{SH}^2(R)} \frac{\left(\fint_{5P}|f(\xi)-f_P|\,d\xi\right)^p\ell(P)^d}{\ell(P)^{\frac{\varepsilon p}{q}}}\ell(R)^{d\left(1-\frac{q}{p}\right)\frac{p}{q}}\ell(R)^{d-sp-\frac{dp}{q}+\frac{\varepsilon p}{q}}.$$

Note that for $p < q$ one can use subadditivity in the last step, and the sum will converge.

The end

Moltes gràcies!