Riemann mapping and regularity in the Sobolev and Triebel-Lizorkin scale

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Function spaces

Measuring smoothness and integrability in $\mathbb{R}^d$

Lebesgue spaces → integrability.

- $\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}$,
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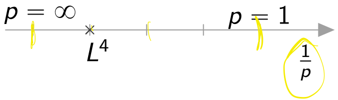

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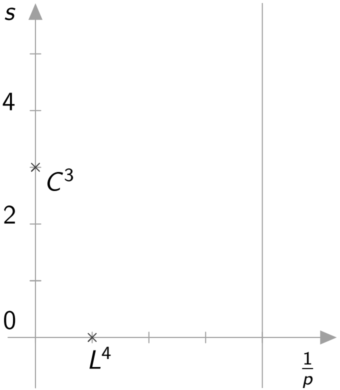

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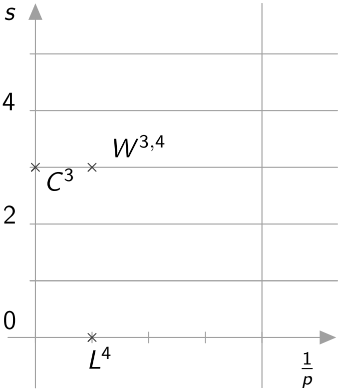

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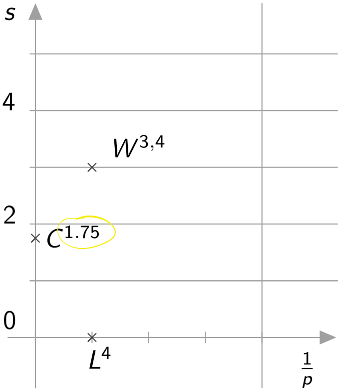

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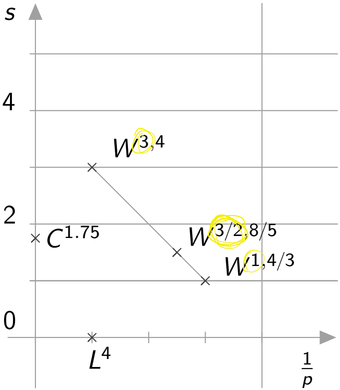

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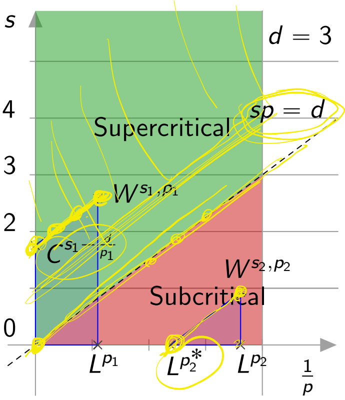

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By means of Sobolev embeddings, we have either continuity or extra integrability.

Composition properties of bi-Lipschitz Sobolev functions

Theorem (P, preprint)

Let $s > 1 + d/p$ and $1 < p < \infty$. Given bounded Lipschitz domains $\Omega_j \subset \mathbb{R}^d$ and functions $f_j \in W^{s,p}(\Omega_j)$ with $f_1(\Omega_1) \subset \Omega_2$ and f_1 bi-Lipschitz, then

$$f_2 \circ f_1 \in W^{s,p}(\Omega_1),$$

and if $f_1(\Omega_1) = \Omega_2$, then

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The result also works for $1 < s \leqslant 1 + d/p$ replacing the function space $W^{s,p}$ by $W^{s,p} \cap C^{0,1}$. Same is true for the Triebel-Lizorkin scale $F^s_{p,q}$ whenever $s \notin \mathbb{N}$ or $d = 1$ and $p = q$ (see [Bourdaud, Moussai, Sickel] for composition, [Astala, P., Saksman] for inversion, the other cases remain open).

Interpolation for Lipschitz Sobolev functions

Lemma (P, preprint)

Let $s > 1$, *let* $1 \leqslant p < \infty$, *and* $d \in \mathbb{N}$ *and let* $f \in W^{s,p}(\Omega) \cap C^{0,1}(\Omega)$ *where* $\Omega \subset \mathbb{R}^d$ *is a bounded Lipschitz domain. Then, for every positive index* $j \leqslant k$

$$\left\|\nabla^j f\right\|_{L^{p\frac{s-1}{j-1}}(\Omega)} \lesssim_{s,p,q,j,\Omega} \|f\|_{W^{s,p}(\Omega)}^{\frac{j-1}{s-1}} \|\nabla f\|_{L^\infty(\Omega)}^{\frac{s-j}{s-1}}.$$

Runst, Sickel + Rychkov's extension

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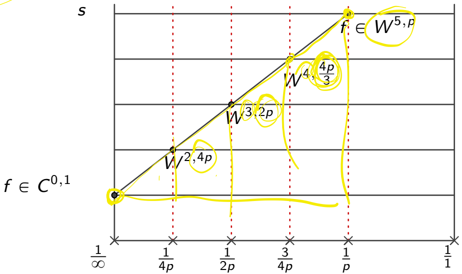

Chain rule and Faà di Bruno's formula

By the chain rule, in $\mathbb{R}$ we have

$$D(g \circ f) = Dg(f)Df$$
$$D^2(g \circ f) = D^2 g(f)DfDf + Dg(f)D^2 f$$
$$D^3(g \circ f) = D^3 g(f)DfDfDf + 3D^2 g(f)DfD^2 f + Dg(f)D^3 f$$
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$$\dots$$

$$D^s(g \circ f) = \sum_{\substack{1 \leqslant i \leqslant s \\ \{\alpha_j\}_{j=1}^i \subset \mathbb{N} : \sum \alpha_j = s}} C_{s,i,\{\alpha_j\}} D^i g(f) \prod_{\ell=1}^{i} D^{\alpha_\ell} f.$$

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For $f:\mathbb{R}^d\to\mathbb{R}^d$, $|\vec{k}| = s$ we have

$$D^{\vec{k}}(g\circ f) = \sum_{\substack{1\leqslant|\vec{i}|\leqslant s\\ \{\alpha_j\}_{j=1}^{|\vec{i}|}\subset\mathbb{N}_0^d\setminus\{\vec{0}\}:\sum|\alpha_j|=s}} C_{\vec{k},\vec{i},\{\alpha_j\}} D^{\vec{i}} g(f)\prod_{\ell=1}^{|\vec{i}|} D^{\alpha_\ell} f^{m(\vec{i})_\ell}$$

Idea of the proof

$\nabla^s(f_2 \circ f_1)$

$$\textcircled{1} := \left\| \sum_{1\leqslant i\leqslant s} \sum_{\alpha\in\mathbb{N}^i:|\alpha|=s} \nabla^i f_2(f_1) \prod_{\ell=1}^{i} \nabla^{\alpha_\ell} f_1 \right\|_{L^p}$$

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$$\lesssim_{d,s} \sum_{1\leqslant i\leqslant s} \sum_{\alpha\in\mathbb{N}^i:|\alpha|=s} \left\| \nabla^i f_2(f_1) \right\|_{p_0} \prod_{\ell=1}^{i} \left\| \nabla^{\alpha_\ell} f_1 \right\|_{p_\ell},$$

where $\sum_0^i \frac{1}{p_\ell} = \frac{1}{p}$.

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where $\sum_0^i \frac{1}{p_\ell} = \frac{1}{p}$. Let $p_0 = \frac{p(s-1)}{i-1}$ and $p_\ell = \frac{p(s-1)}{\alpha_\ell - 1}$: using

$$\left\|\nabla^i f\right\|_{L^{p\frac{s-1}{i-1}}(\Omega)} \lesssim \left\|f\right\|_{W^{s,p}(\Omega)}^{\frac{i-1}{s-1}} \left\|\nabla f\right\|_{L^\infty(\Omega)}^{\frac{s-i}{s-1}}$$

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$$\textcircled{1} \lesssim \sum_i \|\nabla f_1^{-1}\|_\infty^{\frac{d(i-1)}{p(s-1)}} \|f_2\|_{W^{s,p}(\Omega_2)}^{\frac{i-1}{s-1}} \|\nabla f_2\|_{L^\infty(\Omega_2)}^{\frac{s-i}{s-1}} \|f_1\|_{W^{s,p}(\Omega_1)}^{\frac{s-i}{s-1}} \|\nabla f_1\|_{L^\infty(\Omega_1)}^{\frac{is-s}{s-1}}$$

Idea of the proof

$$\text{①} := \left\| \sum_{1\leqslant i\leqslant s} \sum_{\alpha\in\mathbb{N}^i:|\alpha|=s} \nabla^i f_2(f_1) \prod_{\ell=1}^{i} \nabla^{\alpha_\ell} f_1 \right\|_{L^p}$$

$$\lesssim_{d,s} \sum_{1\leqslant i\leqslant s} \sum_{\alpha\in\mathbb{N}^i:|\alpha|=s} \left\| \nabla^i f_2(f_1)\right\|_{p_0} \prod_{\ell=1}^{i} \|\nabla^{\alpha_\ell} f_1\|_{p_\ell},$$

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$$\lesssim C_{f_1}\left(\|f_2\|_{W^{s,p}(\Omega_2)} \|\nabla f_1\|_{L^\infty(\Omega_1)}^{s} + \|\nabla f_2\|_{L^\infty(\Omega_2)} \|f_1\|_{W^{s,p}(\Omega_1)}\right),$$

Triebel-Lizorkin scale

From [P. (JLMS), Theorem 1.2, Corollary 1.4] and [Seeger'89, Corollary 2], given a uniform domain $\Omega \subset \mathbb{R}^d$, $0 < \sigma < 1$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ with $\sigma > \frac{d}{p} - \frac{d}{q}$, for $s = k + \sigma$ we define $\|f\|_{F^s_{p,q}(\Omega)}$ as

$$\|f\|_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \left(\int_\Omega \left(\int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^q}{|x-y|^{\sigma q + d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

with the usual modification for $q = \infty$. Then, the norm defined above is equivalent to the restriction norm for the classical Triebel Lizorkin scale.

Triebel-Lizorkin scale

More generally, in [P, preprint] we drop the restriction $\sigma > \frac{d}{p} - \frac{d}{q}$, and consider the auxiliary index $1 \leqslant u \leqslant \infty$ so that $\sigma > \frac{d}{p \wedge q} - \frac{d}{u}$. Then we define $\|f\|_{F^s_{p,q}(\Omega)}$ as

$0 < \sigma < 1$

$$\|f\|_p + \left(\int_\Omega \left(\int_0^1 \frac{\left(\int_{B(x,t)\cap\Omega} |\nabla^k f(x) - \nabla^k f(y)|^u dy \right)^{\frac{q}{u}}}{t^{\left(\sigma + \frac{d}{u}\right)q}} \frac{dt}{t} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

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Then use

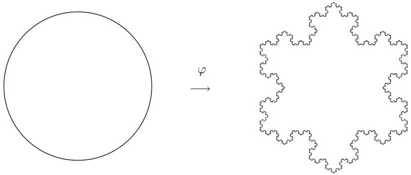
$$|\nabla^k (g \circ f)(x) - \nabla^k (g \circ f)(y)|$$

$$\lesssim \sum_{1 \leqslant i \leqslant k} |\nabla^i g(f(x)) - \nabla^i g(f(y))| \sum_\alpha \prod_{j=1}^{i} |\nabla^{\alpha_j} f(x)| +$$

$$\sum_{1 \leqslant i \leqslant k} |\nabla^i g(f(y))| \sum_\alpha \sum_{\ell=1}^{i} |\nabla^{\alpha_\ell} f(x) - \nabla^{\alpha_\ell} f(y)| \prod_{j=1}^{\ell-1} |\nabla^{\alpha_j} f(y)| \prod_{j=\ell+1}^{i} |\nabla^{\alpha_j} f(x)|,$$

Regularity of the Riemann mapping in the Sobolev scale

Implications of the geometry of a domain



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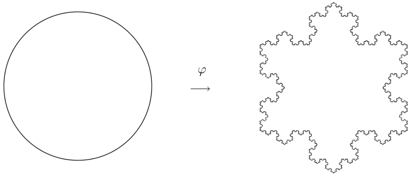

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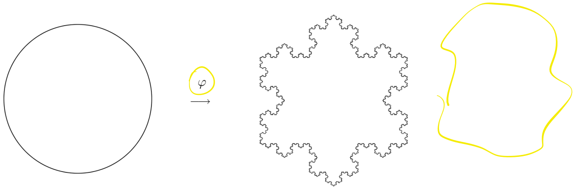

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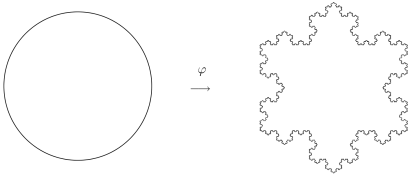

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Theorem (Astala, P, Saksman)

Let $s > 0$ and $1 < p < \infty$ with $sp > 2$, and suppose Ω is a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$.

Then Ω is a $B^{s+1-\frac{1}{p}}_{p,p}$-domain if and only if $\varphi \in W^{s+1,p}(\mathbb{D})$ and $\varphi^{-1} \in W^{s+1,p}(\Omega)$.

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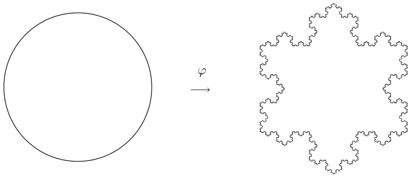

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The Herglotz formula

Let $\Phi(z) := \log \varphi'(z)$, well defined since $\mathbb{D}$ is simply connected. In particular, $\gamma := (\arg \varphi')_{|\mathbb{T}} = \mathrm{Im}\, \Phi_{|\mathbb{T}}$, the argument of φ' on the boundary, is a well defined continuous function.

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Note that the Poisson extension maps $B^{s-1/p}_{p,p}(\mathbb{T}) \to W^{s,p}(\mathbb{D})$.

Idea of the proof

Assume that Ω is a $B^{s+1-\frac{1}{p}}_{p,p}$-domain. We prove that

$$\gamma \in B^{s-1/p}_{p,p}(\mathbb{T}). \tag{1}$$

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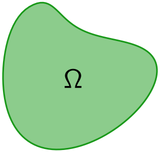
Linking the boundary parameterization and the Riemman mapping



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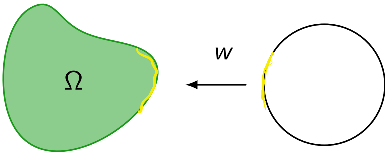

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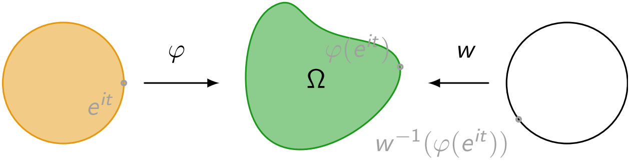

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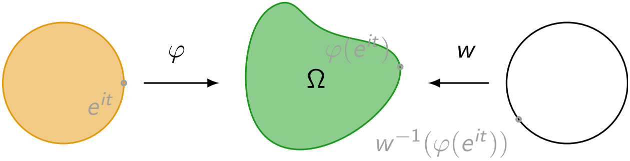

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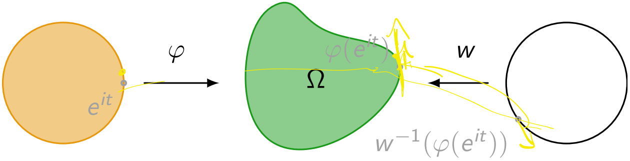

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That is (see [Pommerenke, Theorem 3.2]),

$$\gamma(e^{it}) = -t - \frac{\pi}{2} + [\mathrm{Arg}\, w'] \circ [w^{-1} \circ \varphi](e^{it}),$$

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If $2/p < s < 1 + 1/p$. Since φ, w are bi-Lipschitz, also $w^{-1} \circ \varphi$ is a bi-Lipschitz homeomorphism of $\mathbb{T}$. Thus

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Assume that $s > 1$ and $s' < s < s' + 1$ with $s'p > 2$. By induction, we can assume that $\varphi \in B^{s'+1-1/p}_{p,p}(\mathbb{T})$. Moreover, $w \in B^{s'+1-1/p}_{p,p}$. Since both w and φ are admissible $B^{s'+1-1/p}_{p,p}$-parametrizations of $\partial\Omega$ we deduce that

$$w^{-1} \circ \varphi \in B^{s'+1-1/p}_{p,p}(\mathbb{T}), \quad \text{and} \quad w^{-1} \circ \varphi \text{ is bi-Lipschitz.}$$

Since $s < s' + 1$, then

$$\operatorname{Arg} w' \circ w^{-1} \circ \varphi \in B^{s-1/p}_{p,p}(\mathbb{T}).$$

QC mappings of $\mathbb{C}$

Quasiconformal mappings

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Conformal mappings

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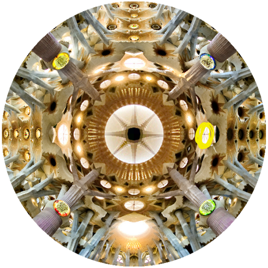

Quasiconformal mappings
Angle distortion bounded.
"Circles to ellipses".
$|\bar{\partial} f| \leqslant \kappa |\partial f|$
$W^{1,2}_{\mathrm{loc}}$-homeo

The Beurling transform

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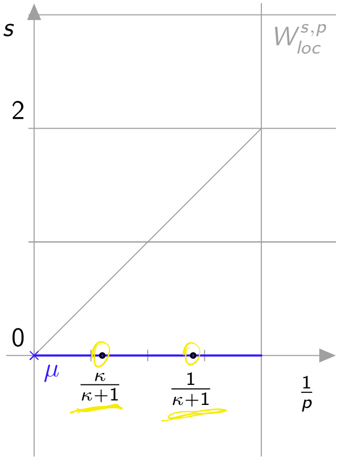
To study regularity on $\mathbb{C}$, study invertibility of $Id - \mu\mathcal{B}$ in different function spaces.

Results without boundaries



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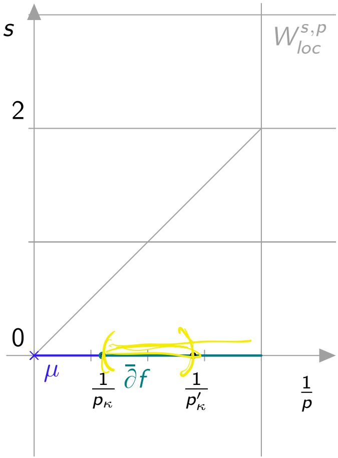

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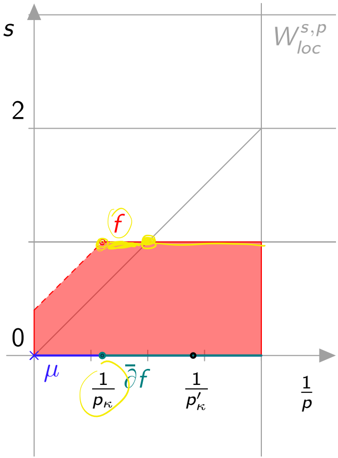

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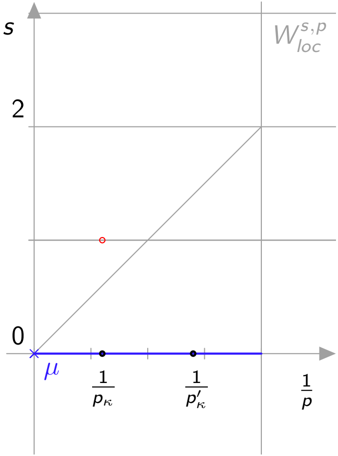

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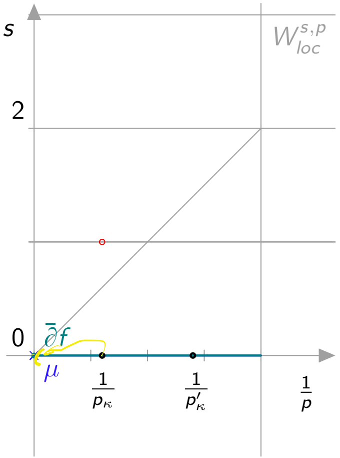

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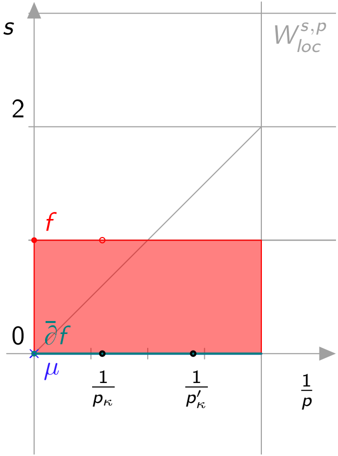

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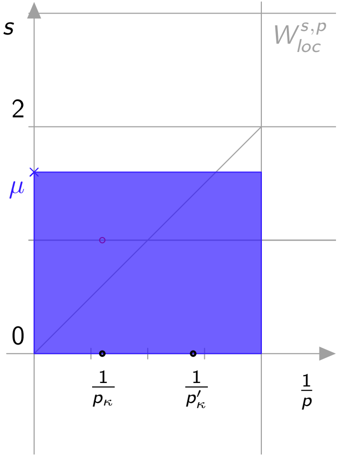

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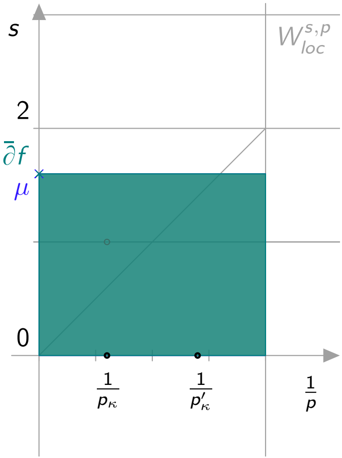

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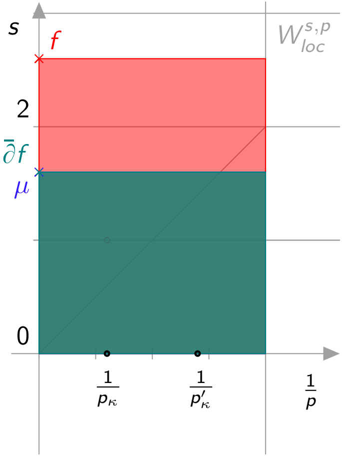

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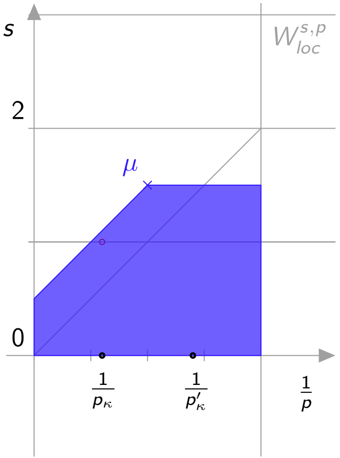

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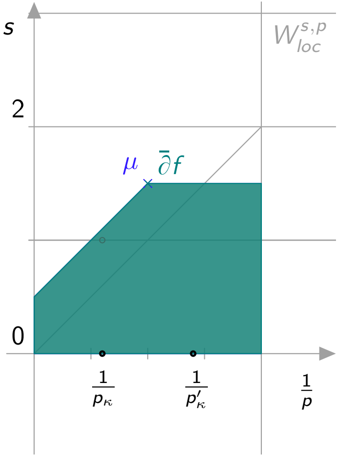

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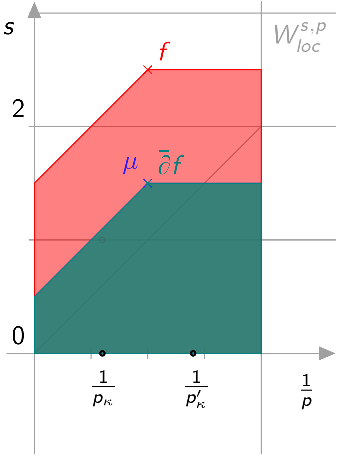

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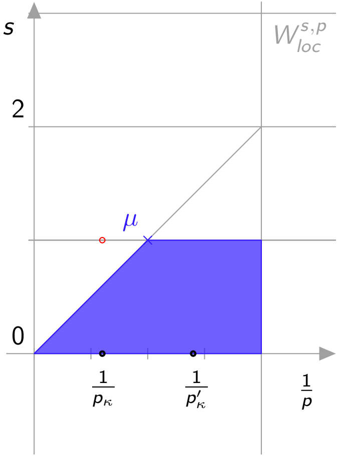

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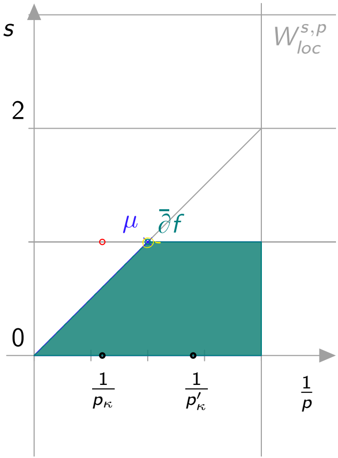

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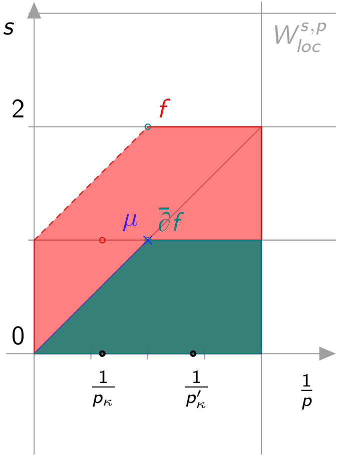

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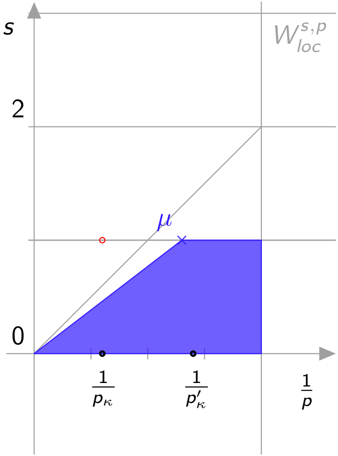

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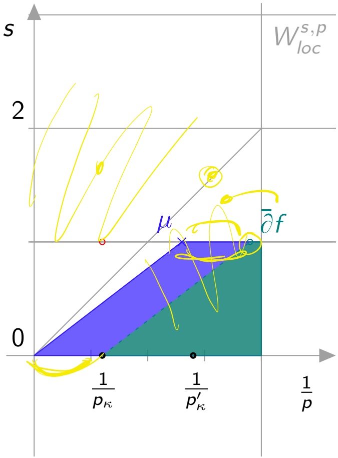

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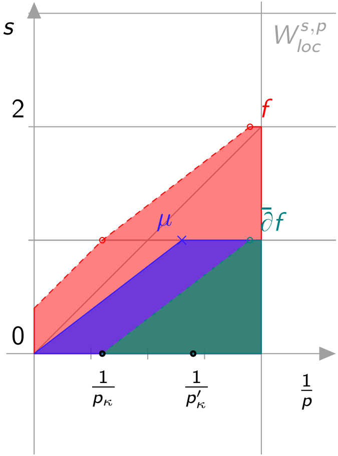

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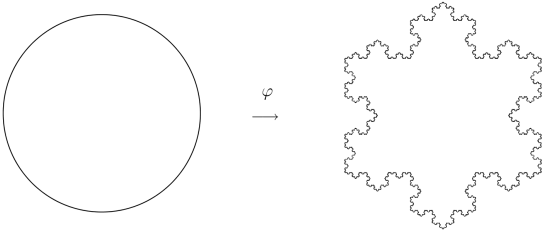
QC mappings on domains

What about quasiconformal mappings on domains?



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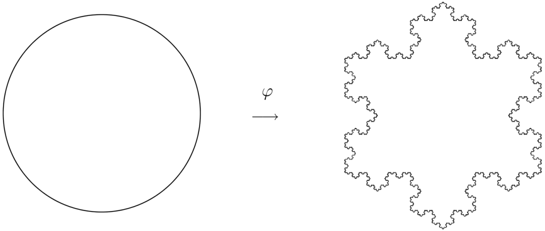

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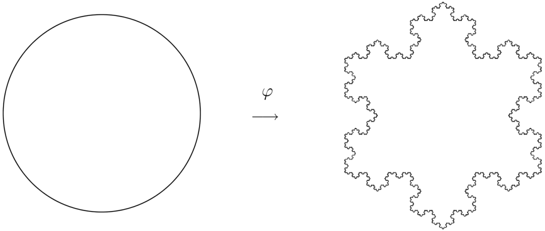

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The moral is that in order to study the regularity of μ-quasiconformal mappings between domains we must take into account both the regularity of the boundary and the regularity of μ.

Result

Theorem (Astala, P., Saksman)

Let $sp > 2$ with $1 < p < \infty$, let Ω_1, Ω_2 be simply connected bounded domains and let $g : \Omega_1 \to \Omega_2$ be a μ-quasiconformal map. If both Ω_j are $B^{s+1-\frac{1}{p}}_{p,p}$-domains and $\mu \in W^{s,p}(\Omega_1)$, then $g \in W^{s+1,p}(\Omega_1)$ and g is bi-Lipschitz.

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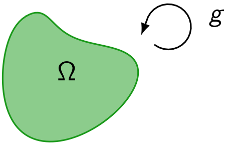
- If $\Omega_1 = \mathbb{D}$, then the statement is if and only if.
- Simply connected can be replaced by finitely connected.
- For $s \notin \mathbb{N}$, we can substitute $\mu \in F^s_{p,q}(\Omega_1)$ and $g \in F^{s+1}_{p,q}(\Omega_1)$ in the conclusion of the theorem.

$q \neq 2$, $s = 2$

Idea of the proof



Let $g : \Omega \to \Omega$ to be μ-QC, with $\mu \in W^{s,p}(\Omega)$ and $\partial\Omega$ regular enough.
Can we say that $\partial g \in W^{s,p}(\Omega)$??

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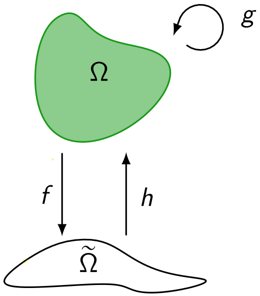

Extend μ to $\mathbb{C}$. By Stoilow factorization, $g = h \circ f$ where $f : \mathbb{C} \to \mathbb{C}$ is the μ-principal mapping and $h : \widetilde{\Omega} \to \Omega$ is conformal.

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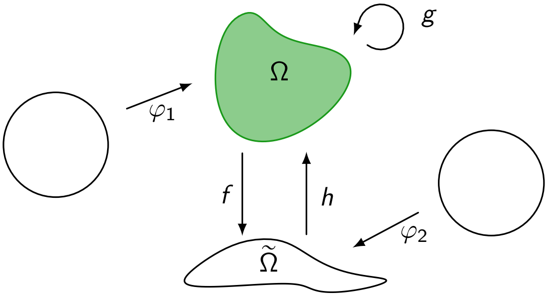
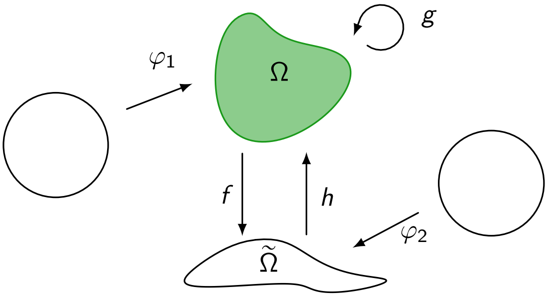


We can find Riemann mappings (conformal) if the domains are simply connected.

Idea of the proof (2)



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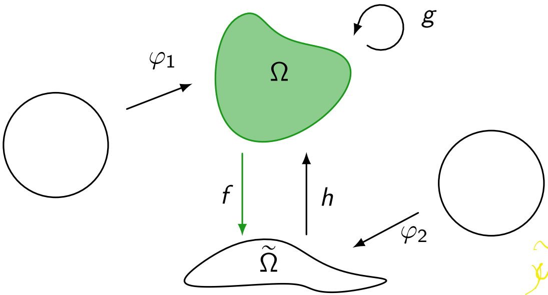

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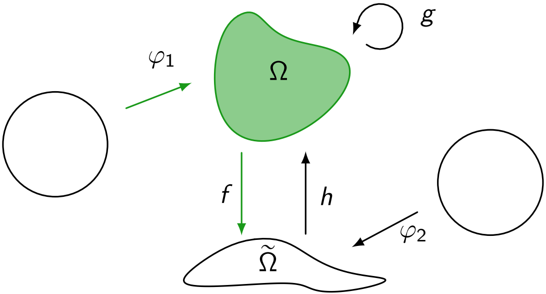

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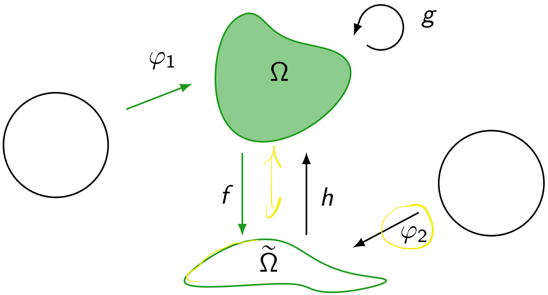

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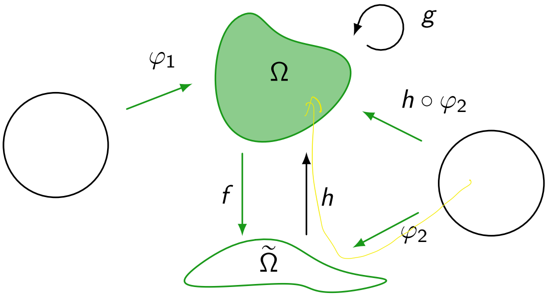

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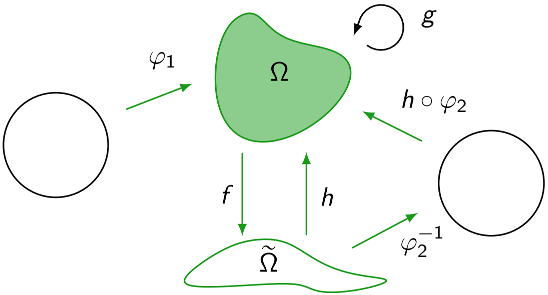

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Then, $g = (h \circ \varphi_2) \circ (\varphi_2^{-1}) \circ f \in W^{s+1,p}(\Omega)$.

Kolaj

The principal mapping

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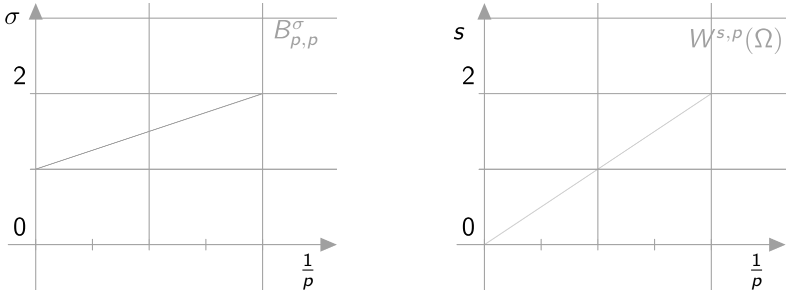

To study regularity of f principal solution to $\bar{\partial} f = \mu \partial f$ on Ω, study invertibility of $\chi_\Omega \cdot - \mu \mathcal{B}(\chi_\Omega \cdot)$ in $W^{s,p}(\Omega)$.

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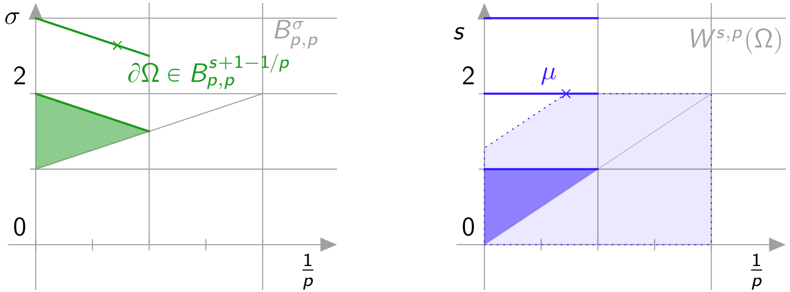

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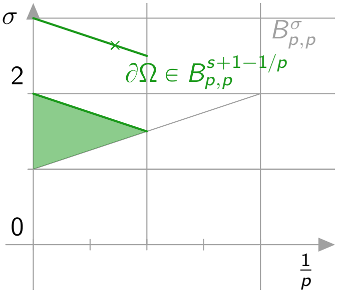

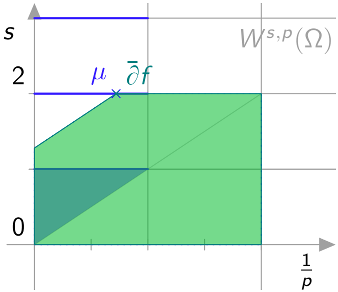

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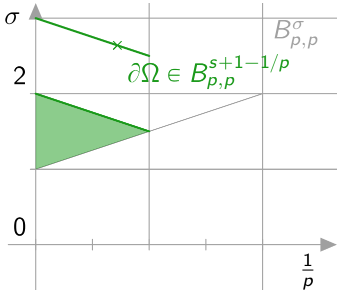

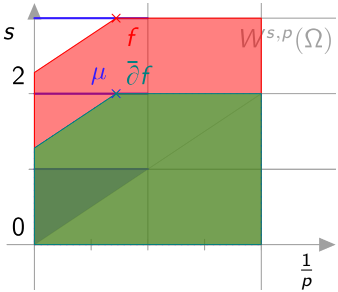

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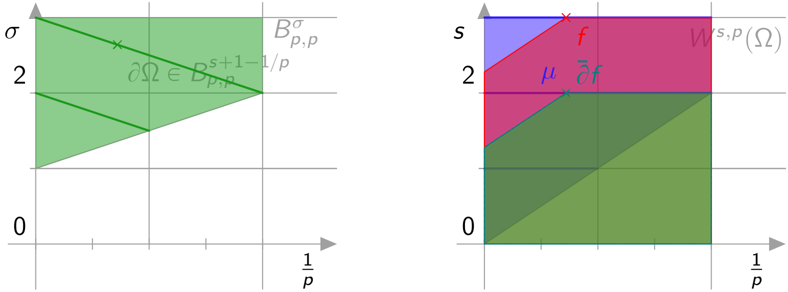

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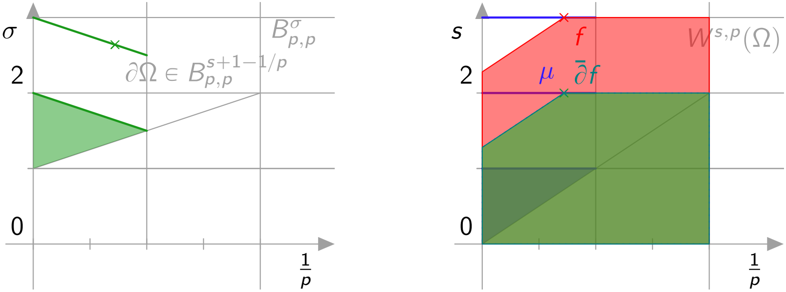

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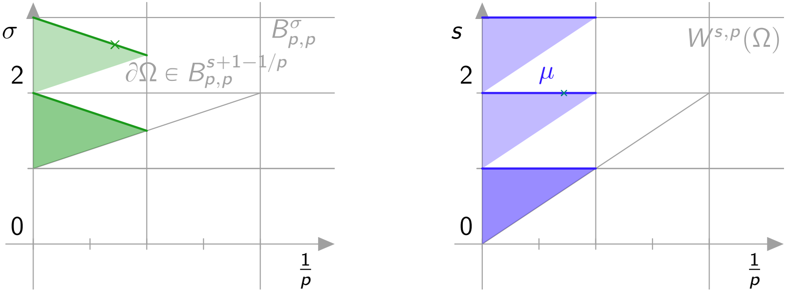

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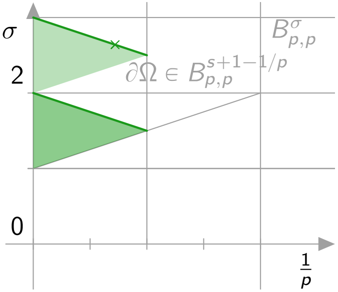

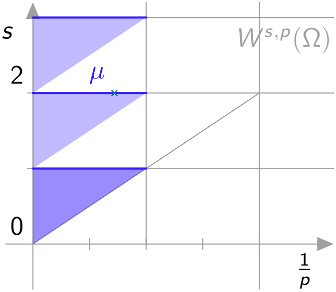

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- Can the principal mapping result be extended to the whole supercritical region?
- (Sub)critical situation: is there any condition on $\partial\Omega$ which can lead to analogous results?

The end

Moltes gràcies!!

Muchas gracias!!