Regularity of planar quasiconformal self-maps

Martí Prats (joint work with K. Astala, E. Saksman and X. Tolsa)

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ICMAT INSTITUTO DE CIENCIAS MATEMÁTICAS

erc European Research Council

September 5th, 2017

Introduction

Measuring smoothness and integrability in $\mathbb{R}^d$

Lebesgue spaces → integrability.

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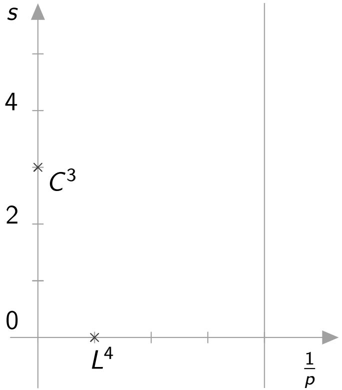

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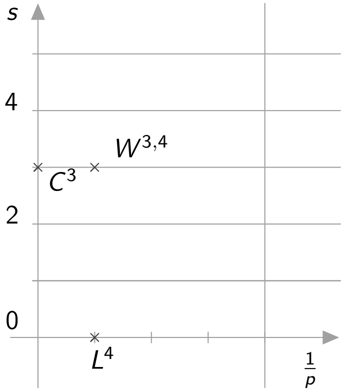

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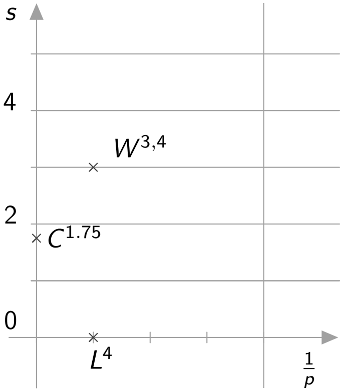

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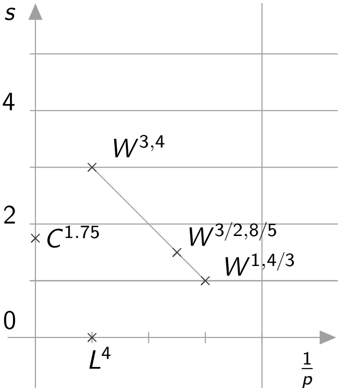

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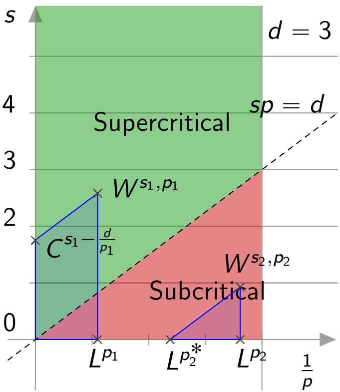

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By means of Sobolev embeddings, we have either continuity or extra integrability.

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Quasiconformal mappings
Angle distortion bounded.
"Circles to ellipses".
$|\overline{\partial} f| \leqslant \kappa |\partial f|$
$W^{1,2}_{\mathrm{loc}}$-homeo

The Beurling transform

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

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Recall that $\mathcal{B} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$.
Also $\mathcal{B} : W^{s,p}(\mathbb{C}) \to W^{s,p}(\mathbb{C})$ is bounded for $1 < p < \infty$ and $s > 0$.

QC mappings of the whole plane

The Beltrami equation: the principal solution

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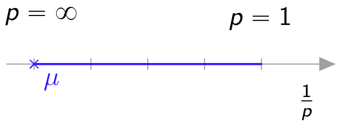

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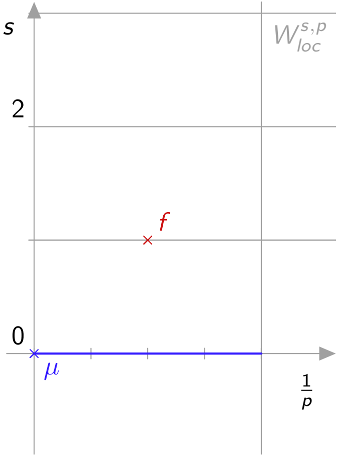

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The *Beltrami equation*

$$\bar\partial f(z) = \mu(z)\partial f(z)$$

has a unique solution $f \in W^{1,2}_{loc}$ such that $f(z) = z + \mathcal{O}(1/z)$ as $z \to \infty$.

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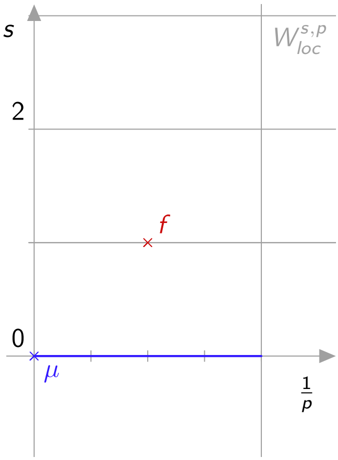

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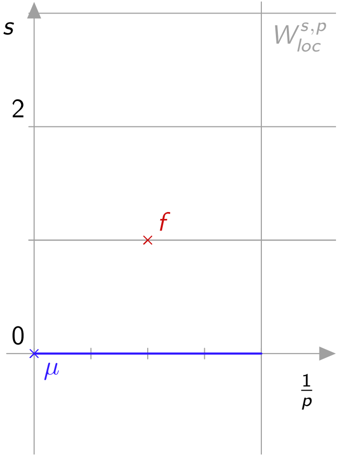

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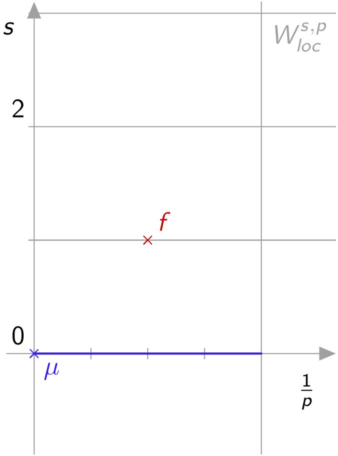

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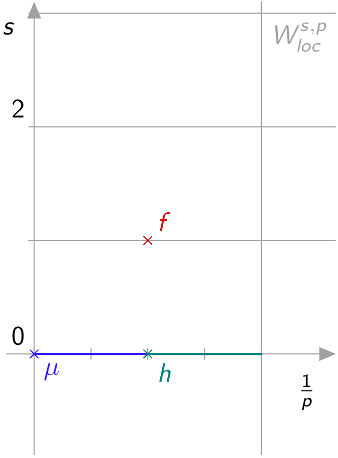

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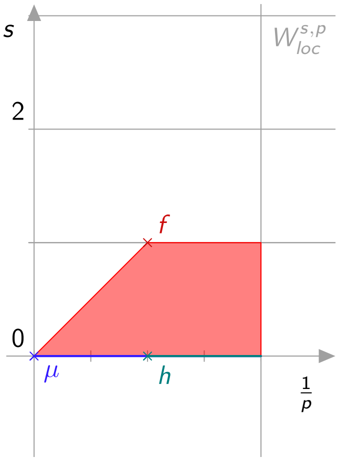

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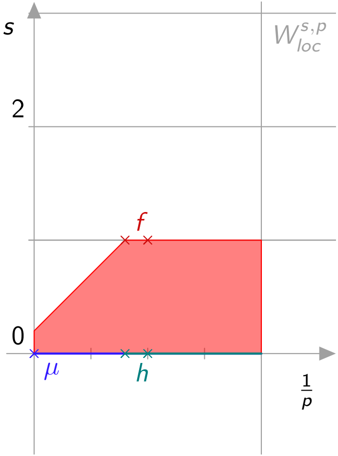

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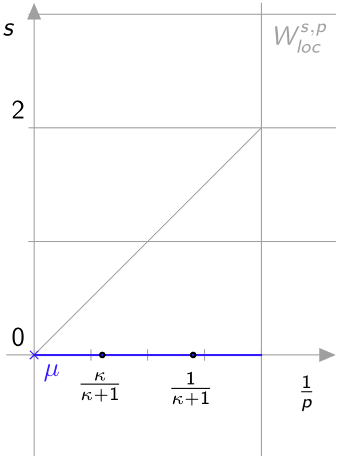
This remains true if $\|\mathcal{B}\|_{(p,p)} < 1/\kappa$.

Results without boundaries



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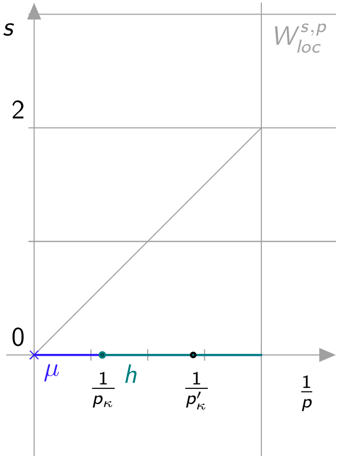

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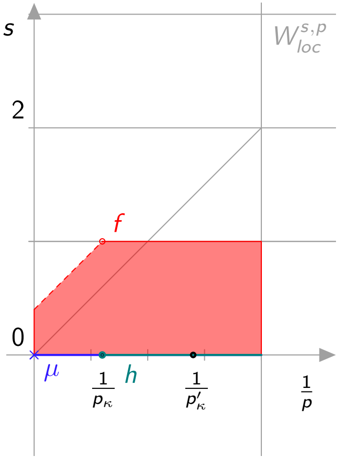

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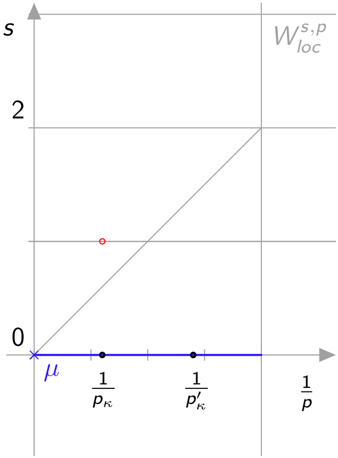

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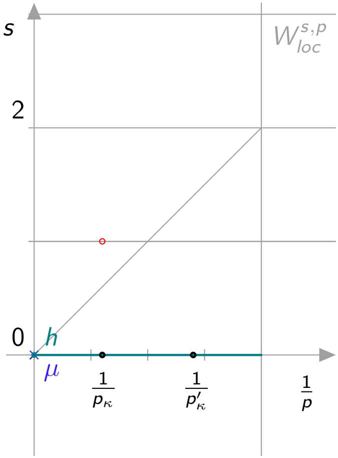

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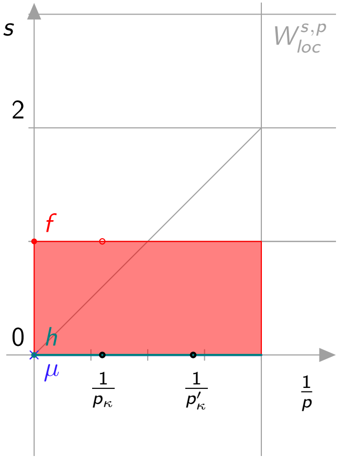

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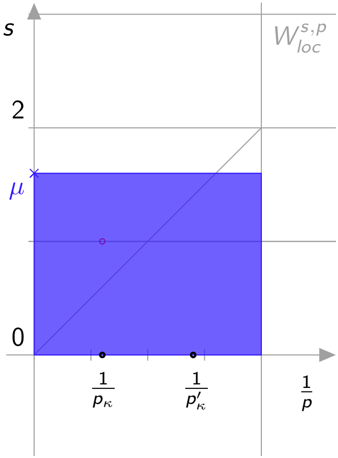

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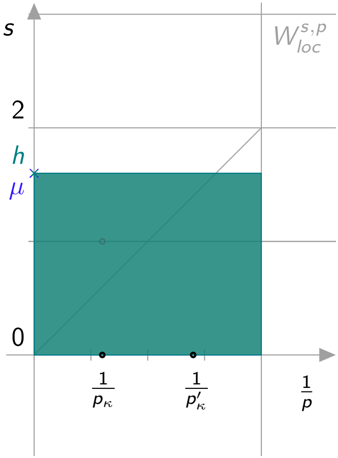

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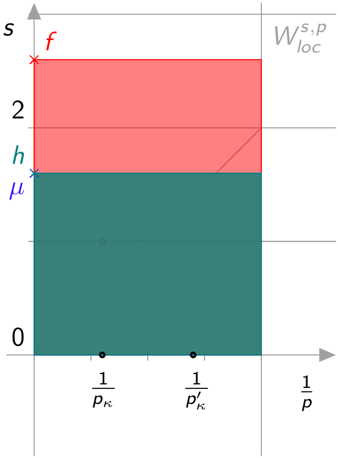

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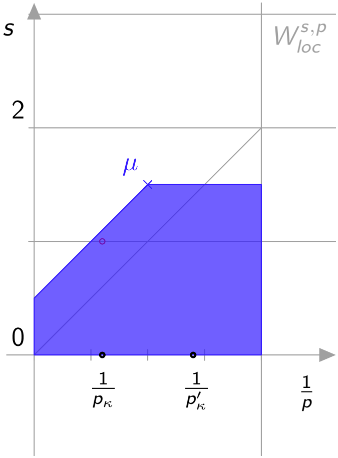

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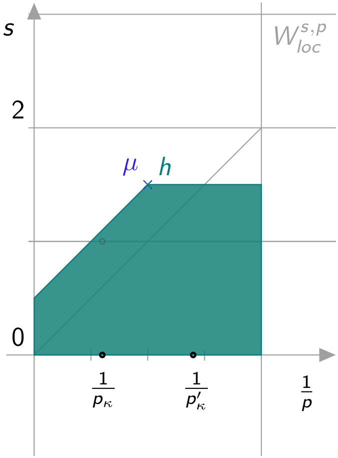

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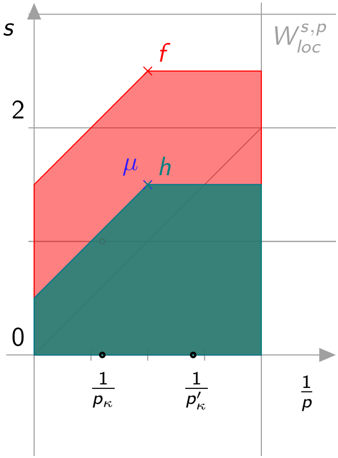

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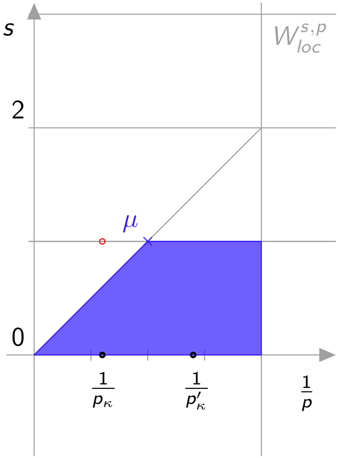

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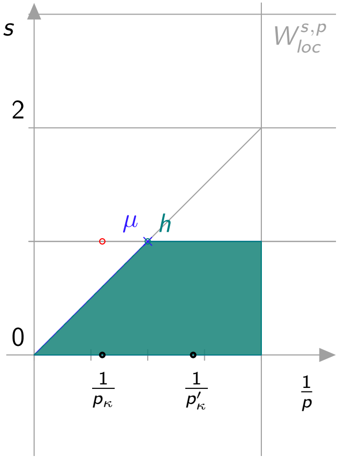

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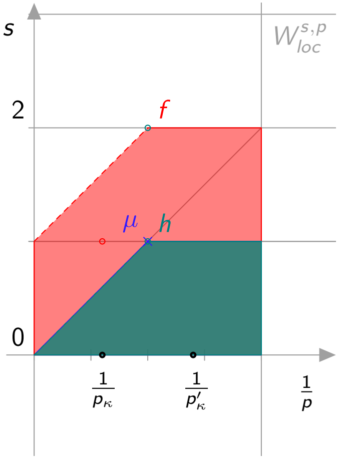

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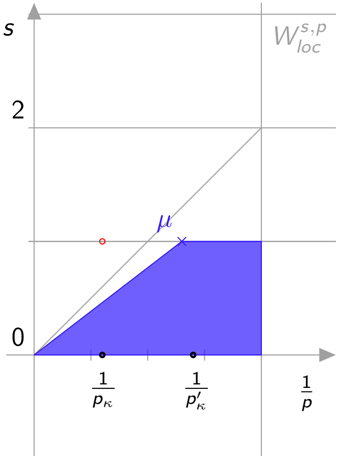

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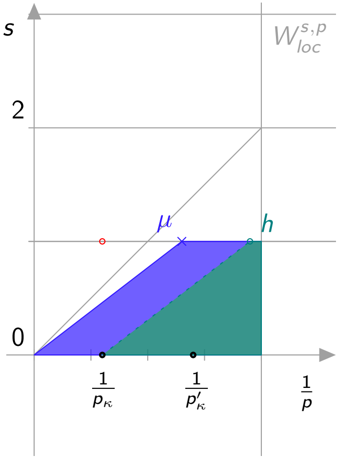

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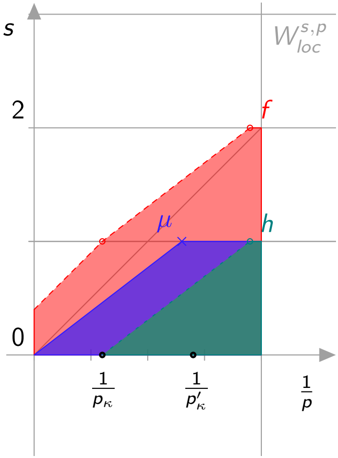

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Recent progress

Theorem (P.)

Let $0 < s < 2$, $1 < p < \infty$, *let* $\mu \in W^{s,p} \cap L^\infty$, *with* $\mu \leqslant \kappa\chi_{\mathbb{D}}$ *and let* f *be the principal solution to the Beltrami equation* $\bar\partial f = \mu \partial f$.

If $s = \frac{2}{p}$, *then*

$$\bar\partial f \in W^{s,q} \qquad \text{for every } \frac{1}{q} > \frac{1}{p}.$$

If $s < \frac{2}{p}$ *and* $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa} = \frac{1-\kappa}{1+\kappa}$, *then*

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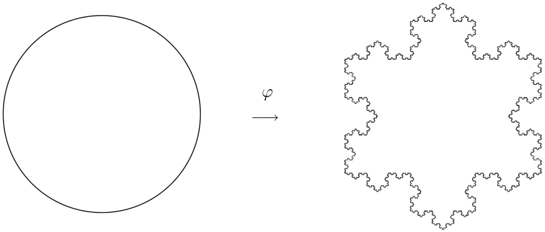
See [Clop, Faraco, Ruiz] for previous weaker results and Baisón's thesis for a stronger result in the critical setting with $s > 1/2$.
It remains unclear if the condition $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$ can be replaced by $\frac{1}{p} < \frac{1}{p'_\kappa}$, which is more natural and is achieved for $s = 1$.

QC mappings on domains

What about quasiconformal mappings on domains?



Consider a Riemann mapping from $\mathbb{D}$ to the Koch Snowflake.
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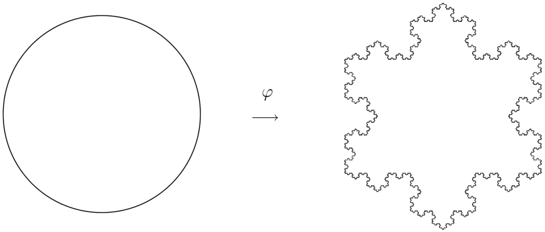

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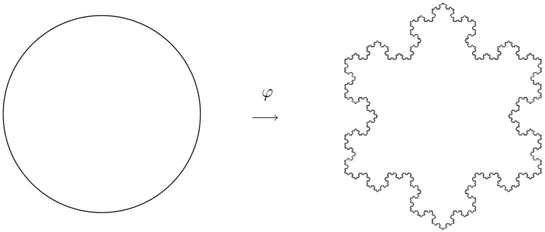

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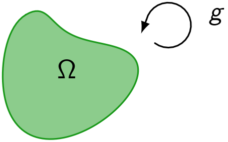
The moral is that in order to study the regularity of μ-quasiconformal mappings between domains we must take into account both the regularity of the boundary and the regularity of μ.

Idea

Let $g : \Omega \to \Omega$ to be μ-QC, with $\mu \in W^{s,p}(\Omega)$ and $\partial\Omega$ regular enough.
Can we say that $\partial g \in W^{s,p}(\Omega)$??



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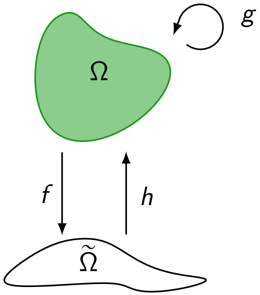
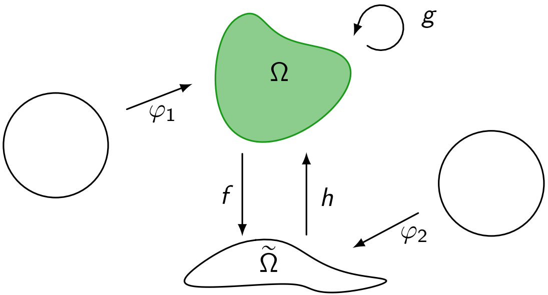
By Stoilow factorization, $g = h \circ f$ where $f : \mathbb{C} \to \mathbb{C}$ is the μ-principal mapping and $h : \widetilde{\Omega} \to \Omega$ is conformal.

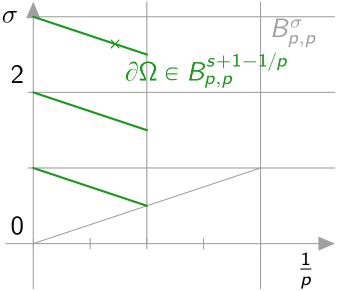


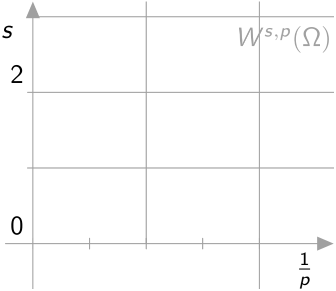
Idea

We can find Riemann mappings (conformal) if the domains are simply connected.



The principal mapping





We study supercritical case.

Theorem (Principal mapping condition, P)

Let $\Omega \subset \mathbb{C}$ be a bounded $B^{s+1-1/p}_{p,p}$-domain, $s \in \mathbb{N}$ and $p > 2$.

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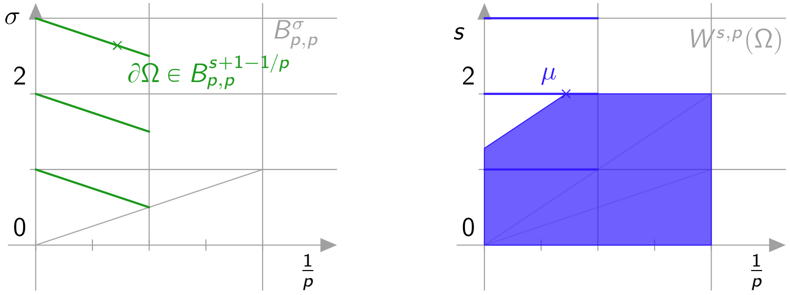

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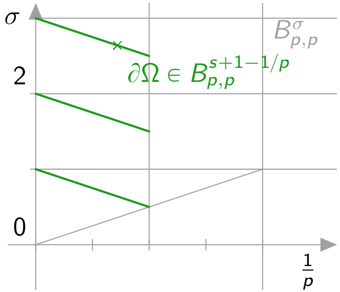

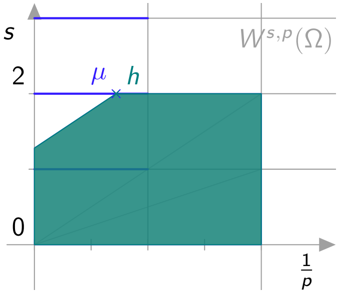

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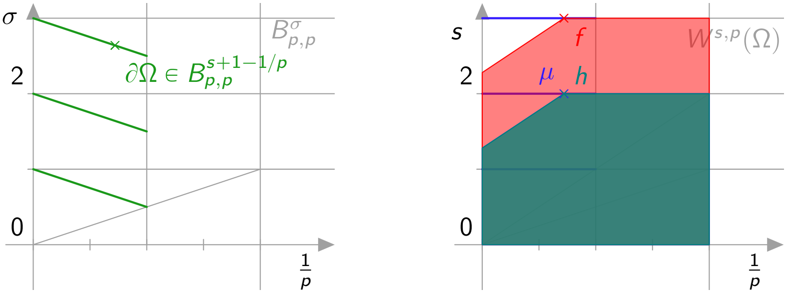

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General case

Riemann mapping condition (in progress, Astala, P, Saksman)

Let $s \in \mathbb{N}$ and $p > 2$. If Ω is a simply connected $B^{s+1-\frac{1}{p}}_{p,p}$-domain, then any Riemann mapping $\varphi : \mathbb{D} \to \Omega$ satisfies that $\varphi \in W^{s+1,p}(\mathbb{D})$ and it is bi-Lipschitz.

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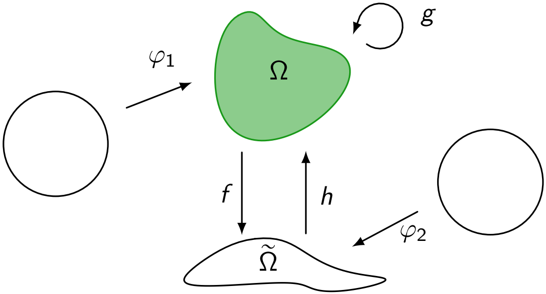
Let $s \in \mathbb{N}$ and $p > 2$, let Ω be a simply connected $B^{s+1-\frac{1}{p}}_{p,p}$-domain and let $g : \Omega \to \Omega$ be a μ-quasiconformal self-map with $\mu \in W^{s,p}(\Omega)$. Then $g \in W^{s+1,p}(\Omega)$.

Idea of the proof



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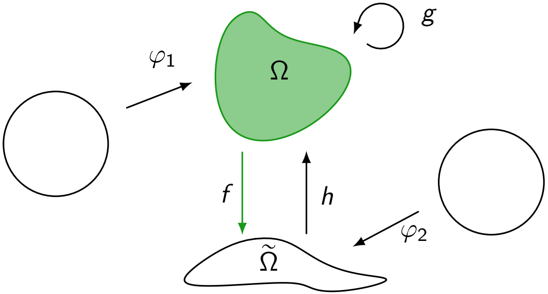

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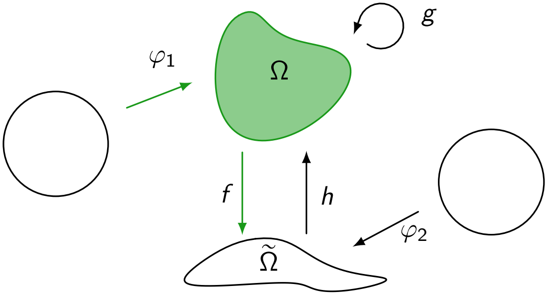

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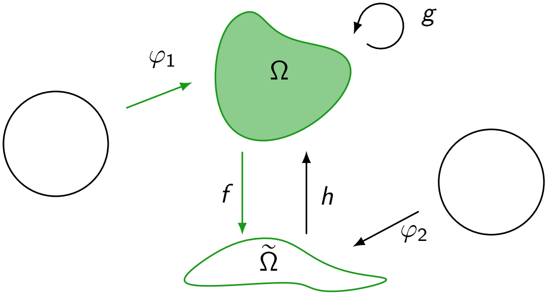

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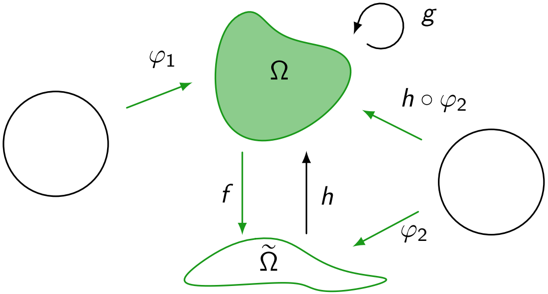

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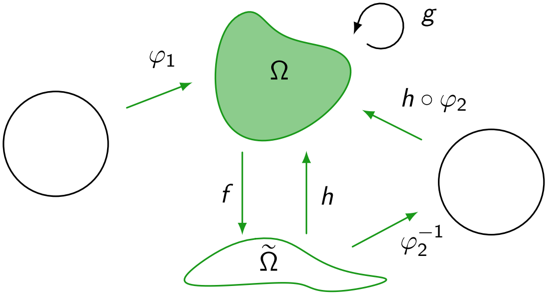

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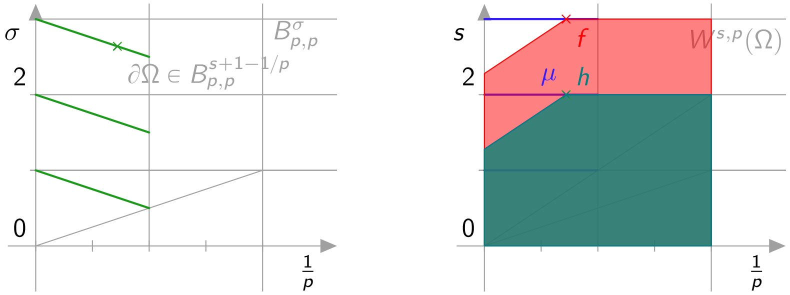
Then, $g = (h \circ \varphi_2) \circ (\varphi_2^{-1}) \circ f \in W^{s+1,p}(\Omega)$.

Conclusions



- In the complex plane, if $N \in B^{s-1/p}_{p,p}(\partial\Omega)$ with $s \in \mathbb{N}$ and $p > 2$, then $\mu \in W^{s,p}(\Omega) \implies f, g \in W^{s+1,p}(\Omega)$.

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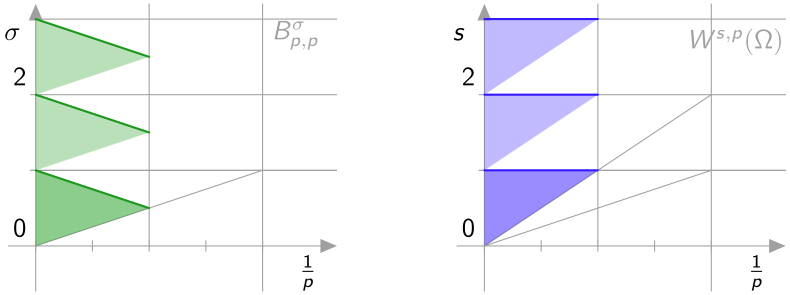

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- Expected further results:
  - The results hold apparently for $0 < s < 1$, $sp > 2$ (work in progress with K. Astala, E. Saksman) and for Hölder spaces with $0 < s < 1$.
  - Subcritical situation: is there any condition on $\partial\Omega$ which can lead to analogous results?

The end

Moltes gràcies!!
Muchas gracias!!