Bounding Calderón-Zygmund operators in Sobolev spaces on Lipschitz domains

PHD thesis in progress, directed by Xavier Tolsa

Martí Prats

Universitat Autònoma de Barcelona

May 24th, 2014

Introduction.

The Beurling transform.

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

$$\mathcal{B}f(z) = c_0 \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} dm(w).$$

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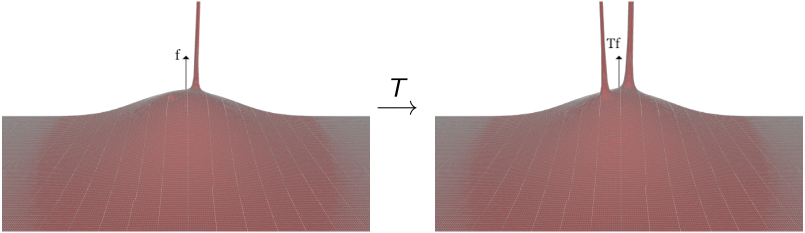
In general, if $x \notin \operatorname{supp}(f) \subset \mathbb{R}^d$ then a convolution CZO of order n is

$$Tf(x) = \int K(x-y)f(y)$$

with

$$|\nabla^j K(x)| \leq \frac{1}{|x|^{d+j}} \qquad \text{for } j \leq n.$$

The problem we face.



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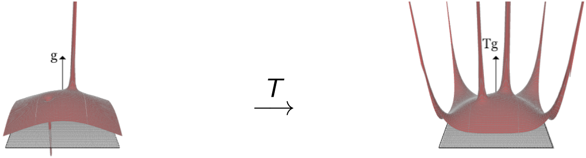

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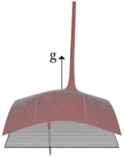
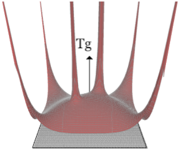


$\xrightarrow{T}$



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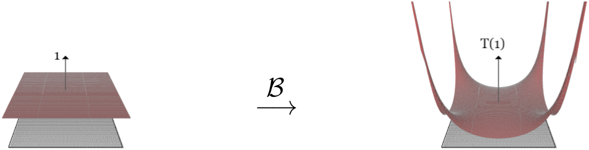

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For Ω a rectangle, $\mathcal{B}\chi_\Omega$ is in every $L^p(\Omega)$ but not in $W^{1,p}(\Omega)$ for $p \geq 2$.

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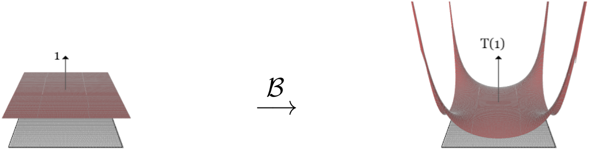

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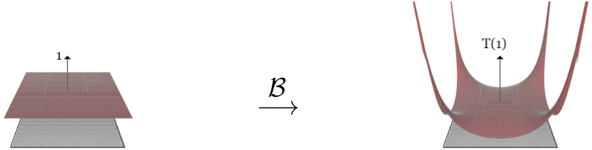

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When is $T : W^{n,p}(\Omega) \to W^{n,p}(\Omega)$ bounded?
We seek for answers in terms of test functions and in terms of the geometry of the boundary.

Results.

Theorem (Cruz, Mateu, Orobitg, 2013)

Given a $C^{1+\epsilon}$ domain $\Omega \subset \mathbb{R}^d$, T even and $p > d$.
If $T(\chi_\Omega) \in W^{1,p}(\Omega)$, then T is bounded in $W^{1,p}(\Omega)$.

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Given a Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $p > d$. If $T(P) \in W^{n,p}(\Omega)$ for polynomials $P \in \mathcal{P}^{n-1}(\Omega)$, then T is bounded in $W^{n,p}(\Omega)$.

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Results.

Theorem (P., Tolsa 2013)

For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to the boundary of Ω is in the Besov space $B^{n-\frac{1}{p}}_{p,p}(\partial\Omega)$ then $\mathcal{B}(\chi_\Omega) \in W^{n,p}(\Omega)$, with

$$\|\nabla^n \mathcal{B}(\chi_\Omega)\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)} + C_{\mathrm{length}(\partial\Omega)}.$$

Sufficient conditions on test functions.

The Whitney covering.

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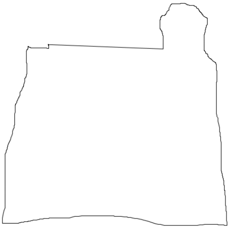

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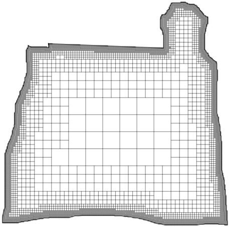

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We perform a Whitney covering $\mathcal{W}$ such that

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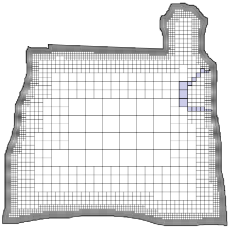

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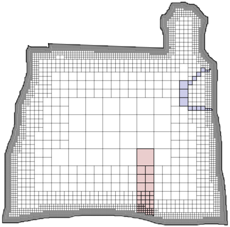

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We can think on Harnack chains.
We can think on Carleson boxes (or shadows).

The key point: approximating by polynomials.

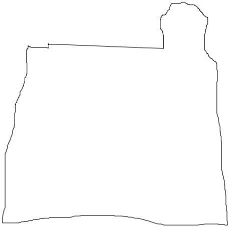
A new approach for the case $n = 1$:

Key Lemma

The following are equivalent:

- $\|\nabla Tf\|_{L^p(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}$.
- $\sum_{Q\in\mathcal{W}} \|\nabla T(f_{3Q}\,\chi_\Omega)\|^p_{L^p(Q)} \leq C\|f\|^p_{W^{1,p}(\Omega)}$.

Proof of the T(P) theorem ($p > d$).



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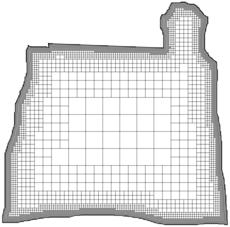

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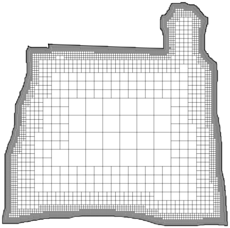

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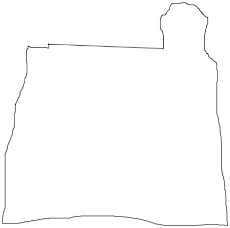

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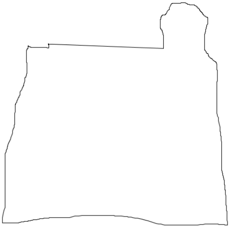

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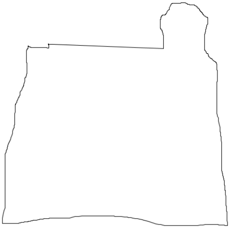

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Since $p > d$, by the Sobolev Embedding Theorem

$$\|f\|_{L^\infty} \leq C\|f\|_{W^{1,p}(\Omega)}.$$

$\square$

Carleson measures in the Besov space of analytic functions.

Consider $\rho(z) = \mathrm{dist}(z, \partial\mathbb{D})^{2-p}$. For analytic functions in $\mathbb{D}$,

$$\|f\|^p_{B_p(\rho)} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \rho(z) \frac{dm(z)}{(1-|z|^2)^2} \approx \|f\|^p_{W^{1,p}(\mathbb{D})}.$$

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Theorem (Arcozzi, Rochberg and Sawyer, 2002)

The following are equivalent:

- *μ is Carleson for $B_p(\rho)$.*
- *For every Whitney cube P,* $\sum_{Q\subset \mathbf{Sh}(P)} \mu(\mathbf{Sh}(Q))^{p'} \rho(Q)^{1-p'} \le C\mu(\mathbf{Sh}(P))$.

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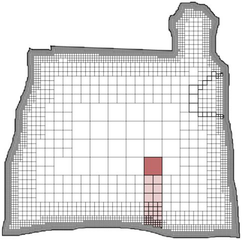

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We say that μ is p-Carleson for $\Omega \subset \mathbb{R}^d$ iff for every Whitney cube P,

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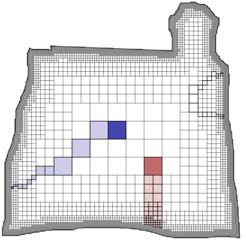

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Assume that $n = 1$ and

$$\mu(x) = |\nabla T\chi_\Omega(x)|^p dx$$

is p-Carleson for Ω. We want

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Idea of the proof of the Key Lemma.

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Idea of the proof of the Key Lemma.

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Break the local part and non-local part.

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Local part is a good function, in $W^{1,p}(\mathbb{R}^d)$.

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Ingredients: bounds for the kernel, Poincaré inequality and Hölder. back

What about $n \geq 2$?

- We need to iterate the Poincaré inequality to get derivatives of higher order. Thus, we approximate f in $3Q$ by polynomials $\mathbf{P}_{3Q}^{n-1} f$ instead of the mean value f_{3Q}. The conditions for those polynomials are

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- When we use the Harnack chain we don't compare numbers but functions evaluated at a certain distance. Thus new polynomially growing terms will appear.

The converse implication holds for $n = 1$.

A duality argument ($n = 1$).

Hypothesis: T bounded in $W^{1,p}(\Omega)$. Then the averaging function

$$\mathcal{A}f(x) := \sum_{Q\in\mathcal{W}} \chi_Q(x)\, f_{3Q},$$

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For $g = \chi_{\mathbf{Sh}(P)}$,

$$\|\mathcal{A}^* g\|^2_{W^{1,2}(\Omega)} \lesssim \|g\|^2_{L^2(\mu)} = \mu(\mathbf{Sh}(P))$$

The Neuman problem ($n = 1$).

To get

$$\sum_{Q \subset \mathbf{Sh}(P)} \mu(\mathbf{Sh}(Q))^2 \lesssim \|\mathcal{A}^* g\|^2_{W^{1,2}(\Omega)} + \text{error terms}$$

we need to estimate $\|\mathcal{A}^* g\|_{W^{1,2}(\Omega)}$ from below.

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Thus, h is the solution of the Neuman problem

$$\begin{cases} -\Delta h = \widetilde{g} & \text{in } \Omega, \\ \partial_\nu h = 0 & \text{in } \partial\Omega. \end{cases}$$

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A geometric condition.

Ingredients for the proof.

Theorem (P., Tolsa 2013)

For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to $\partial\Omega$ is in the Besov space $B^{n-\frac{1}{p}}_{p,p}(\partial\Omega)$ then $\mathcal{B}(\chi_\Omega) \in W^{n,p}(\Omega)$, with

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- Equivalence between Besov $B^s_{p,p}$ norm and a sum of betas (Dorronsoro).
- Beurling of characteristic functions of circles, half-planes, polynomials, ...

See details

Conclusions.

- For $p > d$ we have a $T(P)$ theorem for any Calderon-Zygmund operator of convolution type in any ambient space as long as we have uniform bounds in the derivatives of its kernel.

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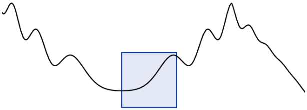
- Next steps:
  - Proving analogous results for any $s \in \mathbb{R}_+$.
  - Looking for a more general set of operators where the Besov condition on the boundary implies Sobolev boundedness.
  - Sharpness of all those results.

The end.

Moltes gràcies!!

Defining some generalized betas of David-Semmes.



A measure of the flatness of a set Γ:

▸ Ending

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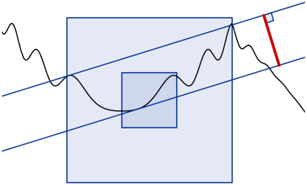

A measure of the flatness of a set Γ:

Definition (P. Jones)

$$\beta_\Gamma(Q) = \inf_V \frac{w(V)}{\ell(Q)}$$

▸ Ending

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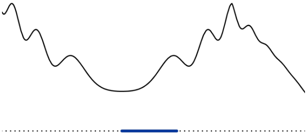
Defining some generalized betas of David-Semmes.



The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

Ending

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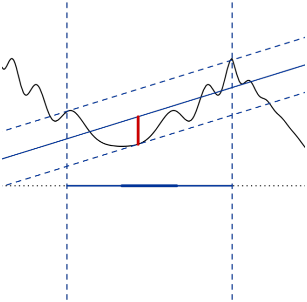

Ending

The graph of a function $y = A(x)$:
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Definition

$$\beta_\infty(I, A) = \inf_{P \in \mathcal{P}^1} \left\| \frac{A-P}{\ell(I)} \right\|_\infty$$

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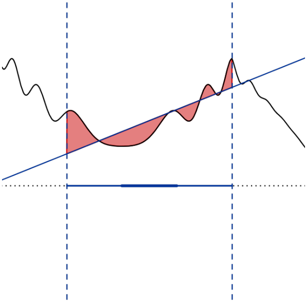

The graph of a function $y = A(x)$:
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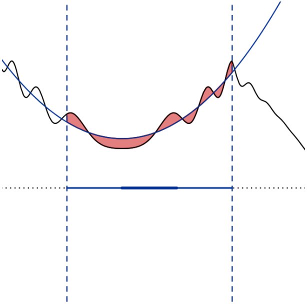
Definition

$$\beta_p(I, A) = \inf_{P \in \mathcal{P}^1} \frac{1}{\ell(I)^{\frac{1}{p}}} \left\| \frac{A-P}{\ell(I)} \right\|_p$$

Ending



The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

Definition

$$\beta_{(n)}(I, A) = \inf_{P \in \mathcal{P}^n} \frac{1}{\ell(I)} \left\| \frac{A-P}{\ell(I)} \right\|_1$$

If there is no risk of confusion, we will write just $\beta_{(n)}(I)$.

Ending

Geometric condition in terms of betas: The Besov space.

Definition

For $0 < s < \infty$, $1 \leq p < \infty$, $f \in B^s_{p,p}(\mathbb{R})$ if

$$\|f\|_{B^s_{p,p}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\Delta_h^{[s]+1} f(x)}{h^s} \right|^p \frac{dm(h)}{|h|} dm(x) \right)^{1/p} < \infty.$$

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Theorem (Dorronsoro)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function in the Besov space $B^s_{p,p}$. Then, for any $n \ge [s]$,

$$\|f\|^p_{B^s_{p,p}} \approx \|f\|_{L^p} + \sum_{I \in \mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{s-1}} \right)^p \ell(I).$$

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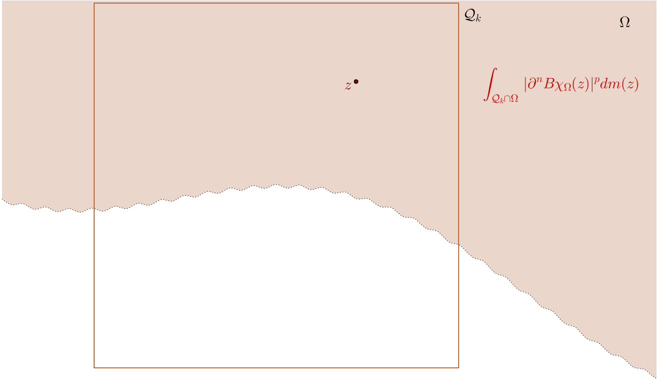
Main idea: projecting cubes to the boundary.



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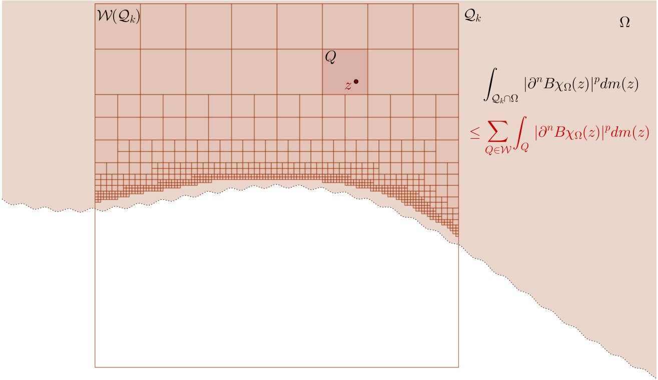

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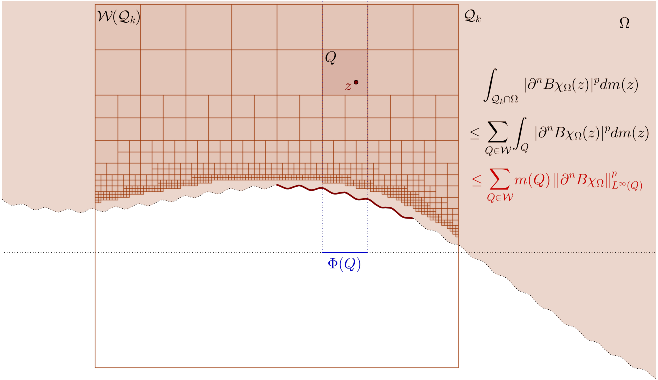

$$\int_{\mathcal{Q}_k\cap\Omega} |\partial^n B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial^n B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} m(Q)\, \|\partial^n B\chi_\Omega\|^p_{L^\infty(Q)}$$

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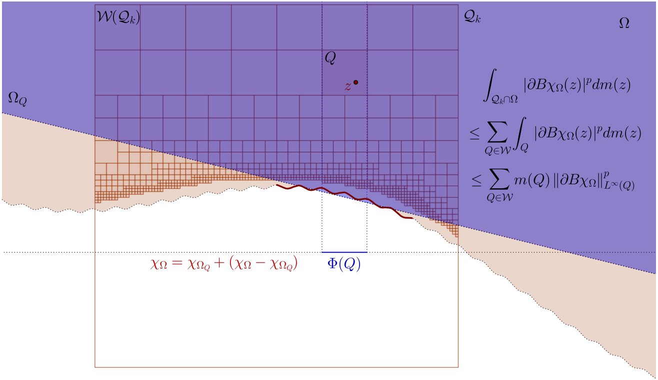

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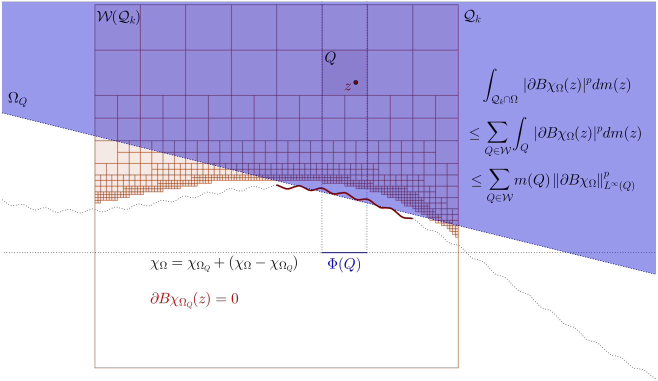

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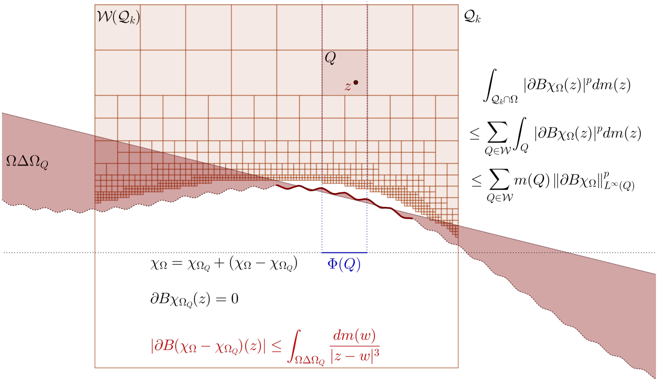

$$\int_{\mathcal{Q}_k\cap\Omega} |\partial B\chi_\Omega(z)|^p dm(z)$$
$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial B\chi_\Omega(z)|^p dm(z)$$
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$$\chi_\Omega = \chi_{\Omega_Q} + (\chi_\Omega - \chi_{\Omega_Q})$$

$$\partial B\chi_{\Omega_Q}(z) = 0$$

$$|\partial B(\chi_\Omega - \chi_{\Omega_Q})(z)| \leq \int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^3}$$

Ending