Singular integral operators on Sobolev spaces on domains and quasiconformal mappings

PHD dissertation, directed by Xavier Tolsa

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October 16th, 2015

Introduction

Measuring smoothness and integrability in $\mathbb{R}^d$

Lebesgue spaces → integrability.

- $\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}$,
  $\|f\|_{L^\infty} = \operatorname{ess\,sup}|f|$

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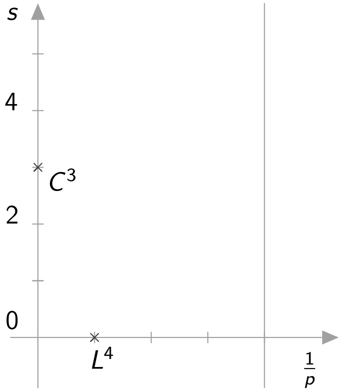

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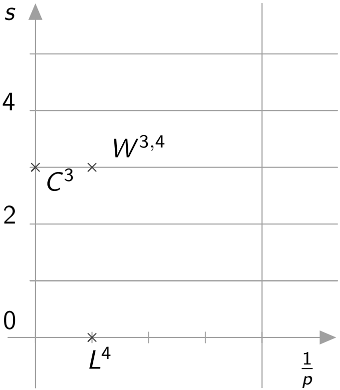

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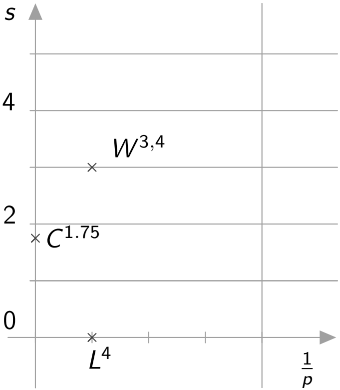

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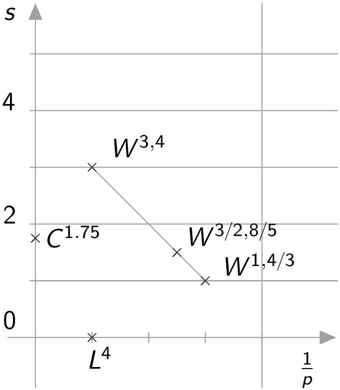

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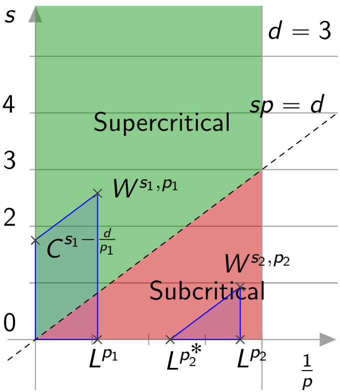

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By means of Sobolev embeddings, we have either continuity or extra integrability.

Singular integral operators

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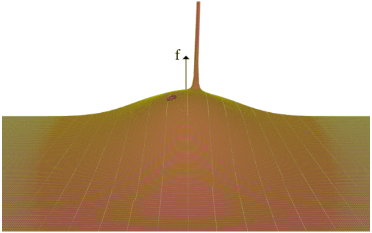
In general a convolution CZO of order s is defined as

$$Tf(x) = \int K(x-y) f(y)\, dm(y)$$

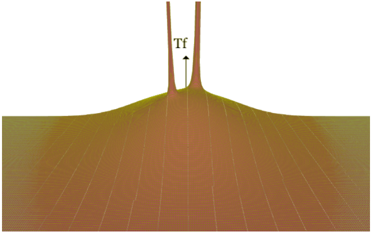
if $x \notin \operatorname{supp}(f) \subset \mathbb{R}^d$, with some cancellation property and some size and smoothness conditions, say

$$|\nabla^j K(x)| \leqslant |x|^{-d-j} \qquad \text{for } j \leqslant s$$

The problem we face



$\xrightarrow{T}$



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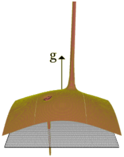
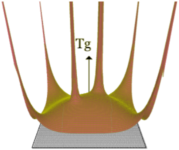


$$\xrightarrow{T_\Omega}$$



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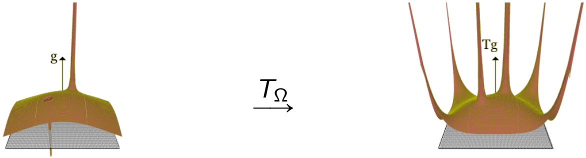

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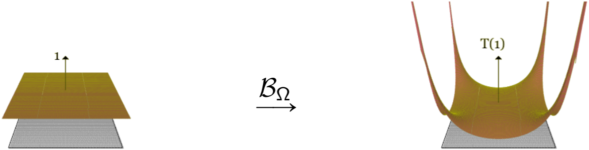

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For Ω a rectangle, $\mathcal{B}\chi_\Omega$ is in every $L^p(\Omega)$ but not in $W^{1,p}(\Omega)$ for $p \geqslant 2$.

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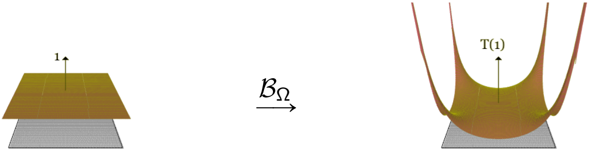

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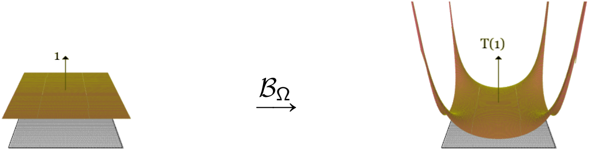

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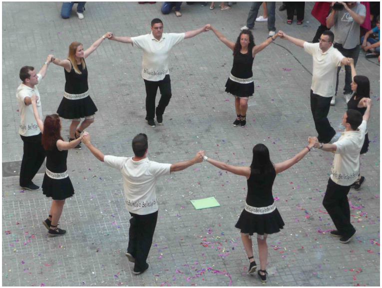
We seek for answers in terms of test functions and in terms of the geometry of the boundary.

Lipschitz domains vs Uniform domains

Lipschitz domains vs Uniform domains



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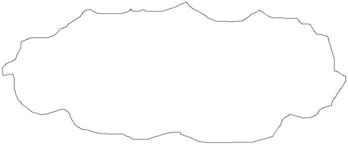

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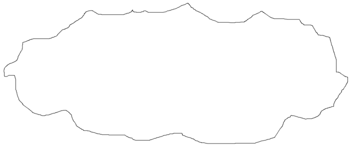

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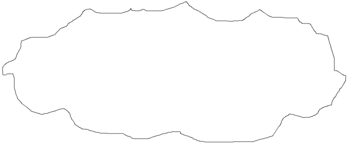

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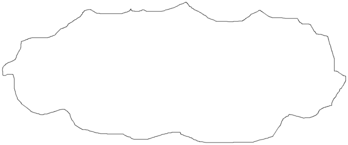

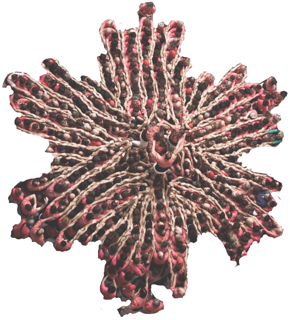

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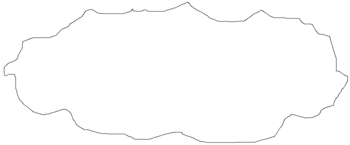

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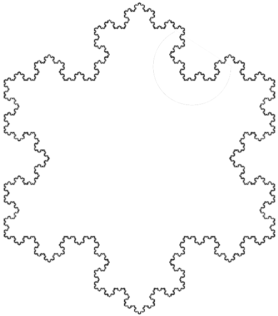

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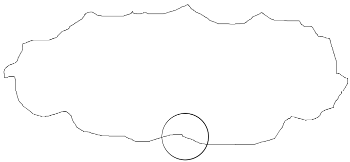
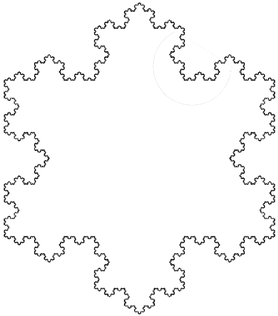


Lipschitz domain
Local parameterizations of $\partial\Omega$.



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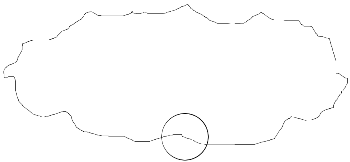

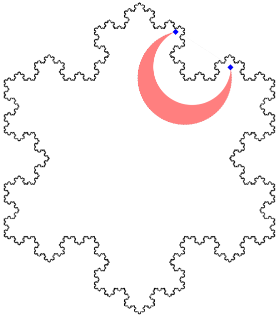

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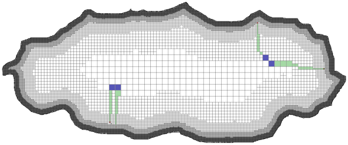
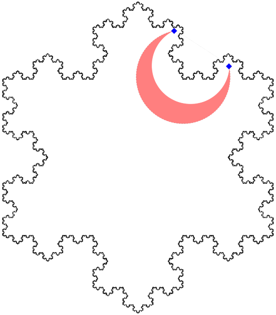
Uniform domain
Cigars joining pairs of points

Lipschitz domains vs Uniform domains





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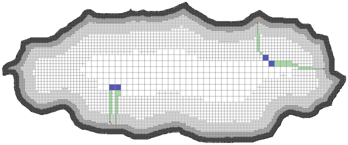

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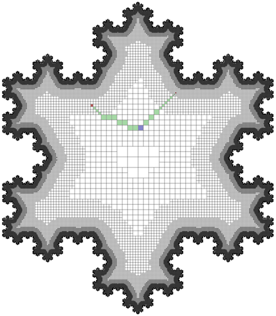

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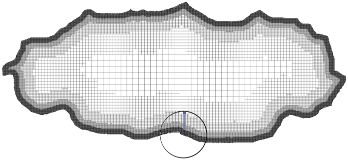
Lipschitz domains vs Uniform domains



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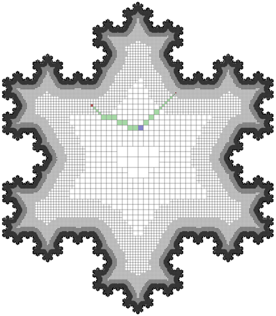
Vertical shadow



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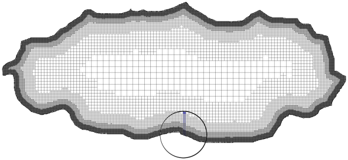

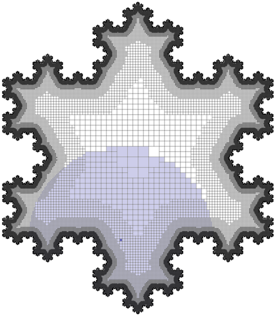

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Uniform domain
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Whitney covering with 'cigar' paths
Spherical shadow

T(P) theorems

Results

Theorem (Cruz, Mateu, Orobitg, 2013)

Given a bdd $C^{1+\epsilon}$ domain $\Omega \subset \mathbb{R}^d$, a convolution CZO T with homogeneous even kernel, $0 < s \leqslant 1$ and $sp > d$.
$T_\Omega(1) \in W^{s,p}(\Omega) \iff T_\Omega$ is bounded in $W^{s,p}(\Omega)$.

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A new approach for the case $s = 1$:

Key Lemma

The following are equivalent:

- $\|\nabla T_\Omega f\|^p_{L^p(\Omega)} \leqslant C\|f\|^p_{W^{1,p}(\Omega)}$.
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back

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For the non-local part, we use a Harnack chain of cubes.

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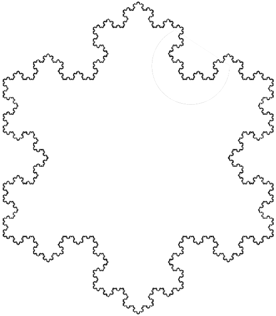
Local part is a good function, in $W^{1,p}(\mathbb{R}^d)$.
For the non-local part, we use a Harnack chain of cubes.
Ingredients: bounds for the kernel, Poincaré inequality and Hölder.

Proof of the T(P) theorem ($p > d$)



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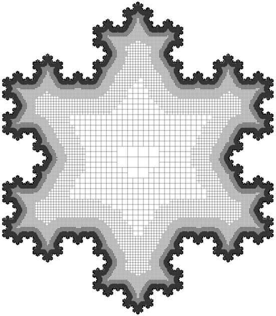

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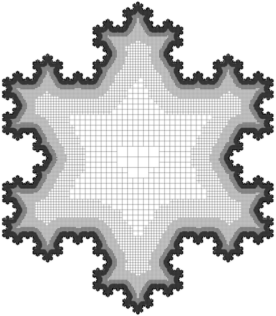

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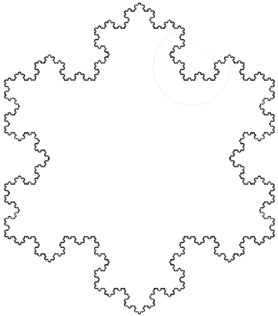

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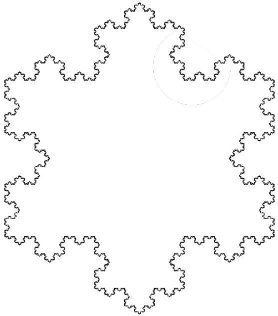

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$$\leqslant \|f\|^p_{L^\infty} \|\nabla T(\chi_\Omega)\|^p_{L^p(\Omega)}$$
$$\leqslant C \|f\|^p_{L^\infty}.$$

Since $p > d$, by the Sobolev Embedding Theorem

$$\|f\|_{L^\infty} \leqslant C \|f\|_{W^{1,p}(\Omega)}.$$

□

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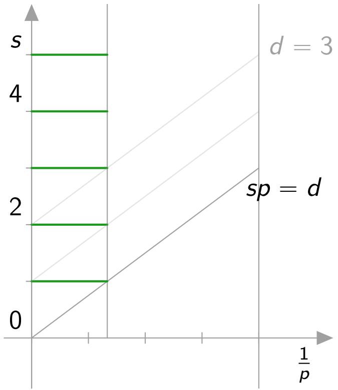
- Some new results (Triebel-Lizorkin norms in terms of differences, extension theorems for that situation, ...) arose to prove this particular result.
- These results have applications to PDE's, in particular quasiconformal mappings, as we will see.

Conclusions



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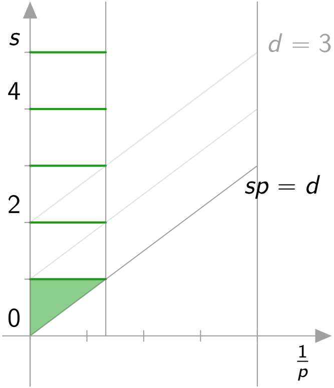

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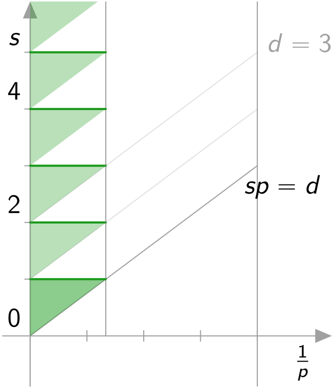

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Expected further results:

- Proving analogous results for $s \in \mathbb{R}$.
- Other characterizations of $W^{s,p}(\Omega)$ may lead to wider range of indices.

The Beurling transform on planar domains

Results

Theorem (P., 2015)

For $\Omega \subset \mathbb{C}$ smooth enough, if the vector normal to the boundary of Ω is in the Besov space $B^{s-\frac{1}{p}}_{p,p}(\partial\Omega)$ with $s \in \mathbb{N}$, $1 < p < \infty$, then $\mathcal{B}(\chi_\Omega) \in W^{s,p}(\Omega)$, and

$$\|\nabla^s \mathcal{B}(\chi_\Omega)\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{s-1/p}_{p,p}(\partial\Omega)}.$$

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V. Cruz and X. Tolsa proved the case $\frac{1}{p} < s \leqslant 1$.
Tolsa proved a converse for $s = 1$ and Ω flat enough.

Ingredients for the proof

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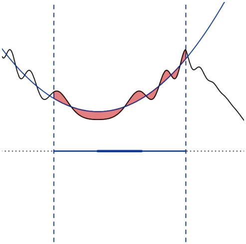

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$$\|\nabla^s \mathcal{B}(\chi_\Omega)\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{s-1/p}_{p,p}(\partial\Omega)}.$$

$$\|f\|^p_{B^{s+1-1/p}_{p,p}} \approx \|f\|^p_{L^p} + \sum_{I\in\mathcal{D}} \left(\frac{\beta_{(s)}(I)}{\ell(I)^s}\right)^p \ell(I)^2$$

Ingredients:

- Generalized Peter Jones' betas (using polynomials instead of lines).
- Equivalence between $B^{s-1/p}_{p,p}$ norm and a sum of betas (Dorronsoro).

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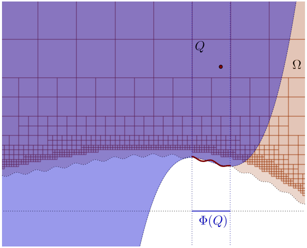
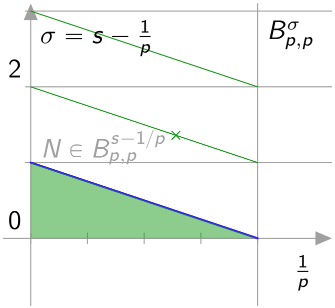
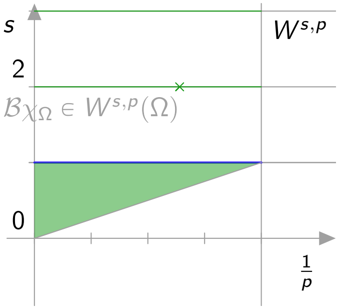


Ingredients:

- Generalized Peter Jones' betas (using polynomials instead of lines).
- Equivalence between $B^{s-1/p}_{p,p}$ norm and a sum of betas (Dorronsoro).
- Beurling of characteristic functions of circles, half-planes, polynomials.

Conclusions





- In the complex plane, the Besov regularity $B^{s-1/p}_{p,p}$ of the vector normal to the boundary of the domain gives us a bound of $\mathcal{B}(\chi_\Omega)$ in $W^{s,p}(\Omega)$ ($s \in \mathbb{N}$ and $\frac{1}{p} < s < 1$).

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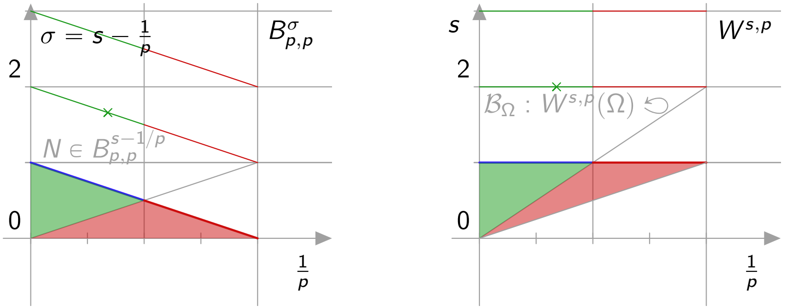

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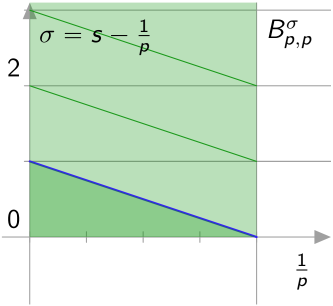
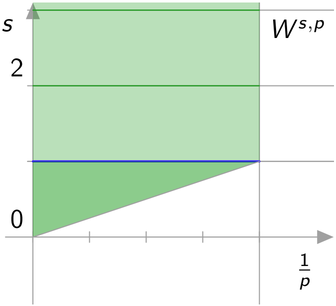
- Combined with the previous results, when $sp > 2$ and $p > 2$ we get that $\mathcal{B}_\Omega$ is bounded in $W^{s,p}(\Omega)$.

Conclusions



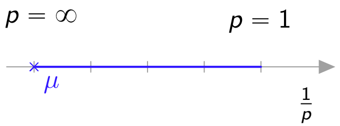


- Expected further results:
  - Proving analogous results for any $s \in \mathbb{R}_+$.
  - Studying higher dimensions.
  - Sharpness of all those results for $s \neq 1$.

Planar quasiconformal mappings

The Beltrami equation

Let $\mu \in L^\infty_c(\mathbb{C})$ with $k := \|\mu\|_\infty < 1$.



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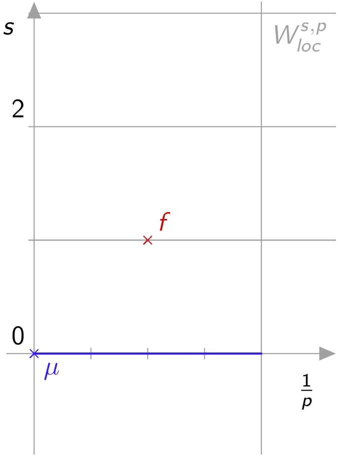

Let $\mu \in L^\infty_c(\mathbb{C})$ with $k := \|\mu\|_\infty < 1$.
The *Beltrami equation*

$$\bar\partial f(z) = \mu(z)\partial f(z)$$

has a unique solution $f \in W^{1,2}_{loc}$ such that $f(z) = z + \mathcal{O}(1/z)$ as $z \to \infty$.

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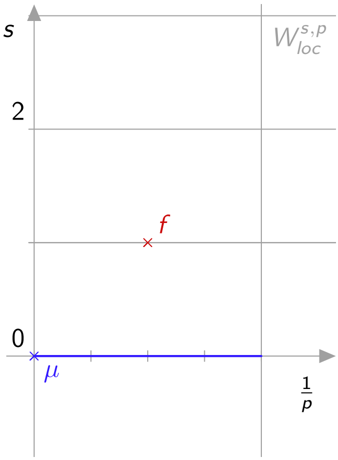

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Consider

$h := \mu + \mu\mathcal{B}(\mu) + \mu\mathcal{B}(\mu\mathcal{B}(\mu)) + \cdots$

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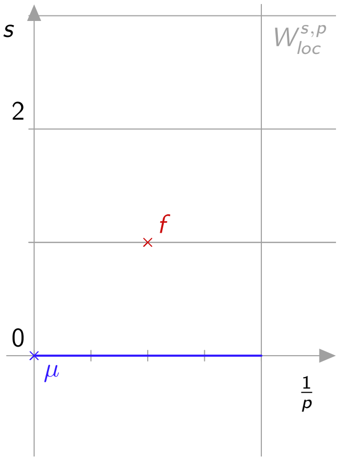

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$= (I - \mu\mathcal{B})^{-1}(\mu),$

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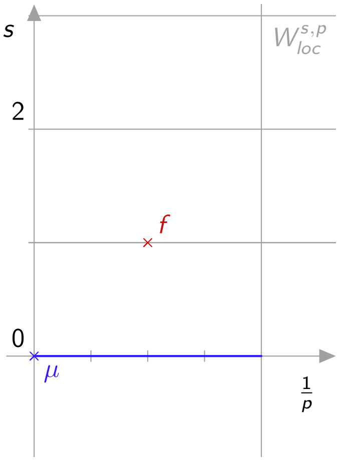

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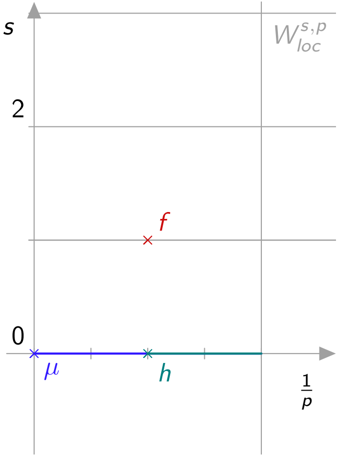

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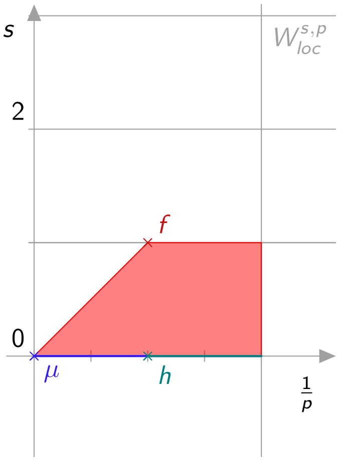

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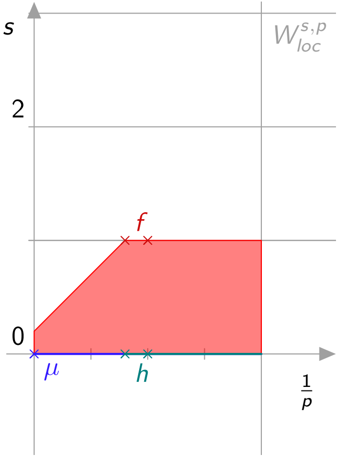

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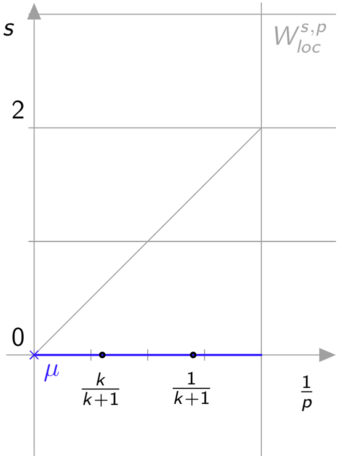
This remains true if $\|\mathcal{B}\|_{(p,p)} < 1/k$.

Results without boundaries



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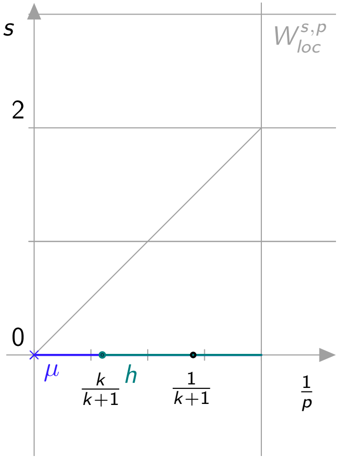

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- $h \in L^p$ for $\frac{k}{k+1} < \frac{1}{p}$ [A92, AIS01].

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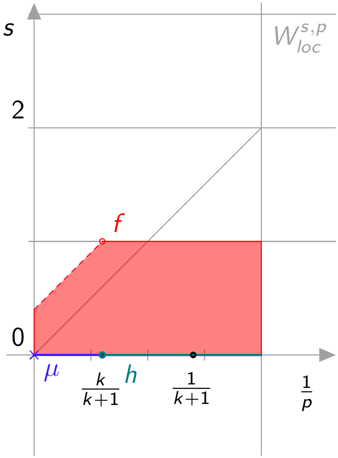

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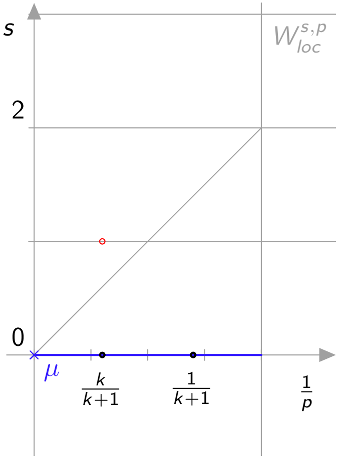

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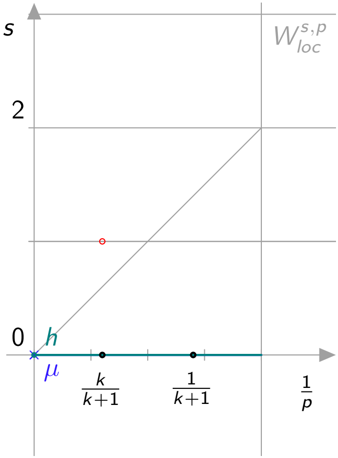

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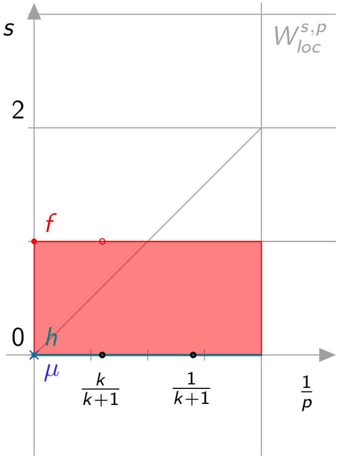

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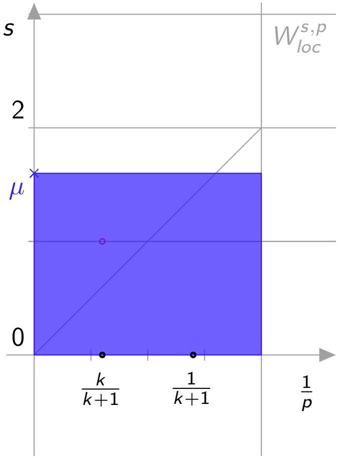

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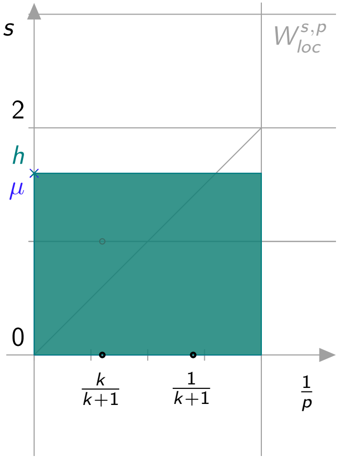

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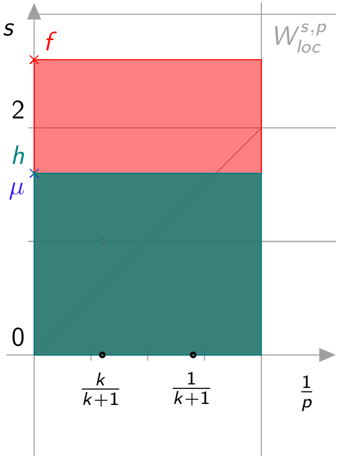

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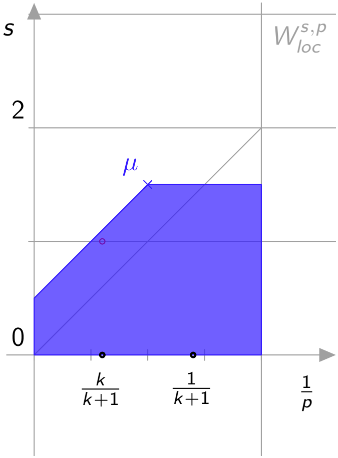

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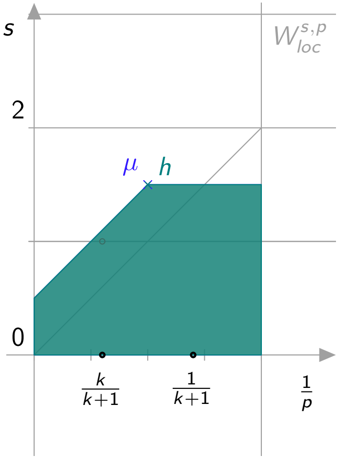

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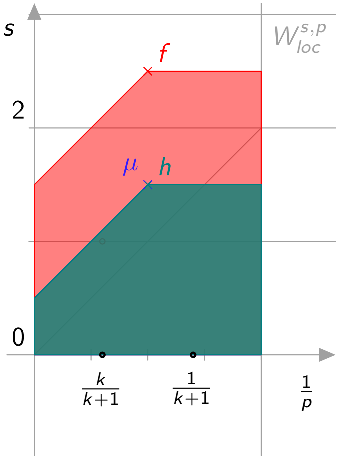

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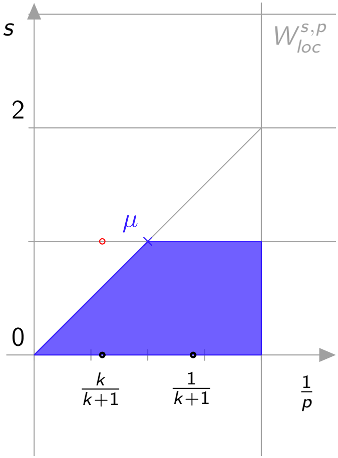

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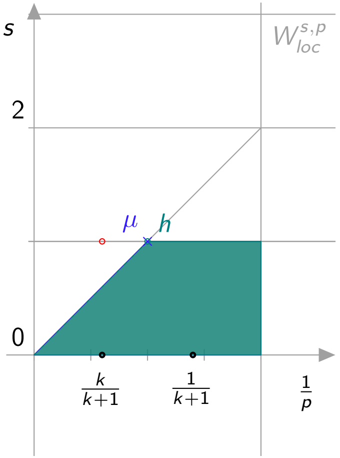

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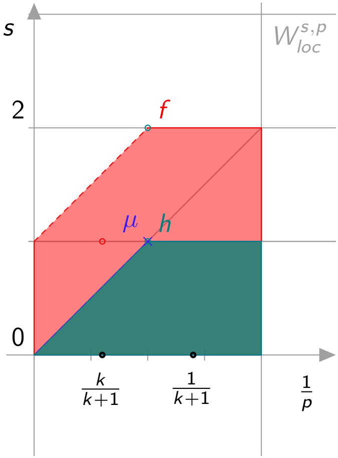

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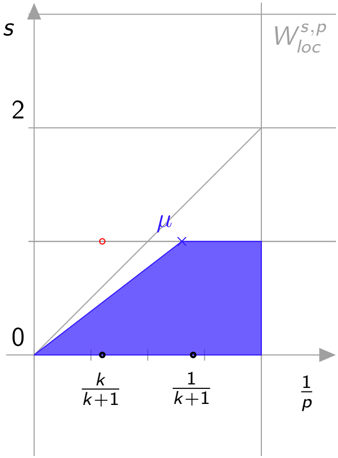

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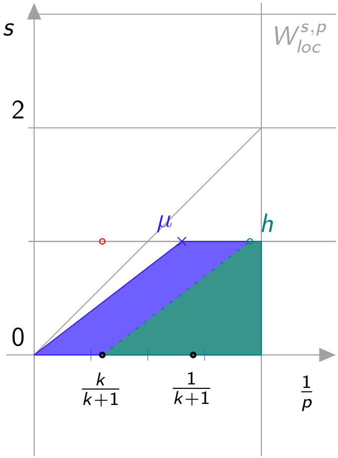

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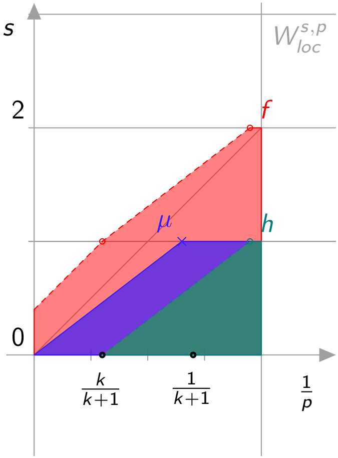

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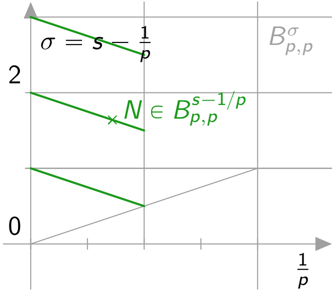

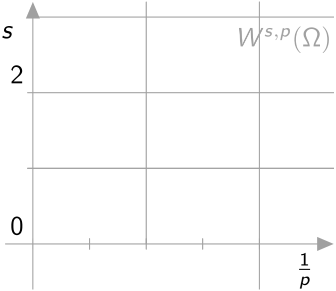

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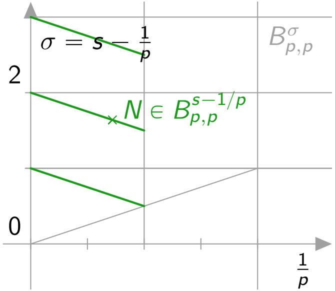

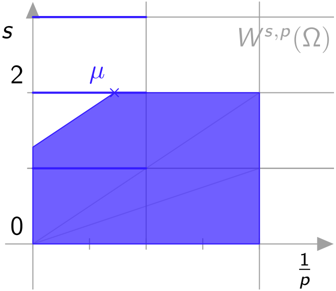

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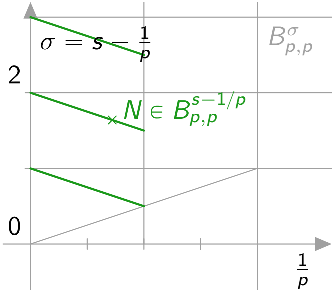

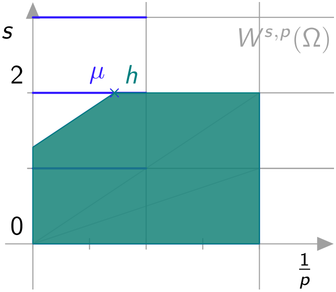

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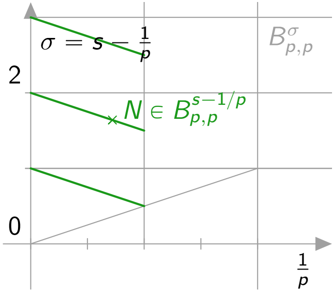

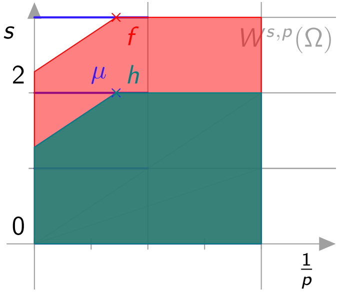

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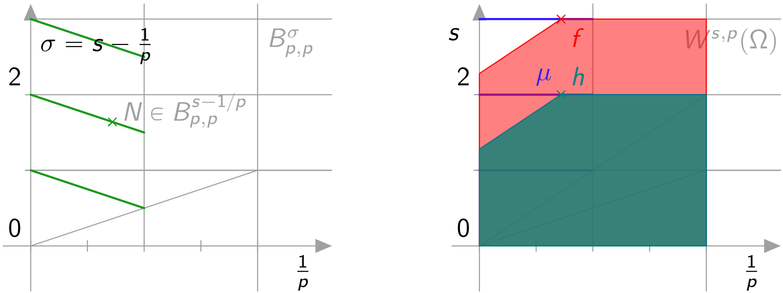
Conclusions



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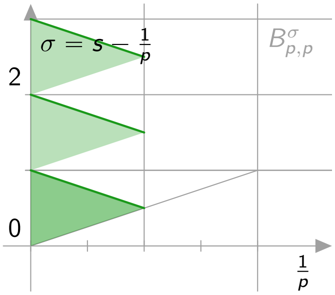

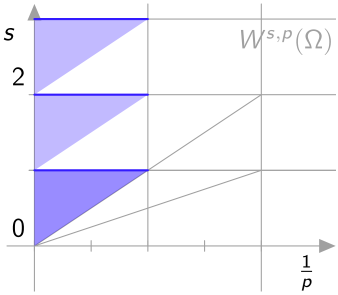

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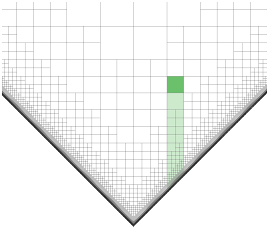
- Expected further results:
  - Proving analogous results for any $s \in \mathbb{R}_+$. $0 < s < 1$, $sp > 2$ seems ready to be done.
  - Subcritical situation: is there any condition on $\partial\Omega$ which can lead to analogous results?

Carleson measures

The Carleson measures



According to [Arcozzi, Rochberg, Sawyer], i.e., Carleson measures for Besov space of analytic functions $B_p(\rho)$,

Definition

We say that ν is p-Carleson for $\Omega \subset \mathbb{R}^d$ iff for every Whitney cube P,

$$\sum_{Q \subset \mathbf{Sh}(P)} \nu(\mathbf{Sh}(Q))^{p'} \ell(Q)^{\frac{p-d}{p-1}} \leqslant C\nu(\mathbf{Sh}(P)).$$

Results

Lipschitz domains, $s \in \mathbb{N}$.

Theorem (P., Tolsa, 2014)

Given a domain $\Omega \subset \mathbb{R}^d$ and $p > d$. If $T_\Omega(P) \in W^{s,p}(\Omega)$ for polynomials $P \in \mathcal{P}^{s-1}(\Omega)$, then T_Ω is bounded in $W^{s,p}(\Omega)$.

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If $s = 1$, the converse is true.

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$$\begin{aligned}\sum_{Q \in \mathcal{W}} |f_{3Q}|^p \nu(Q) &\leqslant \sum_{Q \in \mathcal{W}} \left(\sum_{P:\, Q \subset \mathbf{Sh}(P)} |f_{3P} - f_{3\mathcal{N}(P)}| \right)^p \nu(Q) \\ &\leqslant \sum_{Q \in \mathcal{W}} \left(\sum_{P:\, Q \subset \mathbf{Sh}(P)} \|\nabla f\|_{L^p(5P)} \ell(P)^{1-\frac{d}{p}} \right)^p \nu(Q) \\ &\leqslant C \sum_{Q \in \mathcal{W}} \|\nabla f\|_{L^p(5Q)}^p \leqslant C \|f\|_{W^{1,p}(\Omega)}^p\end{aligned}$$

The converse is true for $s = 1$: a duality argument

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Further comments

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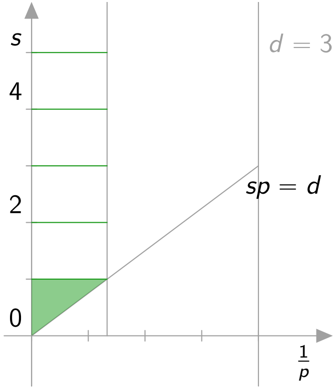
- To avoid some cancellation issues, the Neumann problem is solved in the half-space.
- The necessity of the Carleson condition for $W^{1,p}$ with $p \neq 2$ is shown with the same scheme, but the lack of self-duality makes the proof trickier.
- A sufficient Carleson condition for Triebel-Lizorkin spaces $F^s_{p,q}$ with $0 < s < 1$ and $s > \frac{d}{p} - \frac{d}{q}$ is also obtained in the thesis.

Conclusions



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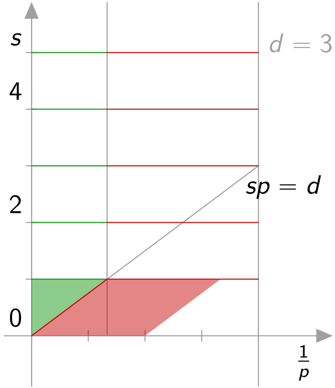

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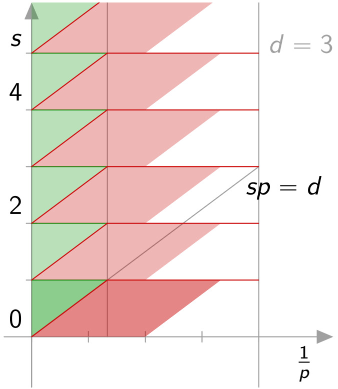

- For $p > d$ and $s \in \mathbb{N}$ or $0 < s < 1$, $sp > d$ we have obtained a $T(P)$ theorem.
- For $p \leqslant d$ it is not enough to have the images of polynomials bounded, but it suffices that they are Carleson measures. When $s = 1$, this yields a complete characterization.
- Expected further results:
    - Proving analogous results for any $s \in \mathbb{R}_+$.
    - Sharpness of all those results.

The end

Moltes gràcies!!

Muchas gracias!!

Kiitos paljon!!