Characterization for stability in planar conductivities

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Introduction

Uniformly strongly elliptic boundary value problems

Let $K \geqslant 1$, $\Omega \subset \mathbb{C}$ bounded domain. We say $\gamma \in \mathcal{G}(K,\Omega)$ when

- Compactly supported: $\mathrm{supp}(\gamma - 1) \subset \overline{\Omega}$.
- Strongly elliptic: $\|\gamma\|_\infty \leqslant K$, $\|\gamma^{-1}\|_\infty \leqslant K$.
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Neumann BVP: prescribed electric current in the boundary, find voltage

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, \\ (\gamma \partial_\nu u)\,|_{\partial\Omega} = g. \end{cases} \tag{2}$$

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Note that we map Dirichlet boundary data to Neumann boundary data of the function. The "forward map"

$$\begin{aligned} \Lambda : \quad \mathcal{G}(K,\Omega) &\to \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right), \\ \gamma &\mapsto \Lambda_\gamma, \end{aligned}$$

is continuous for the distance $\|\gamma_1 - \gamma_2\|_\infty$.

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Calderón's CIP is severely "ill-posed".

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The proof depends on the topology of $\mathbb{C}$. Not useful for higher dimensions (for $\mathbb{R}^n$, with $n \geqslant 3$, see [Caro, Rogers '15], [Haberman'15], ...).

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- Includes all previous results.
- Valid for every bounded domain.
- Yields a characterization for conductivities supported away from the boundary.
- Settles Alessandrini's 2007 conjecture.

Regularization strategy

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Define $\Gamma : \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right) \to \mathcal{G}(K,\Omega)$ so that if $\left\|\tilde{\Lambda} - \Lambda_\gamma\right\| \to 0$ then $\left\|\Gamma(\tilde{\Lambda}) - \gamma\right\| \to 0$.

Moduli of continuity

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$$\left|\langle(\Lambda_{\gamma_0} - \Lambda_{\gamma_j})\varphi, \varphi\rangle\right| = \left|\int_{\Omega} (\gamma_0 - \gamma_j)\nabla u_0 \cdot \nabla u_j\right|$$

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We have continuity of the forward map.
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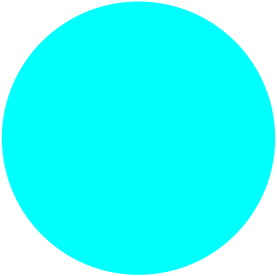
First stability counterexample

Characteristic functions: L^∞ stability fails!

Take a constant conductivity in $\mathbb{C}$.



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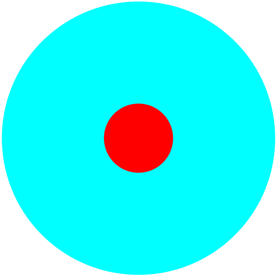

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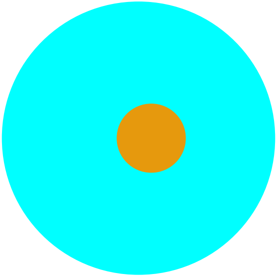

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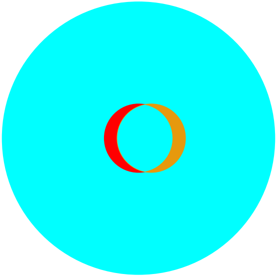

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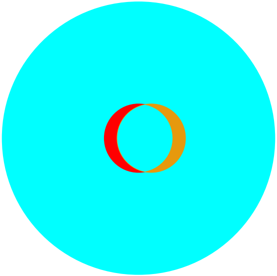

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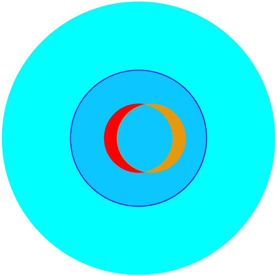

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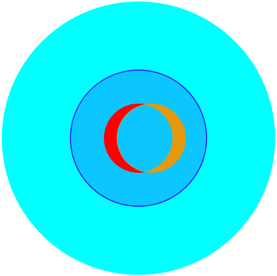

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Thus, $\Lambda_\varepsilon \to \Lambda_0$, but $\gamma_\varepsilon \nrightarrow \gamma_0$. We must seek L^p stability.

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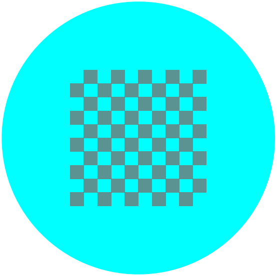

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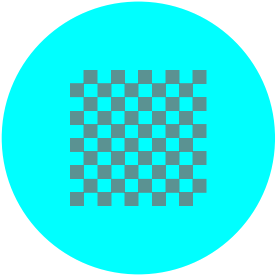

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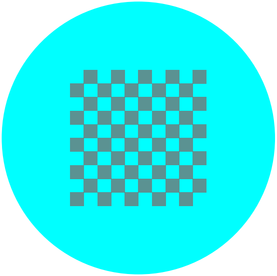

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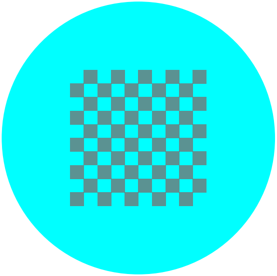

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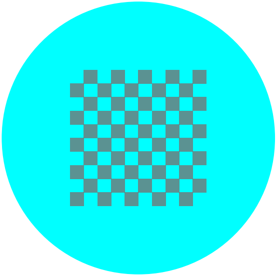
But $\{\gamma_j\}$ has no partial L^p-convergent!!

Second stability counterexample



But take $\gamma_j \in \mathcal{G}(2, \mathbb{D})$ defined by
$\gamma_j(z) = 1 + \frac{1}{2}\chi_{\mathbb{Q}}(z)\chi_{chessboard}(jz)$.
Now, γ_j G-converge to γ [Alessandrini, Cabib]
As a consequence, the flux u_j weakly converge to u_0. The DtN maps converge as well [Faraco, Kurylev, Ruiz].
But $\{\gamma_j\}$ has no partial L^p-convergent!!
L^p stability fails in general! Thus, we seek a priori conditions.

Compactness issues

Lemma (Alessandrini'07)

Let $\mathcal{F} \subset\subset \mathcal{G}(K, \tilde{\Omega})$ in the L^p distance, with $\tilde{\Omega} \subset\subset \Omega$. Then, $\mathcal{F}$ is L^p-stable for Ω.

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$\Lambda(\mathcal{G}(K, r_0\mathbb{D}))$ is a pre-compact subset of $\mathcal{L}(H^{1/2}(\partial\mathbb{D}), H^{-1/2}(\partial\mathbb{D}))$.

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Theorem

Let $K \geqslant 1$, let $r_0 < 1$ and let $\mathcal{F} \subset \mathcal{G}(K, r_0\mathbb{D})$. The family $\mathcal{F}$ is L^2-stable for $\mathbb{D}$ if and only if it is pre-compact.

Kolmogorov-Riesz criterion

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$$\omega_p f(t) := \sup_{|y|\leqslant t} \|f - \tau_y f\|_{L^p} \qquad \text{for } 0 \leqslant t \leqslant \infty,$$

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Given a modulus of continuity ω, we say that $u \in B_p^\omega$ if

$$\|u\|_{B_p^\omega} := \|u\|_{L^p} + \sup_{t>0} \frac{\omega_p u(t)}{\omega(t)} < \infty.$$

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Questions

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Problem: Quantify continuity of inverse mapping for any ω.

Our result

Theorem

Let $K \geqslant 1$, let $0 < p < \infty$, let Ω be a bounded domain and let ω be a modulus of continuity. Then the family $\mathcal{G}(K, \Omega, p, \omega)$ is L^2-stable for Ω.

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$$\|\gamma_1 - \gamma_2\|_2 \leqslant C\ \eta\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\mathcal{L}(\partial\Omega)}\right)^{\frac{1}{2}}$$

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$$\eta(\rho) \lesssim_{K,p} (Id + \omega) \left(C_{K,p}\, \omega \left(\frac{C_K}{|\log(\rho)|^{\frac{1}{K}}} \right)^{b_{K,p}} + \frac{C_K}{|\log(\rho)|^{\alpha_K}} \right).$$

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We have gotten every bounded domain and every modulus of continuity. No "compactly supported" condition!! Every conductivity has an integral modulus of continuity.

Tools

Complex Geometric Optics Solution

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Interesting behavior in k: for every z

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Hodge-* conjugation

Dictionary of divergence equation and Beltrami equation:

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We have Lipschitz continuity on the mapping

$$\begin{aligned} \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right) &\to W^{1,p}(\mathbb{D}^c), \\ \Lambda_\gamma &\mapsto M_\mu(\cdot,k) \end{aligned}$$

with $\|M_1(\cdot,k) - M_2(\cdot,k)\|_{W^{1,p}(\mathbb{D}^c)} \lesssim e^{C|k|}\rho$ ([BFR'07])

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Tools: Hilbert transform, principle of the argument.

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Recover the scattering transform

The scattering transform can be computed with the first terms of the Laurentz series of M_μ and $M_{-\mu}$:

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Subexponential behavior in k

The logarithm $\varphi_\mu := \frac{\log(f_\mu)}{ik}$ is a quasiconformal principal mapping of $\mathbb{C}$.

$$\overline{\partial}\varphi_\mu(\cdot,k) = -\frac{\overline{k}}{k}\mu(\cdot)\, e_{-k}(\varphi_\mu(\cdot,k))\,\overline{\partial\varphi_\mu(\cdot,k)}.$$

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We show that $\|\varphi_\mu(\cdot, k) - Id\|_{L^\infty} \leqslant \upsilon(|k|^{-1})$.

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Tools: interaction of modulus of continuity with translation invariant operators and Fourier transform, control of the Neumann series in k, interaction of the modulus of continuity when composing with qc-maps,...

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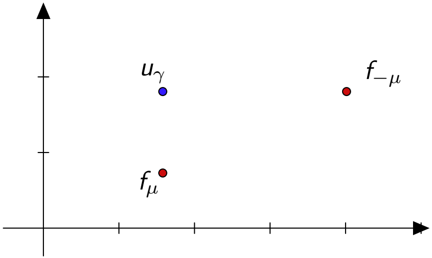
Back to the conductivity



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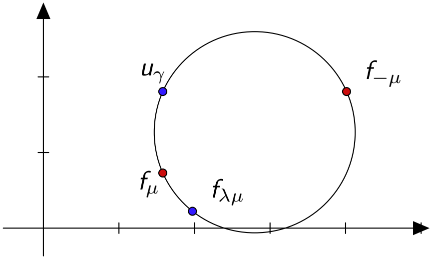

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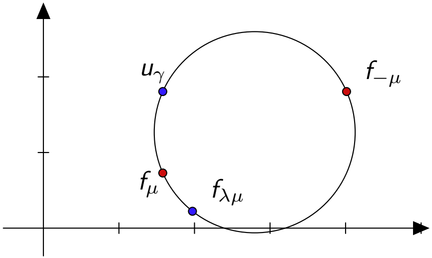

We see that $\log(u_\gamma) = \log(f_{\lambda\mu})$ for a $\lambda : \mathbb{C} \times \mathbb{C} \to \partial\mathbb{D}$ depending on the point. We infer the same asymptotic behavior

$$|\log(u_\gamma)(z,k) - izk| \leqslant |k|\upsilon(|k|^{-1}).$$

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A Cauchy problem

Next we need to solve the Cauchy problem

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Tools: delicate topological argument using both the control on $|\tau_1 - \tau_2|$ and $|\log(u_\gamma)(z,k) - izk|$.

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$\|\Delta u_\gamma\|_{L^\infty(\mathbb{D})} \leqslant \iota(\rho)$

$\Delta\gamma$

Final interpolation

By the preceding ideas, we obtain a control like

$$\|u_1 - u_2\|_\infty \leqslant \iota(\|\Lambda_1 - \Lambda_2\|_{\mathcal{L}}).$$

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Tools: Caccioppoli inequalities for moduli of continuity, interaction of the Fourier transform with the integral moduli.

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Quasiconformal mappings and moduli

Lemma

Let ϕ be K-qc, and let $\mu \in L^\infty_c$. Consider $0 < p \leqslant \infty$ and $\frac{1}{q} > \frac{K}{p}$. For t small enough

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Theorem

Let $\mu \in L^\infty_c$ with $\|\mu\|_{L^\infty} \leqslant \kappa < 1$ and support in $\mathbb{D}$. Let f be a quasiregular solution to

$$\bar{\partial} f = \mu\, \overline{\partial f}.$$

Let $1 < p < p_\kappa$ satisfy that $\kappa \|\mathcal{B}\|_{L^p \to L^p} < 1$, let $r \in [p, p_\kappa)$ and let q be defined by $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, we have that

$$\omega_p(\bar{\partial} f)(t) \lesssim_{\kappa,r,p} \|f\|_{L^r(2\mathbb{D})} \omega_q \mu(t) + \|f\|_{W^{1+p}(2\mathbb{D})} |t|^{1-\frac{2}{p}}.$$

The end

Moltes gràcies!!

Muchas gracias!!