The two-phase problem for harmonic measure in VMO via jump formulas for the Riesz transform

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Introduction

Harmonic measure

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geqslant 2$ be a domain. Consider the Dirichlet problem

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$$u(z) = \int_{\partial\Omega} f \, d\omega^z.$$

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We call ω^z the harmonic measure of Ω with pole z. Different poles give rise to mutually absolutely continuous measures. For this reason z is often neglected.

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- When is $\mathcal{H}^n \approx \omega$?
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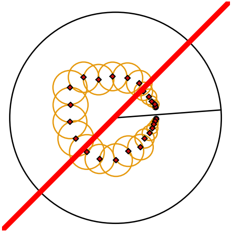
Some answers:

- In the plane, if Ω is simply connected with $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^1 \approx \omega$ (F. and M. Riesz)
- Other results in $\mathbb{C}$ using complex analysis (Carleson, Makarov, Jones, Bishop, Wolff, Garnett,...)
- Analogue of Riesz theorem fails in higher dimensions (Wu, Ziemer)
- Real analysis techniques are needed in $\mathbb{R}^{n+1}$.

NTA domain

- Harnack chain condition:

If $|x-y| \leqslant \Lambda(\mathrm{d}(x,\partial\Omega) \wedge \mathrm{d}(y,\partial\Omega)) \leqslant R$ then $\exists$ a chain $B_1, \ldots, B_m \subset \Omega$, $m \leqslant C(\Lambda)$, with $x \in B_1$, $y \in B_m$, and $\mathrm{d}(B_k, \partial\Omega) \approx \mathrm{diam}(B_k)$.



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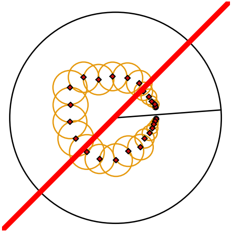
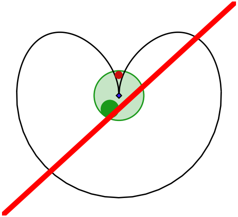
- C-corkscrew domain:

$\forall \xi \in \partial\Omega$ and $r \in (0, R)$ there are two balls of radius r/C contained in $B(\xi, r) \cap \Omega$ and $B(\xi, r) \backslash \Omega$ respectively.





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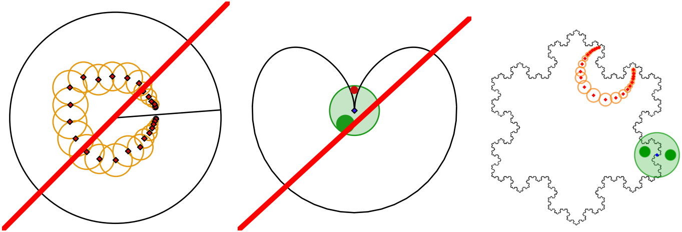

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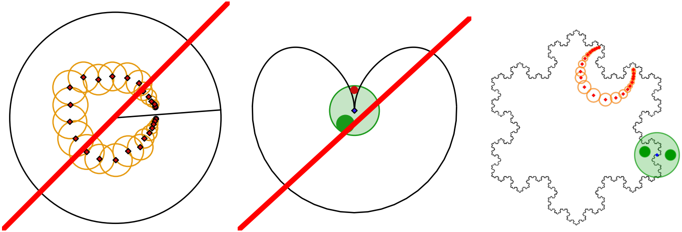

Harmonic measure is doubling in NTA domains, and its support coincides with the whole boundary [Jerison, Kenig'82]

One-sided results

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Theorem (Dahlberg, ARMA'77)

If Ω is a Lipschitz domain, then $\frac{d\omega}{d\sigma} \in RH_2(\sigma)$ and, thus, $\omega \in A_\infty(\sigma)$

Here, the $RH_2(\sigma)$ condition means for balls B centered at $\partial\Omega$

$$\left(⨍_b \left(\frac{d\omega}{d\sigma} \right)^2 d\sigma \right)^{\frac{1}{2}} \leqslant C \frac{\omega(B)}{\sigma(B)}$$

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Recent big break-through: geometric characterization of weak-A_∞, related to Dirichlet solvability [Hofmann, Martell'18]+[Azzam,Mourgoglou,Tolsa'18].

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(a) $\omega^- \in A_\infty(\omega^+)$.

(b) *Either ω^+ or ω^- have very big pieces of uniformly n-rectifiable measures*

c) $\Omega^{\pm}$ *have joint big pieces of chord-arc subdomains*

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Non-quantitative

$(\omega^+|_E \approx \omega^-|_E \implies \exists F \, s.t. \, \omega^+|_F \approx \mathcal{H}^n|_F \,\&\, \omega^\pm(E \setminus F) = 0)$

- Jordan arcs in the plane [Bishop, Carleson, Garnett, Jones'89].
- General domains in the plane [Bishop; Ark. Mat.'91]
- NTA domains in $\mathbb{R}^{n+1}$ [Kenig, Preiss, Toro; JAMS'08]
- CDC domains in $\mathbb{R}^{n+1}$ [Azzam, Mourgoglou, Tolsa; CPAM'17]
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Reifenberg flatness

Given $E \subset \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$, $r > 0$, $B = B(x, r)$ and P an n-plane, we set

$$D_E(x, r, P) = \frac{\sup_{E \cap B} \mathrm{d}(y, P) \vee \sup_{P \cap B} \mathrm{d}(y, E)}{r}.$$

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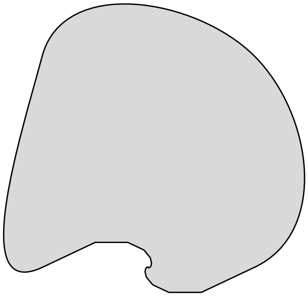

Ω is a (δ, R)-Reifenberg flat domain if:

(a) $\forall x \in \partial\Omega$, $0 < r \leqslant R$ we have $\inf_P D_{\partial\Omega}(x,r,P) < \delta$

(b) $\forall x \in \partial\Omega$, $0 < r \leqslant R$, for the minimizing P, one of the connected components of

$$B \cap \{x \in \mathbb{R}^{n+1} : \mathrm{d}(x,P) \geqslant 2\delta\, r\}$$

is contained in Ω and the other is contained in Ω^c.

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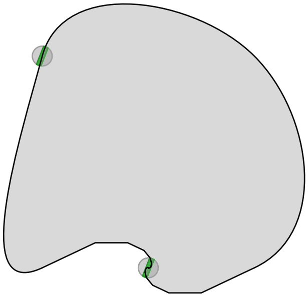

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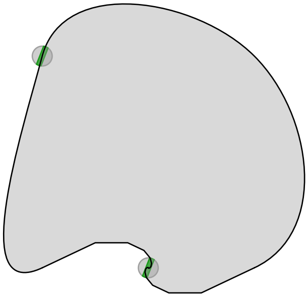

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Small δ implies that Ω is NTA [Kenig, Toro; Duke'97].

Ω is *vanishing Reifenberg flat* if, Ω is a (δ, R_δ)-Reifenberg flat for every $\delta > 0$.

VMO

Given a Radon measure μ in $\mathbb{R}^{n+1}$, $f \in L^1_{loc}(\mu)$, and $A \subset \mathbb{R}^{n+1}$, we write

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Assume μ to be doubling. We say $f \in VMO(\mu)$ if

$$\lim_{r\to 0} \sup_{x\in \operatorname{supp}\mu} ⨍_{B(x,r)} \left| f - m_{\mu,B(x,r)} f\, d\mu \right|^2 d\mu = 0. \tag{1}$$

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It is well known that the space VMO coincides with the closure of the set of bounded uniformly continuous functions on $\operatorname{supp}\mu$ in the BMO norm.

Asymptotic absolute continuity

Given a weight w in a doubling measure space, Korey shows that the following asymptotic weight conditions are equivalent for every $p > 0$

- $\limsup_{\ell(Q)\to 0} \|\log w\|_{*,Q,\mu} = 0$ (BMO norm inside Q wrt μ).
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First condition is $\log w \in VMO(\mu)$. The second can be understood as a "vanishing reverse Hölder space" $w \in VRH_p(\mu)$. Also a vanishing $A_q(\mu)$ condition and some vanishing $A_\infty(\mu)$ conditions are equivalent. The weight w is called asymptotically absolutely continuous by Korey, written $w \in A_{\infty,as}(\mu)$.

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Theorem (Kenig, Toro '97,99,03)

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(a) $\log \frac{d\omega}{d\sigma} \in VMO(\sigma)$. *(i.e. $\omega \in A_{\infty,as}(\sigma)$)*

(b) *The inner normal N to $\partial\Omega$ exists σ-a.e. and it belongs to $VMO(\sigma)$.*

(c) *Ω is vanishing Reifenberg flat and the inner normal N to $\partial\Omega$ exists σ-a.e. and it belongs to $VMO(\sigma)$.*

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where N_B is interior normal to the plane L from RF property.

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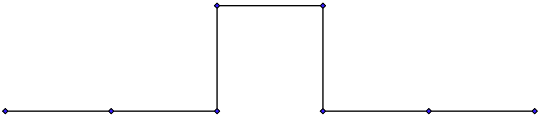
This does not imply $N \in VMO(\omega^+)$:



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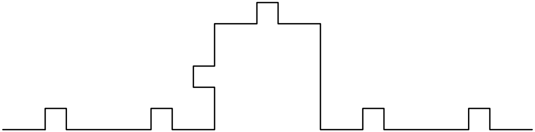

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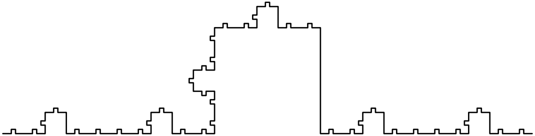
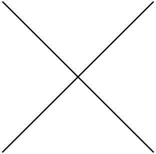


Here N_B is "vertical" for all the balls whose diameter is a horizontal segment of an iteration, while the harmonic measure is concentrated in vertical lines so $⨍_B N d\omega^+ \equiv (1,0)$ and $|N_B - ⨍_B N d\omega^+| \approx \sqrt{(}2)$.

Reifenberg flatness is necessary



The Reifenberg flatness condition on the domain is necessary in the theorem. This can be easily seen by taking a suitable smooth truncation of the cone $\Omega^+ = \{x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 < x_3^2 + x_4^2\}$, for which the harmonic measures ω^+ and ω^- with pole at ∞ coincide:

$\log \dfrac{d\omega^-}{d\omega^+} \in VMO(\omega^+)$, but $N \notin VMO(\omega^+)$!

Preliminaries

Some notation

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Being n-rectifiable means that it is $\mathcal{H}^n$-a.e. contained in a countable union of C^1 n-dimensional manifolds.

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A recent work by Tolsa in arXiv provides jump formulas for n-rectifiable sets. In our setting, we get the following:

Lemma

For ω-a.e. x we have that

$$\mathcal{R}^+\omega^+(x) - \mathcal{R}^-\omega^+(x) = c_n\Theta(x)N(x)$$
$$\mathcal{R}^+\omega^+(x) + \mathcal{R}^-\omega^+(x) = 2p.v.\mathcal{R}\omega^+(x) =: 2\mathcal{R}\omega^+(x)$$

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Uniform n-rectifiable means that Γ is n-AD regular and there are $M, \theta > 0$ so that for all $x \in E$, $0 < r < \mathrm{diam}(\Gamma)$, $\exists g : B = B_r^{\mathbb{R}^n} \to \mathbb{R}^d$ M-Lipschitz with

$$\mathcal{H}^n(\Gamma \cap g(B) \cap B(x,r)) \geqslant \theta r^n$$

Proof of (a) implies (b)

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If ε goes to zero uniformly on $\ell(Q)$ then $N \in VMO(\omega)$ and we are done.

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- "Riesz transform" does not oscillate much in the good set $⨍_{Q \cap G} |\Theta N - m_Q(\Theta N)|^2 d\omega \leqslant \varepsilon_3 \Theta(Q)^2$

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Proof by induction: $\tau = \lambda^M$, $0 < \lambda < 1$ and $M(\tau) >> 1$. Then, writing $\mathcal{LD}^k := \mathcal{LD}_{\lambda^k}$, $LD^k := LD_{\lambda^k}$, we prove

Lemma

Let $\lambda(n)$ *be small.* $\exists\eta \in (0,1)$ *st* $\forall k \geqslant 0$, *if* $P \in \mathcal{LD}_k$, *then* $\omega(P \cap LD^{k+1}) \leqslant \eta\omega(P)$.

Thus, $\varepsilon_2 = \eta^M$.

Proof of the claim

Theorem (Girela-Sarrión, Tolsa)

$\forall C_0, C_1 > 1$, $\exists \delta_0, \tau_0, \theta$ st. given a ball $B \subset \mathbb{R}^{n+1}$ satisfying

a) $\inf_{L \ni 0} \fint_B \frac{dist(x,L)}{r(B)} d\omega \leqslant \delta_0$.

b) $P(B) := \sum_j 2^{-j} \Theta(2^j B) \leqslant C_0 \Theta(B)$.

c) *There exists a good set with* $\omega(B \setminus G_B) \leqslant \delta_0 \omega(B)$, *with*

d) $\mathcal{M}_n(\chi_{2B}\omega) + \mathcal{R}_*(\chi_{2B}\omega) \leqslant C_1 \Theta(B)$ *in* G_B *and*

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Our assumption is that $\log h \in VMO$ for $h := \frac{d\omega^+}{d\omega^-}$, that is, the oscillation of $\log h$ vanishes uniformly as $\ell(Q) \to 0$.

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For better estimates, we consider a slightly modified set $\widetilde{G}_Q$ so that, if $\ell(Q) \leqslant \ell_1(\delta_1, VMO)$ then

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- $\Theta_\pm(B(x,r)) \lesssim \Theta_\pm(Q)$, so $\mathcal{M}_n(\chi_Q \omega)(x) \lesssim \Theta(Q)$.

Control of maximal operators

By the definition of harmonic measure, we have that

$$\begin{aligned} \text{for } x \in \chi_{\Omega^-}, \quad & \mathcal{R}\omega^+(x) = K(x - p^+) \\ \text{for } x \in \chi_{\Omega^+}, \quad & \mathcal{R}\omega^-(x) = K(x - p^-) \end{aligned}$$

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(this shows that condition (d) in [GS] is satisfied). By T(b)-theorem of Nazarov, Trail and Volberg, this implies that

$$\|\mathcal{R}_\omega\|_{L^2(\omega|_{\tilde{G}_Q})\circlearrowleft} \lesssim \Theta(Q)$$

and also weak-(1,1) boundedness

$$\mathcal{R} : \{\text{finite Radon measures in } \mathbb{R}^{n+1}\} \to L^{1,\infty}(\omega).$$

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Thus, to control the oscillation of ΘN in the main proof and the oscillation of $\mathcal{R}\omega$ in the nondegeneracy, it is enough to control oscillation of $\mathcal{R}^+\omega$.

Oscillation of $\mathcal{R}^+\omega$

We define in the main argument $G := \widetilde{G}_{\Lambda Q}$ for Λ big enough, while in the nondegeneracy argument we choose $G_B := \widetilde{G}_{\Lambda P}$.

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$\forall \varepsilon'$, if $\Lambda = \Lambda(\varepsilon')$ is big enough and $\delta_1(\varepsilon', \Lambda)$ small enough, whenever $\ell(Q) \leqslant \ell_2(\delta_1, \Lambda, \varepsilon')$ we have that

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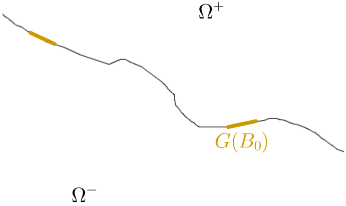
The proof is obtained by combining jump formulas, the weak boundedness of the Riesz transform, the pointwise control of the maximal operators, and the estimates on the good set introduced before. The scaling parameter Λ is used to separate the local part from the non-local part, and CZ "off-diagonal" estimates appear which are small for Λ big.

Proof of (b) implies (a)

Approximating domains

(from [Azzam,Mourgoglou,Tolsa IMRN'17])

- Consider $\Omega^{\pm}$ and the good set $G \subset \partial\Omega^{+}$ to be defined.



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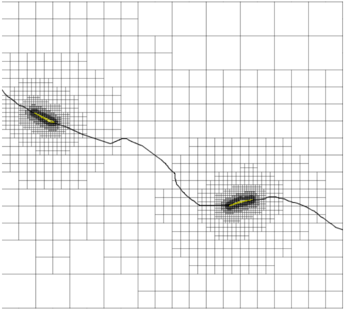

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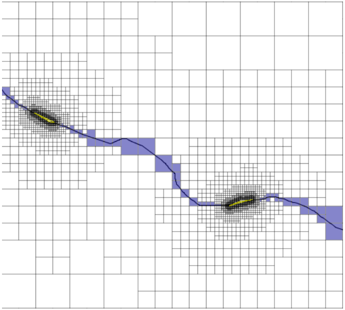

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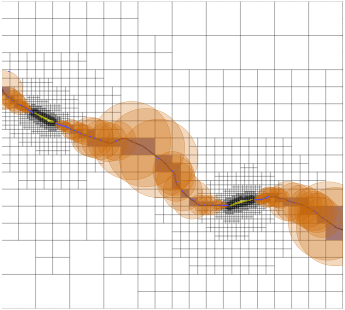

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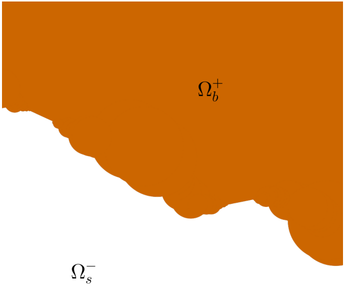

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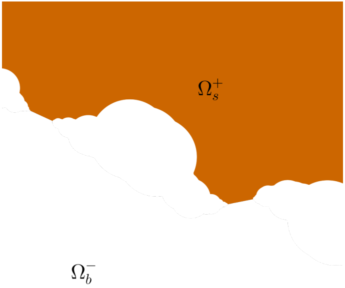

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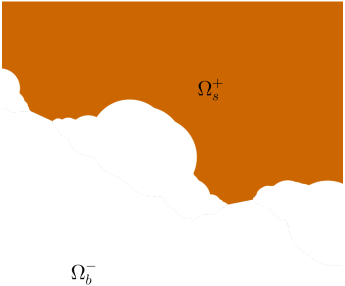

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If Ω^+ is (δ_1, r_0)-RF with $\delta_1(\delta_0)$ small enough, then $\Omega^\pm_{b/s}$ are also $(c\delta_0^{\frac{1}{2}}, r_0/2)$-RF and $\partial\Omega \cap 10B_S$ is a Lipschitz graph.

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$$\boxed{\text{II}}^{\frac{1}{p}} := \left(\fint_B \left(\frac{d\omega^-}{d\omega^+} \right)^p d\omega^+ \right)^{\frac{1}{p}} \leqslant (1+\varepsilon) \frac{\omega^-(B)}{\omega^+(B)}.$$

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Select a good set $G_0 = G(B) = B \setminus (LD_\tau(\Lambda B) \cup HD_A(\Lambda B))$.

$$\boxed{\text{II}} \leqslant \int_{B \cap G_0} \left(\frac{d\omega^-}{d\omega^+} \right)^p d\omega^+ + \int_{B \setminus G_0} \left(\frac{d\omega^-}{d\omega^+} \right)^p d\omega^+$$

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Then by the maximum principle,

we get

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$$\leqslant \int_{B \cap G_0} \left(\frac{d\omega_b^-}{d\omega_s^+} \right)^{1+\beta} d\omega_s^+ + C \left(\frac{\omega^+(B \backslash G_0)}{\omega(B_0)} \right)^{\frac{\beta}{1+2\beta}} \left(\frac{\omega^-(B)}{\omega^+(B)} \right)^{1+\beta}$$

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$$\begin{aligned}
\boxed{\text{II}} &\leqslant \int_{B \cap G_0} \left(\frac{d\omega^-}{d\omega^+} \right)^p d\omega^+ + \int_{B \backslash G_0} \left(\frac{d\omega^-}{d\omega^+} \right)^p d\omega^+ \\
&\leqslant \int_{B \cap G_0} \left(\frac{d\omega_b^-}{d\omega_s^+} \right)^{1+\beta} d\omega_s^+ + C \left(\frac{\omega^+(B \backslash G_0)}{\omega(B_0)} \right)^{\frac{\beta}{1+2\beta}} \left(\frac{\omega^-(B)}{\omega^+(B)} \right)^{1+\beta} \\
&\leqslant \int_{B \cap G_0} \left(\frac{d\omega_b^-}{d\omega_s^+} \right)^{1+\beta} d\omega_s^+ + C(\Lambda)(\varepsilon')^{\frac{\beta}{1+2\beta}} \left(\frac{\omega^-(B)}{\omega^+(B)} \right)^{1+\beta}
\end{aligned}$$

Reverse Hölder for the approximate domains' measures

The key identity to prove is the following:

$$\left\| N_{\Omega_s^+} \right\|_{*,\Lambda r(B),\sigma} \lesssim C(A,\tau) \| N_{\Omega^+} \|_{*,10\Lambda r(B),\omega^+} + \varepsilon_1(\delta_0) = \varepsilon_2.$$

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$$⨍_B \left(\frac{d\omega_b^-}{d\omega_s^+}\right)^2 d\omega_s^+ = \frac{1}{\omega_s^+(B)} \int_B \left(\frac{d\omega_b^-}{d\sigma}\right)^2 \frac{d\sigma}{d\omega_s^+}\, d\sigma$$

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$$⨍_B \left(\frac{d\omega_b^-}{d\omega_s^+}\right)^2 d\omega_s^+ \leqslant \frac{1}{\omega_s^+(B)} \left(\int_B \left(\frac{d\omega_b^-}{d\sigma}\right)^4 d\sigma \int_B \left(\frac{d\sigma}{d\omega_s^+}\right)^2 d\sigma \right)^{1/2}$$

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$$\begin{aligned} ⨍_B \left(\frac{d\omega_b^-}{d\omega_s^+}\right)^2 d\omega_s^+ &\leqslant \left(\frac{\sigma(B)}{\omega_s^+(B)} ⨍_B \left(\frac{d\omega_b^-}{d\sigma}\right)^4 d\sigma ⨍_B \left(\frac{d\sigma}{d\omega_s^+}\right)^3 d\omega_s^+ \right)^{1/2} \\ &\leqslant (1+\varepsilon_3) \frac{\sigma(B)^{1/2}}{\omega_s^+(B)^{1/2}} \left(\frac{\omega_b^-(B)}{\sigma(B)}\right)^2 \left(\frac{\sigma(B)}{\omega_s^+(B)}\right)^{3/2} \end{aligned}$$

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$$\begin{aligned} \fint_B \left(\frac{d\omega_b^-}{d\omega_s^+} \right)^2 d\omega_s^+ &\leqslant \left(\frac{\sigma(B)}{\omega_s^+(B)} \fint_B \left(\frac{d\omega_b^-}{d\sigma} \right)^4 d\sigma \fint_B \left(\frac{d\sigma}{d\omega_s^+} \right)^3 d\omega_s^+ \right)^{1/2} \\ &\leqslant (1+\varepsilon_3) \frac{\sigma(B)^{1/2}}{\omega_s^+(B)^{1/2}} \left(\frac{\omega_b^-(B)}{\sigma(B)} \right)^2 \left(\frac{\sigma(B)}{\omega_s^+(B)} \right)^{3/2} \\ &= (1+\varepsilon_4) \left(\frac{\omega_b^-(B)}{\omega_s^+(B)} \right)^2 . \end{aligned}$$

End of the proof

The last RH inequality together with the previous reasoning implies

$$\boxed{\text{II}} \leqslant \left((1+\varepsilon_4)^{\frac{p}{2}} \left(\frac{\omega^+(B)}{\omega_s^+(B)} \right)^{p-1} \left(\frac{\omega_b^-(B)}{\omega^-(B)} \right)^{p} + C_\Lambda \left(\varepsilon'\right)^{\frac{p-1}{2p-1}} \right) \left(\frac{\omega^-(B)}{\omega^+(B)} \right)^{p}.$$

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Finally, we see that $\omega^+(B) \leqslant (1+\varepsilon_4)\omega_s^+(B)$ and $\omega_b^-(B) \leqslant (1+\varepsilon_4)\omega^-(B)$ for $r(\Lambda B)$ small enough, Λ big enough and δ_0 small enough, using the Hölder continuity of harmonic measure and the separation between B and $(\Lambda B)^c$.

The key estimate

The key estimate remaining

$$\left\| N_{\Omega_s^+} \right\|_{*,\Lambda r(B),\sigma} \lesssim C(A,\tau) \| N_{\Omega^+} \|_{*,10\Lambda r(B),\omega^+} + \varepsilon_1(\delta_0)$$

is deduced from

Lemma

Let $\Omega^+ \subset \mathbb{R}^{n+1}$ *be bdd two-sided NTA* (δ_0, r_0)*-Reifenberg flat for some* $\delta_0 > 0$ *and* $r_0 > 0$*. Suppose also that* $\omega^+ \in RH_{3/2}(\omega^-)$ *and that* $N \in VMO(\omega^+)$*. Let* B *be a ball centered in* $\partial\Omega^+$ *with* $\Lambda_0 r(B) \leqslant r_0/4$*. Let* L_B *be a best approximating n-plane for* $\partial\Omega^+ \cap B$ *and* N_B *the unit normal to* L_B *pointing to* Ω^+*. For any* $\varepsilon_1 > 0$,

$$\left| N_B - m_{B,\omega^+} N_{\Omega^+} \right| \leqslant \varepsilon_1 = \varepsilon_1(\delta_0, r(B)),$$

with ε_1 *as small as wished if* δ_0 *is small enough and* $r(B)$ *small enough,*

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To get

$$\left|N_B - m_{B,\omega^+} N_{\Omega^+}\right| \leqslant \varepsilon_1 = \varepsilon_1(\delta_0, r(B)),$$

we show the estimate

$$\left|\int_{G(\Lambda B)} \Theta N_{\Omega^+}\, d\omega - \frac{C_n}{r(B)^n} N_B\right| \leqslant \frac{\varepsilon_0}{r(B)^n},$$

if the constants are big/small enough. Then we argue as in the implication (a) $\Longrightarrow$ (b) with $\left|\frac{u}{|u|} - \frac{v}{|v|}\right| \leqslant 2\frac{|u-v|}{|u|}$.

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The estimate is obtained using again:

- Jump formulas [Tolsa; arXiv '18]
- Hölder continuity of the harmonic measure and
- change of pole formulas from [Jerison, Kenig; Adv. Math.'82]
- Monotonicity formula [Alt, Caffarelli, Friedmann; TAMS'84]
- Refined doubling properties of ω in [Kenig, Toro; Duke'97]
- Hypothesis $\omega^+ \in B_{3/2}(\omega^-)$ is needed in this proof.

The end

Kiitos paljon! Tack! Moltes gràcies!