The Beurling transform in Sobolev spaces of a Lipschitz domain

Martí Prats *

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1 1st session

1.1 Review of the Beurling transform

The *Beurling transform* is defined as the principal value

$$B\varphi(z) := -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|w-z|>\varepsilon}\frac{\varphi(w)}{(z-w)^2}dm(w), \tag{1.1}$$

for $\varphi \in C_0^\infty(\mathbb{C})$. Notice that the principal value is necessary since the integral is not absolutely convergent, and it exists pointwise by Green's formula.

Lemma 1.1 (Properties of the Beurling transform)**.** *One has*

- *$B\varphi = \partial C\varphi$ for $\varphi \in C_0^\infty$.*
- *$B(\overline{\partial} f) = \partial f$ for $f \in L^1_{loc} \cap \dot{W}^{1,2}$.*
- *$B(f) = \mathcal{F}^{-1}\left(\frac{\bar\zeta}{\zeta}\hat{f}\right)$.*
- *B extends as an isometry in L^2, ($B_2 := \|B\|_{L^2\to L^2} = 1$) and the principal value is well defined.*
- *B is a bounded operator in L^p with norm B_p for $1 < p < \infty$ and the principal value is well defined.*
- *The map $p \mapsto B_p$ is Lipschitz continuous.*

Partial proof. By Green's formula,

$$B\varphi(z) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|w-z|>\varepsilon}\frac{\partial\varphi(w)}{(z-w)}dm(w) + \frac{i}{2\pi}\lim_{\varepsilon\to 0}\int_{|w-z|=\varepsilon}\frac{\varphi(w)}{(z-w)}dm(w) = C\partial\varphi(z) = \partial C\varphi(z).$$

From the Cauchy formula, being φ of compact support we have that in C_0^∞, $C \circ \overline{\partial} = I$ and, thus,

$$B(\overline{\partial}\varphi) = \partial\varphi.$$

*Departament de Matemàtiques, Universitat Autònoma de Barcelona, Catalonia: mprats@mat.uab.cat.

By integration by parts one can see that $\left\|\bar{\partial}\varphi\right\|_{L^2} = \|\partial\varphi\|_{L^2}$ and, togehter with some approximation methods one sees that

$$\|B(\varphi)\|_{L^2} = \|\varphi\|_{L^2}$$

for all $\varphi \in C_0^\infty$.

By density again one can define the Beurling transform acting on L^2. That is, the Beurling transform is an isometry in L^2. □

1.2 Relation with quasiconformal mappings: Solving the Beltrami equation

Consider $0 \leqslant k < 1$ and a measurable and compactly supported function μ such that $\|\mu\| = k$ for $z \in \mathbb{C}$ (we call it *Beltrami coeficient). The* Beltrami equation

$$\bar{\partial} f(z) = \mu(z)\partial f(z)$$

has a unique solution $f \in W^{1,2}_{loc}$ such that

$$f(z) = z + \mathcal{O}(1/z) \quad \text{as } z \to \infty.$$

The solution will be of the following form. Consider

$$h := (I - \mu B)^{-1}(\mu),$$

where we consider the mapping $I - \mu B : L^p \to L^p$ with $\|\mu \cdot B\|_{(p,p)} \leqslant k B_p = k < 1$ for $p = 2 + \varepsilon$ and ε small enough. Then,

$$f = C(h) + z.$$

One can check that, indeed,

$$\bar{\partial} f = h$$

and, since $(I - \mu B)(h) = \mu$,

$$\mu \partial f = \mu B(h) + \mu = h$$

With some more effort one can see that this solution is indeed a K-quasiconformal homeomorphism for $K = \frac{1+k}{1-k}$.

What can we say on the regularity of the solution? We consider μ compactly supported. If $\mu \in A$ is then $h \in A$? In which spaces can we invert $I - \mu B$?

1. [AIS01] For p in the critical interval, $p \in (p_k, q_k)$, we have that

$$\mu \in L^p \implies h \in L^p$$

   and it fails otherwise.

2. [AIM09] With Schauder estimates they get for $0 < \alpha < 1$

$$\mu \in C^{\ell,\alpha}_{loc}(\Omega) \implies f \in C^{\ell+1,\alpha}_{loc}(\Omega)$$

   and it fails for $\alpha = 0$ and $\alpha = 1$.

3. [CMO13] For any $1 < p < \infty$ and $1 < q < \infty$ and $sp > 2$ (that is, when we have that $B^s_{p,q}$ and $F^s_{p,q}$ are multiplication algebras of bounded continuous functions),

$$\mu \in A^s_{p,q} \implies h \in A^s_{p,q}.$$

1.3 Known results:

Theorem ([CMO13])**.** *Let Ω be a bounded $C^{1,\varepsilon}$ domain for $\varepsilon > 0$, and let $1 < p < \infty$ and $0 < s \leqslant 1$ such that $sp > 2$. Then the Beurling transform is bounded in the Sobolev space $W^{s,p}(\Omega)$ if and only if $B(\chi_\Omega) \in W^{s,p}(\Omega)$.*

Using this, a Hölder estimate for $C^{1,\varepsilon}$-domains and Fredholm theory they prove the following.

Corollary ([CMO13])**.** *If $\varepsilon > s$,*

$$\mu \in W^{s,p}(\Omega) \implies h \in W^{s,p}(\Omega).$$

Tolsa and Cruz looked for weaker conditions on the regularity of Ω to bound $B(\chi_\Omega)$:

Theorem ([CT12])**.** *Let $\Omega \subset \mathbb{C}$ be a Lipschitz domain and its unitary outward normal vector N is in the Besov space $B^{s-1/p}_{p,p}(\partial\Omega)$ for $s \leqslant 1$, then one has $B(\chi_\Omega) \in W^{s,p}(\Omega)$.*

Notice that, if $sp > 2$, $B^{s-1/p}_{p,p} \subset C^{0,s-2/p}$, so one can use the result in [CMO13]

Corollary ([CT12])**.** *If $sp > 2$ and $N \in B^{s-1/p}_{p,p}(\partial\Omega)$, then B is bounded in $W^{s,p}(\Omega)$.*

Theorem ([Tol13])**.** *This geometric condition is necessary when the Lipschitz constants of $\partial\Omega$ are small.*

1.4 Results found:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, T a smooth convolution Calderón-Zygmund operator of order $n \in \mathbb{N}$ and $p > d$. Then the following statements are equivalent:*

a) The operator T is bounded in $W^{n,p}(\Omega)$.

b) For every polynomial P of degree at most $n-1$, we have that $T(P) \in W^{n,p}(\Omega)$.

Theorem 1.3. *Let T be a smooth convolution Calderón-Zygmund operator of order n, and consider a Lipschitz domain Ω and let $1 < p \leqslant d$. If the measure $|\nabla^n TP(x)|^p dx$ is a p-Carleson measure for every polynomial P of degree at most $n-1$ restricted to the domain, then T is a bounded operator on $W^{n,p}(\Omega)$.*

Theorem 1.4. *This condition is in fact necessary for $n = 1$ and small Lipschitz constant.*

Theorem 1.5. *Let Ω be a $C^{n-1,1}$ domain. Then,*

$$\|B\chi_\Omega\|^p_{\dot{W}^{n,p}(\Omega)} \lesssim \|N\|^p_{\dot{B}^{n-1/p}_{p,p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)^{2-np}$$

with constants depending only on p, n and the Lipschitz character of the domain.

2 2nd session

2.1 Proof of Theorem 1.2

Proof. The implication $a) \Rightarrow b)$ is trivial.

To see the converse, we will use a Whitney covering of Ω.

Recall that the Poincaré inequality tells us that, given a cube Q and a function $f \in W^{1,p}(Q)$ with 0 mean in the cube,

$$\|f\|_{L^p(Q)} \lesssim \ell(Q)\|\nabla f\|_{L^p(Q)}$$

Since we want to iterate that inequality, we also need the gradient of f to have 0 mean on Q and so on.

Given $f \in W^{n,p}(Q)$, we define $\mathbf{P}^n_Q(f) \in \mathcal{P}^n(\Omega)$ as the unique polynomial (restricted to Ω) of degree smaller or equal than n such that

$$\fint_Q D^\beta \mathbf{P}^n_Q f \, dm = \fint_Q D^\beta f \, dm \tag{2.1}$$

for every multiindex $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant n$.

Let x_Q be the center of Q. If we consider the Taylor expansion of $\mathbf{P}^{n-1}_{3Q} f$ at x_Q,

$$\mathbf{P}^{n-1}_{3Q} f(y) = \chi_\Omega(y) \sum_{\substack{\gamma \in \mathbb{N}^d \\ |\gamma| < n}} m_{Q,\gamma} (y - x_Q)^\gamma, \tag{2.2}$$

then the coefficients $m_{Q,\gamma}$ are bounded by

$$|m_{Q,\gamma}| \leqslant c_n \sum_{\substack{\beta \geqslant \gamma \\ |\beta| < n}} \left\| D^\beta f \right\|_{L^\infty(3Q)} \ell(Q)^{j - |\gamma|}.$$

Now, fix a point $x_0 \in \Omega$. We have a finite number of monomials $P_\lambda(x) = (x - x_0)^\lambda \chi_\Omega(x)$ for multiindices $\lambda \in \mathbb{N}^d$ and $|\lambda| < n$, so the hypothesis can be written as

$$\|T(P_\lambda)\|_{W^{n,p}(\Omega)} \leqslant C. \tag{2.3}$$

Assume $f \in W^{n,p}(\Omega)$. We will see later that it is enough to prove that

$$\sum_{Q \in \mathcal{W}} \|D^\alpha T(\mathbf{P}^{n-1}_{3Q} f)\|^p_{L^p(Q)} \lesssim \|f\|^p_{W^{n,p}(\Omega)}.$$

Taking the Taylor expansion of the polynomial $\mathbf{P}^{n-1}_{3Q} f$ in x_0, one has

$$\mathbf{P}^{n-1}_{3Q} f(x) = \chi_\Omega(x) \sum_{|\gamma| < n} m_{Q,\gamma} \sum_{\vec{0} \leqslant \lambda \leqslant \gamma} \binom{\gamma}{\lambda} (x - x_0)^\lambda (x_0 - x_Q)^{\gamma - \lambda}.$$

Thus,

$$D^\alpha T(\mathbf{P}^{n-1}_{3Q} f)(y) = \sum_{|\gamma| < n} m_{Q,\gamma} \sum_{\vec{0} \leqslant \lambda \leqslant \gamma} \binom{\gamma}{\lambda} (x_0 - x_Q)^{\gamma - \lambda} D^\alpha(TP_\lambda)(y). \tag{2.4}$$

Raising (2.4) to the power p, integrating in Q and using the bounds on $|m_{Q,\gamma}|$ we get

$$\left\| D^\alpha T(\mathbf{P}^{n-1}_{3Q} f) \right\|^p_{L^p(Q)} \lesssim \sum_{|\beta| < n} \|D^\beta f\|^p_{L^\infty(\Omega)} \sum_{\vec{0} \leqslant \lambda \leqslant \beta} \operatorname{diam}\Omega^{(|\beta| - |\lambda|)p} \|D^\alpha(TP_\lambda)\|^p_{L^p(Q)}.$$

By the Sobolev Embedding Theorem, we know that $\left\|\nabla^j f\right\|_{L^\infty(\Omega)} \leqslant C \left\|\nabla^j f\right\|_{W^{1,p}(\Omega)}$ as long as $p > d$. If we add with respect to $Q \in \mathcal{W}$ and we use (2.3) we get

$$\sum_{Q \in \mathcal{W}} \left\| D^\alpha T(\mathbf{P}^{n-1}_{3Q} f) \right\|^p_{L^p(Q)} \lesssim \sum_{|\beta| < n} \|D^\beta f\|^p_{W^{1,p}(\Omega)} \sum_{\vec{0} \leqslant \lambda \leqslant \beta} \|D^\alpha(TP_\lambda)\|^p_{L^p(\Omega)} \lesssim \|f\|^p_{W^{n,p}(\Omega)},$$

with constants depending on the diameter of Ω, p, d and n. □

2.2 The Key Lemma

To complete the proof of Theorem 1.2 it remains to prove the following lemma which says that it is equivalent to bound the transform of a function and its approximation by polynomials.

Key Lemma 2.1. *Given a multiindex α with $|\alpha| = n$, we whave*

$$\sum_{Q\in\mathcal{W}} \left\| D^\alpha T(f - \mathbf{P}^{n-1}_{3Q} f)\right\|^p_{L^p(Q)} \lesssim \|\nabla^n f\|^p_{L^p(\Omega)}. \tag{2.5}$$

Proof. For each cube $Q \in \mathcal{W}$ we define a bump function $\varphi_Q \in C^\infty_c$ such that $\chi_{\frac{3}{2}Q} \leqslant \varphi_Q \leqslant \chi_{2Q}$ and $\left\|\nabla^j \varphi_Q\right\|_\infty \approx \ell(Q)^{-j}$ for every $j \in \mathbb{N}$. Then we can break (2.5) into local and non-local parts as follows:

$$\begin{aligned}\sum_{Q\in\mathcal{W}} \left\| D^\alpha T(f - \mathbf{P}^{n-1}_{3Q} f)\right\|^p_{L^p(Q)} &\lesssim \sum_{Q\in\mathcal{W}} \left\| D^\alpha T\left(\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(Q)} \\ &\quad + \sum_{Q\in\mathcal{W}} \left\| D^\alpha T\left((\chi_\Omega - \varphi_Q)(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(Q)} \\ &= \textcircled{1} + \textcircled{2}. \end{aligned} \tag{2.6}$$

First of all we will show that the local term in (2.6) satisfies

$$\textcircled{1} = \sum_{Q\in\mathcal{W}} \left\| D^\alpha T\left(\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(Q)} \lesssim \|\nabla^n f\|^p_{L^p(\Omega)}. \tag{2.7}$$

To do so, notice that $\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f) \in W^{n,p}(\mathbb{R}^d)$ and, by (**??**) and the boundedness of T in L^p,

$$\begin{aligned}\left\| D^\alpha T\left(\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(Q)} &\lesssim \|T\|^p_{(p,p)} \left\| D^\alpha \left(\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(\mathbb{R}^d)} \\ &= C\left\| D^\alpha \left(\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(2Q)},\end{aligned}$$

where $\|\cdot\|_{(p,p)}$ stands for the operator norm in $L^p(\mathbb{R}^d)$. Using first the Leibnitz formula, and then using j times the Poincaré inequality, we get

$$\begin{aligned}\left\| D^\alpha T\left(\varphi_Q(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(Q)} &\lesssim \sum_{j=1}^n \|\nabla^j \varphi_Q\|^p_{L^\infty(2Q)} \left\|\nabla^{n-j}(f - \mathbf{P}^{n-1}_{3Q} f)\right\|^p_{L^p(2Q)} \\ &\lesssim \sum_{j=1}^n \frac{1}{\ell(Q)^{jp}} \ell(Q)^{jp} \left\|\nabla^n(f - \mathbf{P}^{n-1}_{3Q} f)\right\|^p_{L^p(3Q)} = n\|\nabla^n f\|^p_{L^p(3Q)}.\end{aligned}$$

Summing over all Q we get (2.7).

For the non-local part in (2.6),

$$\textcircled{2} = \sum_{Q\in\mathcal{W}} \left\| D^\alpha T\left((\chi_\Omega - \varphi_Q)(f - \mathbf{P}^{n-1}_{3Q} f)\right)\right\|^p_{L^p(Q)},$$

we will argue by duality. We can write

$$\textcircled{2}^{\frac{1}{p}} = \sup_{\|g\|_{L^{p'}}\leqslant 1} \sum_{Q\in\mathcal{W}} \int_Q \left| D^\alpha T\left[(\chi_\Omega - \varphi_Q)(f - \mathbf{P}^{n-1}_{3Q} f)\right](x)\right| g(x)\, dx. \tag{2.8}$$

Note that given $x \in Q$, one has

$$D^\alpha T[(\chi_\Omega - \varphi_Q)(f - \mathbf{P}_{3Q}^{n-1} f)](x) = \int_\Omega D^\alpha K(x-y)\,(1-\varphi_Q(y))\,\Big(f(y) - \mathbf{P}_{3Q}^{n-1} f(y)\Big)\,dy.$$

Taking absolute values, we can bound

$$\begin{aligned}|D^\alpha T[(\chi_\Omega - \varphi_Q)(f - \mathbf{P}_{3Q}^{n-1} f)](x)| &\lesssim \int_{\Omega\setminus\frac{3}{2}Q} \frac{|f(y) - \mathbf{P}_{3Q}^{n-1} f(y)|}{|x-y|^{n+d}} dy \\ &\lesssim \sum_{S\in\mathcal{W}} \frac{\left\| f - \mathbf{P}_{3Q}^{n-1} f\right\|_{L^1(S)}}{D(Q,S)^{n+d}}. \end{aligned} \tag{2.9}$$

We cannot use Poincaré inequality here, but we have

$$\left\| f - \mathbf{P}_{3Q}^{n-1} f\right\|_{L^1(S)} \leqslant \sum_{P\in[S,Q]} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}} \|\nabla^n f\|_{L^1(3P)}.$$

Proof. Consider the chain function $[Q,S]$ connecting Q and S by the shortest hyperbolic path.

$$\left\| f - \mathbf{P}_{3Q}^{n-1} f\right\|_{L^1(S)} \leqslant \| f - \mathbf{P}_{3S}^{n-1} f\|_{L^1(S)} + \sum_{P\in[S,Q)} \left\| \mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1} f\right\|_{L^1(S)} \tag{2.10}$$

where we write $\mathcal{N}(P)$ for the next cube in the chain. For every polynomial $q \in \mathcal{P}^{n-1}$, from the equivalence of norms of polynomials of bounded degree $\mathcal{P}^{n-1}(Q(0,1))$ it follows that

$$\|q\|_{L^1(Q)} \approx \ell(Q)^d \|q\|_{L^\infty(Q)},$$

and for $r > 1$, also

$$\|q\|_{L^\infty(rQ)} \lesssim r^{n-1} \|q\|_{L^\infty(Q)},$$

with constants depending only on d and n. Applying these estimates to $q = \mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1} f$ with $r \approx \frac{\mathrm{D}(P,S)}{\ell(P)}$, it follows that

$$\begin{aligned}\left\| \mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1} f\right\|_{L^1(S)} &\approx \left\| \mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1} f\right\|_{L^\infty(S)} \ell(S)^d \\ &\lesssim \left\| \mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1} f\right\|_{L^\infty(3P\cap 3\mathcal{N}(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1}} \\ &\approx \left\| \mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1} f\right\|_{L^1(3P\cap 3\mathcal{N}(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1}\ell(P)^d}. \end{aligned}$$

Using that and the Poincaré inequality,

$$\begin{aligned}\left\| f - \mathbf{P}_{3Q}^{n-1} f\right\|_{L^1(S)} &\lesssim \sum_{P\in[S,Q]} \|f - \mathbf{P}_{3P}^{n-1} f\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d+n-1}} \\ &\leqslant \sum_{P\in[S,Q]} \|\nabla^n f\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}}. \end{aligned}$$

□

Plugging this expression and (2.9) into (2.8), we get

$$\textcircled{2}^{\frac{1}{p}} \lesssim \sup_{\|g\|_{p'} \leqslant 1} \sum_{Q \in \mathcal{W}} \int_Q g(x)\,dx \sum_{S \in \mathcal{W}} \sum_{P \in [S,Q]} \frac{\ell(S)^d D(P,S)^{n-1} \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q,S)^{n+d}}.$$

Finally, we use that $P \in [S,Q]$ implies $\mathrm{D}(P,S) \lesssim \mathrm{D}(Q,S)$ (one can prove that using the Lipschitz condition) to get

$$\begin{aligned}\textcircled{2}^{\frac{1}{p}} &\lesssim \sup_{\|g\|_{p'} \leqslant 1} \sum_{Q,S \in \mathcal{W}} \sum_{P \in [S,S_Q]} \int_Q g(x)\,dx \frac{\ell(S)^d \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q,S)^{d+1}} \\ &+ \sup_{\|g\|_{p'} \leqslant 1} \sum_{Q,S \in \mathcal{W}} \sum_{P \in [Q_S,Q]} \int_Q g(x)\,dx \frac{\ell(S)^d \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q,S)^{d+1}} \\ &= \textcircled{2.1} + \textcircled{2.2},\end{aligned}$$

where Q_S and S_Q are two neighbor cubes of maximal size in this path. We consider first the term $\textcircled{2.1}$ where $P \in [S, S_Q]$ and, thus, $\mathrm{D}(Q,S) \approx \mathrm{D}(P,Q)$. Rearranging the sum,

$$\textcircled{2.1} \lesssim \sup_{\|g\|_{p'} \leqslant 1} \sum_{P \in \mathcal{W}} \frac{\|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1}} \sum_{Q \in \mathcal{W}} \frac{\int_Q g(x)\,dx}{D(Q,P)^{d+1}} \sum_{S \leqslant P} \ell(S)^d.$$

We have

$$\sum_{S \leqslant P} \ell(S)^d \approx \ell(P)^d,$$

and

$$\sum_{Q \in \mathcal{W}} \frac{\int_Q g(x)\,dx}{D(Q,P)^{d+1}} \lesssim \frac{\inf_{x \in 3P} Mg(x)}{\ell(P)}.$$

An analogous argument can be performed with $\textcircled{2.2}$, leading to

$$\textcircled{2.1} + \textcircled{2.2} \lesssim \sup_{\|g\|_{p'} \leqslant 1} \sum_{P \in \mathcal{W}} \frac{\|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1}} \frac{\inf_{3P} Mg}{\ell(P)} \ell(P)^d \lesssim \sup_{\|g\|_{p'} \leqslant 1} \sum_{P \in \mathcal{W}} \|\nabla^n f \cdot Mg\|_{L^1(3P)}$$

and, by Hölder inequality and the boundedness of the Hardy-Littlewood maximal operator in $L^{p'}$, the theorem is proved. □

2.3 Carleson measures

Theorem 2.2. *[ARS02, Theorem 3] Let $1 < p < \infty$ and let $\rho(x) = \mathrm{dist}(x, \partial\Omega)^{d-p}$ (a weight on Ω), $\rho_{\mathcal{W}}(Q) = \ell(Q)^{d-p}$ (a weight on the tree $\mathcal{T}$ of Whitney cubes). For a nonnegative measure μ on $\mathcal{T}$, the following statements are equivalent:*

i) There exists a constant $C = C(\mu)$ such that

$$\|\mathcal{I}h\|_{L^p(\mu)} \leqslant C \|h\|_{L^p(\rho)}$$

ii) There exists a constant $C = C(\mu)$ such that for every $P \in \mathcal{W}$ one has

$$\sum_{Q \leqslant P} \left(\sum_{S \leqslant Q} \mu(S) \right)^{p'} \rho_{\mathcal{W}}(Q)^{1-p'} \leqslant C \sum_{Q \leqslant P} \mu(Q). \tag{2.11}$$

iii) For every $a \in \Omega$ one has

$$\int_{\widetilde{\mathbf{Sh}}(a)} \rho(x)^{1-p'} (\mu(\mathbf{Sh}(x) \cap \mathbf{Sh}(a)))^{p'} \frac{dx}{\operatorname{dist}(x, \partial\Omega)^d} \leqslant C\mu(\mathbf{Sh}(a)).$$

In virtue of [ARS02, Theorem 1], when $d = 2$ and the domain Ω is the unit disk in the plane, the first condition is equivalent to μ being a Carleson measure for the analytic Besov space $B_p(\rho)$, that is, for every analytic function defined on the unit disc $\mathbb{D}$,

$$\|f\|^p_{L^p(\mu)} \lesssim \|f\|^p_{B_p(\rho)} = |f(0)|^p + \int_{\mathbb{D}} (1 - |z|^2)^p |f'(z)|^p \rho(z) \frac{dm(z)}{(1 - |z|^2)^2}.$$

Theorem 2.3. *If for every multiindex $|\lambda| < n$*

$$d\mu_\lambda(x) = |\nabla^n T P_\lambda(x)|^p dx$$

defines a p-Carleson measure, then T is a bounded operator on $W^{n,p}(\Omega)$.

This proof is very much in the spirit of Theorem 1.2. Again we fix a point $x_0 \in \Omega$ and we use the polynomials $P_\lambda(x) = (x - x_0)^\lambda \chi_\Omega(x)$ for every multiindex $|\lambda| < n$, but now the key point is to use the Poincaré inequality instead of the Sobolev Embedding Theorem. Our hypothesis is reduced to $d\mu_\lambda(x) = |\nabla^n T P_\lambda(x)|^p dx$ being a p-Carleson measure for Ω for every $|\lambda| < n$.

Sketch of the proof. Arguing as in the T(P) theorem, we can bound

$$\sum_{Q \in \mathcal{W}} \left\| \nabla^n T(\mathbf{P}^{n-1}_{3Q} f) \right\|^p_{L^p(Q)} \lesssim \sum_{\substack{|\beta| < n \\ \vec{0} \leqslant \lambda \leqslant \beta}} \sum_{Q \in \mathcal{W}} \left| \fint_{3Q} D^\beta f \, dm \right|^p \mu_\lambda(Q),$$

where $\mu_\lambda(Q) = \|\nabla^n T P_\lambda\|^p_{L^p(Q)}$.

Substracting the mean in the central cube Q_0 if necessary, we have $\fint_{3Q_0} D^\beta f \, dm = 0$. Thus,

$$\fint_{3Q} D^\beta f \, dm = \sum_{P \in [Q, Q_0)} \left(\fint_{3P} D^\beta f \, dm - \fint_{3\mathcal{F}(P)} D^\beta f \, dm \right),$$

and we can use the Poincaré inequality to find that

$$\sum_{Q \in \mathcal{W}} \left\| \nabla^n T(\mathbf{P}^{n-1}_{3Q} f) \right\|^p_{L^p(Q)} \lesssim \sum_{\substack{|\beta| < n \\ \vec{0} \leqslant \lambda \leqslant \beta}} \sum_{Q \in \mathcal{W}} \left(\sum_{P \geqslant Q} \ell(P) \fint_{5P} |\nabla D^\beta f| \, dm \right)^p \mu_\lambda(Q). \tag{2.12}$$

By assumption, μ_λ is a p-Carleson measure for every $|\lambda| < n$. By Theorem 2.2, we have that, for every $h \in l^p(\rho_{\mathcal{W}})$,

$$\sum_{Q \in \mathcal{W}} \left(\sum_{P \geqslant Q} h(P) \right)^p \mu_\lambda(Q) \leqslant C \sum_{Q \in \mathcal{W}} h(Q)^p \ell(Q)^{d-p}, \tag{2.13}$$

where $\rho_{\mathcal{W}}(Q) = \ell(Q)^{d-p}$.

Let us fix β and λ momentarily and take $h(P) = \ell(P) \fint_{5P} |\nabla D^\beta f| \, dm$ in (2.13). Using Jensen's inequality and the finite overlapping of the quintuple cubes, we have

$$\begin{aligned}\sum_{Q \in \mathcal{W}} \left(\sum_{P \geqslant Q} \ell(P) \fint_{5P} |\nabla D^\beta f| \, dm \right)^p \mu_\lambda(Q) &\leqslant C \sum_{Q \in \mathcal{W}} \left(\fint_{5Q} |\nabla D^\beta f| \, dm \right)^p \ell(Q)^d \\ &\lesssim \sum_{Q \in \mathcal{W}} \fint_{5Q} |\nabla D^\beta f|^p \, dm \, \ell(Q)^d \\ &\lesssim \int_\Omega |\nabla D^\beta f|^p \, dm, \end{aligned} \tag{2.14}$$

finishing the proof. □

2.4 The converse implication

Sketch of the proof. We are going to perform a duality argument for the case $n = 1$. Recall that our hypothesis is that our operator T bounded in $W^{1,p}(\Omega)$. Then the averaging function

$$\mathcal{A} f(x) := \sum_{Q \in \mathcal{W}} \chi_Q(x) \, f_{3Q},$$

by the Key Lemma, is also bounded $\mathcal{A} : W^{1,p}(\Omega) \to L^p(\mu)$ for

$$\mu(x) = |\nabla T \chi_\Omega(x)|^p dx.$$

For the sake of simplicity, let us consider the case $p = 2$, $d = 2$. By duality, $\mathcal{A}^* : L^2(\mu) \to W^{1,2}(\Omega)$ is also bounded.

We want to show that for any P,

$$\sum_{Q \subset \mathbf{Sh}(P)} \mu(\mathbf{Sh}(Q))^2 \leqslant C \mu(\mathbf{Sh}(P)).$$

For $g = \chi_{\mathbf{Sh}(P)}$,

$$\|\mathcal{A}^* g\|^2_{W^{1,2}(\Omega)} \lesssim \|g\|^2_{L^2(\mu)} = \mu(\mathbf{Sh}(P))$$

To get

$$\sum_{Q \subset \mathbf{Sh}(P)} \mu(\mathbf{Sh}(Q))^2 \lesssim \|\mathcal{A}^* g\|^2_{W^{1,2}(\Omega)} + \text{error terms}$$

we need to estimate $\|\mathcal{A}^* g\|_{W^{1,2}(\Omega)}$ from below.

For $f \in W^{1,2}(\Omega)$

$$\langle \mathcal{A}^*(g), f \rangle = \int_\Omega g \, \mathcal{A}(f) \, d\mu = \int_\Omega \widetilde{g} \, f \, dx$$

But using Hilbert structure of $W^{1,2}(\Omega)$, $\mathcal{A}^*(g)$ is represented by a function $h \in W^{1,2}(\Omega)$ with

$$\langle \mathcal{A}^*(g), f \rangle = \int_\Omega \nabla h \cdot \nabla f = - \int_\Omega \Delta h \, f \, dx + \int_{\partial\Omega} \partial_\nu h \, f \, d\sigma.$$

Thus, h is the solution of the Neuman problem

$$\begin{cases} -\Delta h = \widetilde{g} & \text{in } \Omega, \\ \partial_\nu h = 0 & \text{in } \partial\Omega. \end{cases}$$

It is well known that

$$h(x) := N[(R_d^{(d-1)} g_0) d\sigma](x) - N g_0(x). \tag{2.15}$$

Claim 2.4. *One has*

$$\sum_{Q\leqslant P}\mu(\mathbf{Sh}(Q))^{p'}\ell(Q)^{\frac{p-d}{p-1}}\lesssim\|\partial_d h\|^{p'}_{L^{p'}(\mathbf{Sh}_\omega(P))}+\sum_{Q\leqslant P}\int_{Q_\omega}\left(\int_{\{z:z_d>x_d\}}\frac{z_d-x_d}{|x-z|^d}\widetilde{g}(\omega(z))dz\right)^{p'}dx$$
$$=\textcircled{1}+\textcircled{2}. \tag{2.16}$$

Finally, we bound the negative contribution of the $(d-1)$-dimensional Riesz transform in (2.16), that is we bound $\textcircled{2}$.

Claim 2.5. *One has*

$$\textcircled{2}=\sum_{Q\leqslant P}\int_{Q_\omega}\left(\int_{\{z:z_d>x_d\}}\frac{z_d-x_d}{|x-z|^d}\widetilde{g}(\omega(z))dz\right)^{p'}dx\lesssim\mu(\mathbf{Sh}(P)). \tag{2.17}$$

□

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