The Beurling transform in Sobolev spaces of a Lipschitz domain

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1 1st session

1.1 Review of the Beurling transform

The Beurling transform is defined as the principal value

\[ B\varphi(z) := -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{\varphi(w)}{(z-w)^2} dm(w), \tag{1.1} \]

for \( \varphi \in C_0^\infty(\mathbb{C}) \). Notice that the principal value is necessary since the integral is not absolutely convergent, and it exists pointwise by Green’s formula.

**Lemma 1.1** (Properties of the Beurling transform). One has

- \( B\varphi - \partial C\varphi \) for \( \varphi \in C_0^\infty \).
- \( B(\overline{f}) - \partial f \) for \( f \in L_{\text{loc}}^1 \cap \dot{W}^{1,2} \).
- \( B(f) = \mathcal{F}^{-1}\left( \frac{z}{\sqrt{1-|z|^2}} \right) \).
- \( B \) extends as an isometry in \( L^2 \), \( B_2 := \|B\|_{L^2 \to L^2} = 1 \) and the principal value is well defined.
- \( B \) is a bounded operator in \( L^p \) with norm \( B_p \) for \( 1 < p < \infty \) and the principal value is well defined.
- The map \( p \mapsto B_p \) is Lipschitz continuous.

**Partial proof.** By Green’s formula,

\[
B\varphi(z) - \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{\partial \varphi(w)}{(z-w)} dm(w) + \frac{i}{2\pi} \lim_{\varepsilon \to 0} \int_{|w-z|<\varepsilon} \frac{\varphi(w)}{(z-w)} dm(w) = C\partial\varphi(z) - \partial C\varphi(z).
\]

From the Cauchy formula, being \( \varphi \) of compact support we have that in \( C_0^\infty \), \( C \circ \overline{\partial} = I \) and, thus,

\[ B(\overline{\partial} \varphi) - \partial \varphi. \]

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By integration by parts one can see that \( \| \partial_\nu \varphi \|_{L^2} - \| \partial_\varphi \|_{L^2} \) and, together with some approximation methods one sees that

\[ \| B(\varphi) \|_{L^2} - \| \varphi \|_{L^2} \]

for all \( \varphi \in C_0^\infty \).

By density again one can define the Beurling transform acting on \( L^2 \). That is, the Beurling transform is an isometry in \( L^2 \). \( \square \)

### 1.2 Relation with quasiconformal mappings: Solving the Beltrami equation

Consider \( 0 \leq k < 1 \) and a measurable and compactly supported function \( \mu \) such that \( \| \mu \| = k \) for \( z \in \mathbb{C} \) (we call it Beltrami coefficient). The Beltrami equation

\[ \overline{\partial} f(z) - \mu(z) \partial f(z) \]

has a unique solution \( f \in W_{\text{loc}}^{1,2} \) such that

\[ f(z) = z + O(1/z) \quad \text{as } z \to \infty. \]

The solution will be of the following form. Consider

\[ h := (I - \mu B)^{-1}(\mu), \]

where we consider the mapping \( I - \mu B : L^p \to L^p \) with \( \| \mu \cdot B \|_{(p,p)} \leq kB - k < 1 \) for \( p = 2 + \varepsilon \) and \( \varepsilon \) small enough. Then,

\[ f = C(h) + z. \]

One can check that, indeed,

\[ \overline{\partial} f = h \]

and, since \( (I - \mu B)(h) = \mu \),

\[ \mu \overline{\partial} f - \mu B(h) + \mu - h \]

With some more effort one can see that this solution is indeed a \( K \)-quasiconformal homeomorphism for \( K = \frac{1+k}{1-k} \).

What can we say on the regularity of the solution? We consider \( \mu \) compactly supported. If \( \mu \in A \) is then \( h \in A \)? In which spaces can we invert \( I - \mu B \)?

1. [AIS01] For \( p \) in the critical interval, \( p \in (p_k, q_k) \), we have that

\[ \mu \in L^p \Longleftrightarrow h \in L^p \]

and it fails otherwise.

2. [AIM09] With Schauder estimates they get for \( 0 < \alpha < 1 \)

\[ \mu \in C^{\alpha,\alpha}_{\text{loc}}(\Omega) \quad \Longleftrightarrow \quad f \in C^{\alpha+1,\alpha}_{\text{loc}}(\Omega) \]

and it fails for \( \alpha = 0 \) and \( \alpha = 1 \).

3. [CMO13] For any \( 1 < p < \infty \) and \( 1 < q < \infty \) and \( sp > 2 \) (that is, when we have that \( B_{p,q}^s \) and \( F_{p,q}^s \) are multiplication algebras of bounded continuous functions),

\[ \mu \in A_{p,q}^s \quad \Longleftrightarrow \quad h \in A_{p,q}^s. \]
1.3 Known results:

**Theorem ([CMO13]).** Let \( \Omega \) be a bounded \( C^{1,\varepsilon} \) domain for \( \varepsilon > 0 \), and let \( 1 < p < \infty \) and \( 0 < s \leq 1 \) such that \( sp > 2 \). Then the Beurling transform is bounded in the Sobolev space \( W^{s,p}(\Omega) \) if and only if \( B(\chi_\Omega) \in W^{s,p}(\Omega) \).

Using this, a Hölder estimate for \( C^{1,\varepsilon} \)-domains and Fredholm theory they prove the following.

**Corollary ([CMO13]).** If \( \varepsilon > s \),
\[
\mu \in W^{s,p}(\Omega) \implies h \in W^{s,p}(\Omega).
\]

Tolsa and Cruz looked for weaker conditions on the regularity of \( \Omega \) to bound \( B(\chi_\Omega) \):

**Theorem ([CT12]).** Let \( \Omega \subset \mathbb{C} \) be a Lipschitz domain and its unitary outward normal vector \( N \) is in the Besov space \( B^{s-1/p}_p(\partial\Omega) \) for \( s \leq 1 \), then one has \( B(\chi_\Omega) \in W^{s,p}(\Omega) \).

Notice that, if \( sp > 2 \), \( B^{s-1/p}_p \subset C^{0,s-2/p} \), so one can use the result in [CMO13]

**Corollary ([CT12]).** If \( sp > 2 \) and \( N \in B^{s-1/p}_p(\partial\Omega) \), then \( B \) is bounded in \( W^{s,p}(\Omega) \).

**Theorem ([Tol13]).** This geometric condition is necessary when the Lipschitz constants of \( \partial\Omega \) are small.

1.4 Results found:

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^d \) be a Lipschitz domain, \( T \) a smooth convolution Calderón-Zygmund operator of order \( n \in \mathbb{N} \) and \( p > d \). Then the following statements are equivalent:

a) The operator \( T \) is bounded in \( W^{n,p}(\Omega) \).

b) For every polynomial \( P \) of degree at most \( n \), we have that \( T(P) \in W^{n,p}(\Omega) \).

**Theorem 1.3.** Let \( T \) be a smooth convolution Calderón-Zygmund operator of order \( n \), and consider a Lipschitz domain \( \Omega \) and let \( 1 < p \leq d \). If the measure \( |\nabla^n T(\chi_\Omega)|^p dx \) is a \( p \)-Carleson measure for every polynomial \( P \) of degree at most \( n \) restricted to the domain, then \( T \) is a bounded operator on \( W^{n,p}(\Omega) \).

**Theorem 1.4.** This condition is in fact necessary for \( n \geq 2 \) and small Lipschitz constant.

**Theorem 1.5.** Let \( \Omega \) be a \( C^{\alpha-1,1} \) domain. Then,
\[
\|B(\chi_\Omega)\|^p_{W^{n,p}(\Omega)} \leq \|\nabla\|^p_{B^{1-n/p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)^2 \eta^{-n/p}
\]
with constants depending only on \( p \), \( n \) and the Lipschitz character of the domain.

2 2nd session

2.1 Proof of Theorem 1.2

**Proof.** The implication \( a) \Rightarrow b) \) is trivial.

To see the converse, we will use a Whitney covering of \( \Omega \).

Recall that the Poincaré inequality tells us that, given a cube \( Q \) and a function \( f \in W^{1,p}(Q) \) with 0 mean in the cube,
\[
\|f\|_{L^p(Q)} \leq \ell(Q) \|\nabla f\|_{L^p(Q)}
\]
Since we want to iterate that inequality, we also need the gradient of $f$ to have 0 mean on $Q$ and so on.

Given $f \in W^{n,p}(Q)$, we define $P^p_Q(f) \in P^n(\Omega)$ as the unique polynomial (restricted to $\Omega$) of degree smaller or equal than $n$ such that

$$
\int_Q D^\beta P^p_Q f \, dm - \int_Q D^\beta f \, dm 
$$

for every multiindex $\beta \in \mathbb{N}^d$ with $|\beta| \leq n$.

Let $x_Q$ be the center of $Q$. If we consider the Taylor expansion of $P^{n-1}_Q f$ at $x_Q$,

$$
P^{n-1}_Q f(y) - \chi_{\Omega}(y) \sum_{\gamma \in \mathbb{N}^d} m_{Q,\gamma}(y - x_Q)^\gamma,
$$

then the coefficients $m_{Q,\gamma}$ are bounded by

$$
|m_{Q,\gamma}| \leq c_n \sum_{\beta \geq \gamma \atop |\beta| < n} \|D^\beta f\|_{L^\infty(3Q)} \ell((Q)^{j-1}).
$$

Now, fix a point $x_0 \in \Omega$. We have a finite number of monomials $P_\lambda(x) - (x - x_0)^\lambda \chi_{\Omega}(x)$ for multiindices $\lambda \in \mathbb{N}^d$ and $|\lambda| < n$, so the hypothesis can be written as

$$
\|T(P_\lambda)\|_{W^{n,p}(\Omega)} \leq C.
$$

Assume $f \in W^{n,p}(\Omega)$. We will see later that it is enough to prove that

$$
\sum_{Q \in \mathcal{W}} \|D^\alpha T(P^{n-1}_Q f)\|_{L^p(Q)} \leq \|f\|_{W^{n,p}(\Omega)}^p.
$$

Taking the Taylor expansion of the polynomial $P^{n-1}_Q f$ in $x_0$, one has

$$
P^{n-1}_Q f(x) - \chi_{\Omega}(x) \sum_{|\gamma| < n} m_{Q,\gamma} \sum_{\beta \leq \gamma \atop \beta,\gamma \in \mathbb{N}^d} \binom{\gamma}{\lambda} (x - x_0)^\lambda (x_0 - x_Q)^{\gamma - \lambda}.
$$

Thus,

$$
D^\alpha T(P^{n-1}_Q f)(y) - \sum_{|\gamma| < n} m_{Q,\gamma} \sum_{\beta \leq \gamma \atop \beta,\gamma \in \mathbb{N}^d} \binom{\gamma}{\lambda} (x_0 - x_Q)^{\gamma - \lambda} D^\alpha(TP_\lambda)(y).
$$

Raising (2.4) to the power $p$, integrating in $Q$ and using the bounds on $|m_{Q,\gamma}|$ we get

$$
\left\|D^\alpha T(P^{n-1}_Q f)\right\|_{L^p(Q)}^p \leq \sum_{|\beta| < n} \|D^\beta f\|_{L^p(\Omega)}^p \sum_{\beta \leq \gamma \atop \beta,\gamma \in \mathbb{N}^d} \text{diam}\Omega^{|\beta| - |\beta|} p \|D^\alpha(TP_\lambda)\|_{L^p(Q)}^p.
$$

By the Sobolev Embedding Theorem, we know that $\|\nabla^j f\|_{L^\infty(\Omega)} \leq C_\alpha \|\nabla^j f\|_{W^{1,p}(\Omega)}$ as long as $p > d$. If we add with respect to $Q \in \mathcal{W}$ and we use (2.3) we get

$$
\sum_{Q \in \mathcal{W}} \|D^\alpha T(P^{n-1}_Q f)\|_{L^p(Q)}^p \leq \sum_{|\beta| < n} \|D^\beta f\|_{W^{1,p}(\Omega)}^p \sum_{\beta \leq \gamma \atop \beta,\gamma \in \mathbb{N}^d} \|D^\alpha(TP_\lambda)\|_{L^p(\Omega)}^p \leq \|f\|_{W^{n,p}(\Omega)}^p,
$$

with constants depending on the diameter of $\Omega$, $p$, $d$ and $n$. \qed
2.2 The Key Lemma

To complete the proof of Theorem 1.2 it remains to prove the following lemma which says that it is equivalent to bound the transform of a function and its approximation by polynomials.

Key Lemma 2.1. Given a multiindex $\alpha$ with $|\alpha| = n$, we have
\[
\sum_{Q \in \mathcal{W}} \left\| D^\alpha T(f - P_{3Q}^{n-1} f) \right\|_{L^p(Q)}^p \lesssim \left\| \nabla^n f \right\|_{L^p(\Omega)}^p.
\] (2.5)

Proof. For each cube $Q \in \mathcal{W}$ we define a bump function $\varphi_Q \in C_c^\infty$ such that $\chi_{\frac{3}{4}Q} \leq \varphi_Q \leq \chi_Q$ and $\|\nabla^j \varphi_Q\|_\infty \approx \ell(Q)^{-j}$ for every $j \in \mathbb{N}$. Then we can break (2.5) into local and non-local parts as follows:
\[
\sum_{Q \in \mathcal{W}} \left\| D^\alpha T(f - P_{3Q}^{n-1} f) \right\|_{L^p(Q)}^p \leq \sum_{Q \in \mathcal{W}} \left\| D^\alpha T(\varphi_Q(f - P_{3Q}^{n-1} f)) \right\|_{L^p(Q)}^p
\]
\[+ \sum_{Q \in \mathcal{W}} \left\| D^\alpha T((\chi_\Omega - \varphi_Q)(f - P_{3Q}^{n-1} f)) \right\|_{L^p(Q)}^p
- (1) + (2).
\] (2.6)

First of all we will show that the local term in (2.6) satisfies
\[
(1) - \sum_{Q \in \mathcal{W}} \left\| D^\alpha T(\varphi_Q(f - P_{3Q}^{n-1} f)) \right\|_{L^p(Q)}^p \lesssim \left\| \nabla^n f \right\|_{L^p(\Omega)}^p.
\] (2.7)

To do so, notice that $\varphi_Q(f - P_{3Q}^{n-1} f) \in W^{n,p}(\mathbb{R}^d)$ and, by (2.2) and the boundedness of $T$ in $L^p$, $D^\alpha T(\varphi_Q(f - P_{3Q}^{n-1} f)) \|_{L^p(Q)} \leq \|T\|_{L^p} D^\alpha(\varphi_Q(f - P_{3Q}^{n-1} f)) \|_{L^p(\Omega)} - C \|D^\alpha(\varphi_Q(f - P_{3Q}^{n-1} f))\|_{L^p(\Omega^d)},$

where $\|\cdot\|_{L^p(\Omega)}$ stands for the operator norm in $L^p(\Omega^d)$. Using first the Leibnitz formula, and then using $j$ times the Poincaré inequality, we get
\[
\|D^\alpha T(\varphi_Q(f - P_{3Q}^{n-1} f))\|_{L^p(Q)} \leq \sum_{j=1}^n \|\nabla^j \varphi_Q\|_{L^1(2Q)} \left\| \nabla^{n-j}(f - P_{3Q}^{n-1} f) \right\|_{L^p(2Q)}
\]
\[\leq \sum_{j=1}^n \frac{1}{\ell(Q)^j} \|f\|_{L^p(2Q)} \left\| \nabla^n (f - P_{3Q}^{n-1} f) \right\|_{L^p(3Q)} - n \left\| \nabla^n f \right\|_{L^p(3Q)}.
\]

Summing over all $Q$ we get (2.7).

For the non-local part in (2.6),
\[
(2) - \sum_{Q \in \mathcal{W}} \left\| D^\alpha T((\chi_\Omega - \varphi_Q)(f - P_{3Q}^{n-1} f)) \right\|_{L^p(Q)}^p
\]
we will argue by duality. We can write
\[
(2)^\frac{1}{2} = \sup_{1 \leq \|\cdot\|_{L^p} \leq 1} \sum_{Q \in \mathcal{W}} \left\| D^\alpha T(\varphi_Q(f - P_{3Q}^{n-1} f)) \right\|_{L^p(Q)} \|g(x)\|_{L^p(\Omega)}.
\] (2.8)
Note that given \( x \in Q \), one has

\[
D^n T[(\chi_N - \varphi_Q)(f - P_{3Q}^{n-1} f)](x) = \int_\Omega D^n K(x - y) (1 - \varphi_Q(y)) \left( f(y) - P_{3Q}^{n-1} f(y) \right) dy.
\]

Taking absolute values, we can bound

\[
|D^n T[(\chi_N - \varphi_Q)(f - P_{3Q}^{n-1} f)](x)| \leq \int_{\Omega} \frac{|f(y) - P_{3Q}^{n-1} f(y)|}{|x - y|^{d+n}} dy
\]

\[
\leq \sum_{S \in \mathcal{W}} \frac{\|f - P_{3Q}^{n-1} f\|_{L^1(S)}}{D(Q,S)^{n+d}}.
\]

(2.9)

We cannot use Poincaré inequality here, but we have

\[
\left\| f - P_{3Q}^{n-1} f \right\|_{L^1(S)} \leq \sum_{P \in [S,Q]} \ell(S)^d D(P,S)^{n-1} \| \nabla^n f \|_{L^1(3P)}.
\]

Proof. Consider the chain function \([Q,S]\) connecting \( Q \) and \( S \) by the shortest hyperbolic path.

\[
\left\| f - P_{3Q}^{n-1} f \right\|_{L^1(S)} \leq \| f - P_{3S}^{n-1} f \|_{L^1(S)} + \sum_{P \in [S,Q]} \left\| P_{3P}^{n-1} f - P_{3N(P)}^{n-1} f \right\|_{L^1(S)}
\]

(2.10)

where we write \( N(P) \) for the next cube in the chain. For every polynomial \( q \in \mathcal{P}^{n-1} \), from the equivalence of norms of polynomials of bounded degree \( \mathcal{P}^{n-1}(Q(0,1)) \) it follows that

\[
\| q \|_{L^1(Q)} \approx \ell(Q)^n \| q \|_{L^\infty(Q)},
\]

and for \( r > 1 \), also

\[
\| q \|_{L^\infty(rQ)} \leq r^{n-1} \| q \|_{L^\infty(Q)},
\]

with constants depending only on \( d \) and \( n \). Applying these estimates to \( q = P_{3P}^{n-1} f - P_{3N(P)}^{n-1} f \) with \( r \approx \frac{D(P,S)}{\ell(P)} \), it follows that

\[
\left\| P_{3P}^{n-1} f - P_{3N(P)}^{n-1} f \right\|_{L^1(S)} \approx \left\| P_{3P}^{n-1} f - P_{3N(P)}^{n-1} f \right\|_{L^\infty(S)} \ell(S)^d
\]

\[
\leq \left\| P_{3P}^{n-1} f - P_{3N(P)}^{n-1} f \right\|_{L^\infty(3P \cap 3N(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1}}
\]

\[
\approx \left\| P_{3P}^{n-1} f - P_{3N(P)}^{n-1} f \right\|_{L^1(3P \cap 3N(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1} \ell(P)^d}.
\]

Using that and the Poincaré inequality,

\[
\left\| f - P_{3Q}^{n-1} f \right\|_{L^1(S)} \leq \sum_{P \in [S,Q]} \left\| f - P_{3P}^{n-1} f \right\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d+n-1}}
\]

\[
\leq \sum_{P \in [S,Q]} \left\| \nabla^n f \right\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}}.
\]
Plugging this expression and (2.9) into (2.8), we get
\[
2^{t/2} \leq \sup_{|Q| \leq 1} \sum_{Q \in W} \sum_{P \in S(Q)} \int_Q g(x) \, dx \frac{\ell(S)^d D(P, S)^{n-1} \|\nabla^n f\|_{L^1(3P)}}{D(Q, S)^{d+1}}.
\]
Finally, we use that \(P \in [S, Q]\) implies \(D(P, S) \leq D(Q, S)\) (one can prove that using the Lipschitz condition) to get
\[
2^{t/2} \leq \sup_{|Q| \leq 1} \sum_{Q \in W} \sum_{P \in S(Q)} \int_Q g(x) \, dx \frac{\ell(S)^d \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d+1} D(Q, S)^{d+1}}
\]
\[
+ \sup_{|Q| \leq 1} \sum_{Q \in W} \sum_{P \in S(Q)} \int_Q g(x) \, dx \frac{\ell(S)^d \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d+1} D(Q, S)^{d+1}}
\]
\[
= 2.1 + 2.2,
\]
where \(Q_S\) and \(S_Q\) are two neighbor cubes of maximal size in this path. We consider first the term 2.1 where \(P \in [S, S_Q]\) and, thus, \(D(Q, S) \approx D(P, Q)\). Rearranging the sum, we have
\[
2.1 \leq \sup_{|Q| \leq 1} \sum_{P \in W} \|\nabla^n f\|_{L^1(3P)} \frac{\int_Q g(x) \, dx}{D(Q, P)^{d+1}} \sum_{S \leq P} \ell(S)^d.
\]
We have
\[
\sum_{S \leq P} \ell(S)^d \approx \ell(P)^d,
\]
and
\[
\sum_{Q \in W} \frac{\int_Q g(x) \, dx}{D(Q, P)^{d+1}} \leq \inf_{x \in \partial P} \frac{Mg(x)}{\ell(P)}.
\]
An analogous argument can be performed with 2.2, leading to
\[
2.1 + 2.2 \leq \sup_{|Q| \leq 1} \sum_{P \in W} \|\nabla^n f\|_{L^1(3P)} \frac{\inf_{x \in \partial P} Mg}{\ell(P)^{d+1}} \ell(P)^d \leq \sup_{|Q| \leq 1} \sum_{P \in W} \|\nabla^n f \cdot Mg\|_{L^1(3P)}
\]
and, by Hölder inequality and the boundedness of the Hardy-Littlewood maximal operator in \(L^p\), the theorem is proved.

2.3 Carleson measures

Theorem 2.2. [ARS02, Theorem 3] Let \(1 < p < \infty\) and let \(\rho(x) = \text{dist}(x, \partial \Omega)^{d-p}\) (a weight on \(\Omega\)), \(\rho_W(Q) - \ell(Q)^{d-p}\) (a weight on the tree \(T\) of Whitney cubes). For a nonnegative measure \(\mu\) on \(\mathcal{T}\), the following statements are equivalent:

i) There exists a constant \(C = C(\mu)\) such that
\[
\|\mathcal{I} h\|_{L^p(\mu)} \leq C \|h\|_{L^p(\rho)}
\]
ii) There exists a constant $C - C(\mu)$ such that for every $P \in \mathcal{W}$ one has
\[
\sum_{Q \subseteq P} \left( \sum_{S \subseteq Q} \mu(S) \right)^{\gamma} \rho_{\mathcal{W}}(Q)^{1-\gamma} \lesssim C \sum_{Q \subseteq P} \mu(Q). \tag{2.11}
\]

iii) For every $a \in \Omega$ one has
\[
\int_{\text{Sh}(a)} \rho(x)^{1-\gamma} \left( \mu(\text{Sh}(x) \cap \text{Sh}(a)) \right)^{\gamma} \frac{dx}{\text{dist}(x, \partial \Omega)^d} \lesssim C \mu(\text{Sh}(a)).
\]

In virtue of [ARS02, Theorem 1], when $d - 2$ and the domain $\Omega$ is the unit disk in the plane, the first condition is equivalent to $\mu$ being a Carleson measure for the analytic Besov space $B_p(\rho)$, that is, for every analytic function defined on the unit disc $\mathbb{D}$,
\[
\|f\|_{L^p(\rho)}^p \leq \|f\|_{B_p(\rho)}^p - |f(0)|^p + \int_{\mathbb{D}} (1 - |z|^2)^p |f'(z)|^p \rho(z) \frac{dm(z)}{(1 - |z|^2)^d}.
\]

**Theorem 2.3.** If for every multiindex $|\lambda| < n$
\[
d\mu_\lambda(x) - |\nabla^n T P_\lambda(x)|^p dx
\]
defines a $p$-Carleson measure, then $T$ is a bounded operator on $W^{n,p}(\Omega)$.

This proof is very much in the spirit of Theorem 1.2. Again we fix a point $x_0 \in \Omega$ and we use the polynomials $P_\lambda(x) - (x - x_0)^\lambda \chi_\Omega(x)$ for every multiindex $|\lambda| < n$, but now the key point is to use the Poincaré inequality instead of the Sobolev Embedding Theorem. Our hypothesis is reduced to $d\mu_\lambda(x) - |\nabla^n T P_\lambda(x)|^p dx$ being a $p$-Carleson measure for $\Omega$ for every $|\lambda| < n$.

**Sketch of the proof.** Arguing as in the $T(P)$ theorem, we can bound
\[
\sum_{Q \subseteq P} \left\| \nabla^n T (P^{n-1}_Q) \right\|_{L^p(Q)}^p \lesssim \sum_{|Q| \leq n} \sum_{0 \leq \lambda \leq |Q|} \int_{3Q} D^\lambda f \, dm \mu_\lambda(Q),
\]
where $\mu_\lambda(Q) = \|\nabla^n T P_\lambda\|_{L^p(Q)}$.

Subtracting the mean in the central cube $Q_0$ if necessary, we have $\int_{3Q_0} D^\lambda f \, dm = 0$. Thus,
\[
\int_{3Q} D^\lambda f \, dm = \sum_{P \subseteq Q \cap Q_0} \left( \int_{3P} D^\lambda f \, dm - \int_{3P(P)} D^\lambda f \, dm \right),
\]
and we can use the Poincaré inequality to find that
\[
\sum_{Q \subseteq P} \left\| \nabla^n T (P^{n-1}_Q) \right\|_{L^p(Q)}^p \lesssim \sum_{|Q| \leq n} \sum_{0 \leq \lambda \leq |Q|} \left( \sum_{P \supseteq Q} \ell(P) \int_{3P} \left| \nabla D^\lambda f \right| \, dm \right)^p \mu_\lambda(Q). \tag{2.12}
\]

By assumption, $\mu_\lambda$ is a $p$-Carleson measure for every $|\lambda| < n$. By Theorem 2.2, we have that, for every $h \in L^p(\rho_{\mathcal{W}})$,
\[
\sum_{Q \subseteq P} \left( \sum_{P \supseteq Q} h(P) \right)^p \mu_\lambda(Q) \lesssim C \sum_{Q \subseteq P} h(Q)^p \ell(Q)^{d-p}, \tag{2.13}
\]
where $\rho_W(Q) - \ell(Q)^{d-p}$.

Let us fix $\beta$ and $\lambda$ momentarily and take $h(P) - \ell(P) \int_{5P} |\nabla D^\beta f| \, dm$ in (2.13). Using Jensen’s inequality and the finite overlapping of the quintuple cubes, we have

$$
\sum_{Q \in W} \left( \sum_{P \supseteq Q} \ell(P) \int_{5P} |\nabla D^\beta f| \, dm \right)^p \mu_\lambda(Q) \leq C \sum_{Q \in W} \left( \int_{5Q} |\nabla D^\beta f| \, dm \right)^p \ell(Q)^d
$$

$$
\leq \sum_{Q \in W} \int_{5Q} |\nabla D^\beta f|^p \, dm \ell(Q)^d
$$

$$
\leq \int_{\Omega} |\nabla D^\beta f|^p \, dm,
$$

(2.14)

finishing the proof.

2.4 The converse implication

Sketch of the proof. We are going to perform a duality argument for the case $n = 1$. Recall that our hypothesis is that our operator $T$ bounded in $W^{1,p}(\Omega)$. Then the averaging function $A_P$ is bounded

$$
\|A_P g\|_{W^{1,2}(\Omega)}^2 \leq \|g\|_{L^2(\mu)}^2 - \mu(\text{Sh}(P)).
$$

To get

$$
\sum_{Q \supseteq \text{Sh}(P)} \mu(\text{Sh}(Q))^2 \leq \|A^* g\|_{W^{1,2}(\Omega)}^2 + \text{error terms}
$$

we need to estimate $\|A^* g\|_{W^{1,2}(\Omega)}$ from below.

For $f \in W^{1,2}(\Omega)$

$$
\langle A^*(g), f \rangle = \int_{\Omega} g A(f) \, d\mu - \int_{\Omega} \tilde{g} f \, dx
$$

But using Hilbert structure of $W^{1,2}(\Omega)$, $A^*(g)$ is represented by a function $h \in W^{1,2}(\Omega)$ with

$$
\langle A^*(g), f \rangle = \int_{\Omega} \nabla h \cdot \nabla f - \int_{\partial \Omega} \Delta h f \, dx + \int_{\partial \Omega} \tilde{\partial}_\nu h f \, d\sigma.
$$

Thus, $h$ is the solution of the Neuman problem

$$
\begin{cases}
-\Delta h - \tilde{g} & \text{in } \Omega, \\
\tilde{\partial}_\nu h - 0 & \text{in } \partial \Omega.
\end{cases}
$$

It is well known that

$$
h(x) := N[(R_{d-1}^\nu g_0) d\sigma](x) - N g_0(x).
$$

(2.15)
Claim 2.4. One has
\[
\sum_{Q \in P} \mu(\text{Sh}(Q))^{q'} \ell(Q) \frac{\bar{\alpha}}{\bar{\gamma}} \lesssim \|\partial_{\delta} \|_{L^{q'}(\text{Sh}(P))}^{q'} + \sum_{Q \in P} \int_{Q} \frac{z_{d} - x_{d}}{|x - \frac{x_{d}}{d}|^{q} g(\omega(z))dz} \ dx
\]
\[- \ (1) + (2). \quad (2.16)
\]
Finally, we bound the negative contribution of the \((d-1)\)-dimensional Riesz transform in (2.16), that is we bound \((2)\).

Claim 2.5. One has
\[
(2) - \sum_{Q \in P} \int_{Q} \left( \int_{\{z: z_{d} > x_{d}\}} \frac{z_{d} - x_{d}}{|x - \frac{x_{d}}{d}|^{q} g(\omega(z))dz} \right)^{q'} \ dx \lesssim \mu(\text{Sh}(P)). \quad (2.17)
\]

References


