RELATIVE INJECTIVE RESOLUTIONS VIA TRUNCATIONS, PART I

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Abstract. In this first part we reinterpret the by now classical result of Spaltenstein or Bökstedt and Neeman of the construction of injective resolutions for unbounded chain complexes. Our point of view is that one can do homotopical algebra with unbounded complexes without knowing that they support a model category structure. There is a model category of towers of bounded chain complexes which forms a model approximation for unbounded chain complexes.

Introduction

The construction of injective resolutions for bounded chain complexes is very classical, but it is only in the late eighties that Spaltenstein gave a first construction for unbounded chain complexes, [10]. A more conceptual interpretation for basically the same construction was then given five years later by Bökstedt and Neeman by studying homotopy limits in derived categories, [2]. It is also known that the category of unbounded chain complexes forms a Quillen model category (there is a proof of this fact for projective resolutions in Hovey's book [8]), but our point of view is that often the existence of a model structure does not help so much to construct explicit resolutions, which is what the approaches of Spaltenstein and Bökstedt–Neeman provide.

Therefore we prefer to interpret the concrete description of the injective resolutions by saying that the category of unbounded chain complexes admits a model approximation by towers of bounded chain complexes. The concept of model approximation has been introduced in [4] in order to construct homotopy limits and colimits in arbitrary model category. It provides a great deal of flexibility at the time of computing derived functors and the structure of the approximation encodes the way the resolutions are constructed. In the present situation we "approximate" an unbounded chain complex by successive truncations, which form a "tower" of bounded chain complexes. Since bounded chain complexes are easy to resolve, this category forms a model category. The algorithm to obtain a resolution for the original unbounded complex is now encoded in a pair of adjoint functor which is the heart of our main result Theorem 3.8 and reads as follows: Form first the tower of truncations, then construct a fibrant replacement, a.k.a injective resolutions, in the category of towers, and finally go back to unbounded chain complexes by taking the limit. This should sound familiar to anyone who has had a look at [10] or [2].

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At the same time as we emphasize the explicitness of the construction, let us also say that the model approximation gives more than that. It yields in fact all the benefits of a model structure, such as the construction of the homotopy category in which the homotopy classes of maps form a set, and of course the possibility to compute derived functors. It is in this spirit that Quillen developed his axiomatic homotopy theory, [9].

In the sequel [3] of this first part our objective will be to relativize, i.e. to alter the choice of injective objects. The idea to alter the choice of projective or injective objects, and hence to do "relative" homological algebra is not new. The idea goes back at least to Adamson [1] for group cohomology and Chevalley-Eilenberg [5] for Lie algebra homology, these where subsumed in a general theory by Hochschild, [7]. The most complete reference for the classical point of view is Eilenberg–Moore, [6]. We will see how far our point of view can be pushed and will thus try to do relative homological algebra for unbounded chain complexes by constructing a suitable model approximation.

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1. Model approximations

In the next two sections we discus our set-up for doing homotopical algebra. We explain how to localize certain categories and construct derived functors. In homotopy theory a convenient framework for doing this is given by Quillen's model categories. We use the term model category as defined in Hovey's book [8] or [4, Section 2]. There are however situations in which either it is very hard to construct a model structure or such a structure might not exist. The aim of this section is to explain how to construct right derived functors in a more general context than model categories. The idea is not to try to impose a model structure on a given category directly but rather use model categories to approximate a given category.

Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a collection of morphisms in $\mathcal{C}$ which contains all isomorphisms and satisfies the "2 out of 3" property: if f and g are composable morphism in $\mathcal{C}$ and 2 out of $\{f, g, gf\}$ belong to $\mathcal{W}$ then so does the third. We call elements of $\mathcal{W}$ weak equivalences and a pair $(\mathcal{C}, \mathcal{W})$ a category with weak equivalences. The concept of model approximation was introduced in [4].

Definition 1.1. A *right Quillen pair* for $(\mathcal{C}, \mathcal{W})$ is a model category $\mathcal{M}$ and a pair of functors $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$ satisfying the following conditions:

(1) l is left adjoint to r;
(2) if f is a weak equivalence in $\mathcal{C}$, then lf is a weak equivalence in $\mathcal{M}$;

(3) if f is a weak equivalence between fibrant objects in $\mathcal{M}$, then rf is a weak equivalence in $\mathcal{C}$;

We say that this Quillen pair forms a *right model approximation* if moreover the following condition is satisfied:

(4) for any weak equivalence $lA \to X$ in $\mathcal{M}$ with X fibrant, the adjoint $A \to rX$ is a weak equivalence in $\mathcal{C}$.

Note that if the condition (4) of the Definition 1.1 is satisfied for one fibrant replacement X, then it is satisfied for all such fibrant X.

Let us fix a right Quillen pair $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$. We recall now the key properties of model approximations, [4, Section 5]:

Proposition 1.2. (1) *The localization* $\mathrm{Ho}(\mathcal{C})$ *of* $\mathcal{C}$ *with respect to weak equivalences exists and can be constructed as follows: objects of* $\mathrm{Ho}(\mathcal{C})$ *are the same as objects of* $\mathcal{C}$ *and* $\mathrm{mor}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(lX, lY)$.

(2) *A morphism in* $\mathcal{C}$ *is a weak equivalence if and only if it induces an isomorphism in* $\mathrm{Ho}(\mathcal{C})$.

(3) *The class of weak equivalence in* $\mathcal{C}$ *is closed under retracts.*

(4) *Let* $F : \mathcal{C} \to \mathcal{T}$ *be a functor. Assume that the composition* $Fr : \mathcal{M} \to \mathcal{T}$ *takes weak equivalences between fibrant objects in* $\mathcal{M}$ *to isomorphisms in* $\mathcal{T}$*. Then the right derived functor of the restriction* $F : \mathcal{C} \to \mathcal{T}$ *exists and is given by* $A \mapsto F(rX)$*, where* X *is a fibrant replacement of* lA *in* $\mathcal{M}$.

For a given category with weak equivalences $(\mathcal{C}, \mathcal{W})$ our strategy is to construct a right model approximation $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$. We can then use it to localize $\mathcal{C}$ with respect to weak equivalences and construct right derived functors as explained in Proposition 1.2. For this strategy to work we need examples of model categories. This is the purpose of the next section in which we show how to glue model categories together to build new model categories.

2. Towers

Our main example of model category is assembled from towers of better known and simpler model categories, very much in the same way as spectra can be seen as “telescopes of spaces” in a dual setting.

A *tower* $\mathcal{T}$ of model categories consists of a sequence of model categories $\{\mathcal{T}_n\}_{n\geq 0}$ and a sequence of Quillen functors $\{l : \mathcal{T}_{n+1} \rightleftarrows \mathcal{T}_n : r\}_{n\geq 0}$ (for any n, l is left adjoint to r and r preserves fibrations and acyclic fibrations). A tower of model categories can be assembled to form a category of towers:

Definition 2.1. Let $\mathcal{T}$ be a tower of model categories. The objects of the *category of towers* $\mathrm{Tow}(\mathcal{T})$ are sequences $\{a_n\}_{n\geq 0}$ of objects $a_n \in \mathcal{T}_n$ together with a sequence of morphisms $\{a_{n+1} \to r(a_n)\}_{n\geq 0}$. We write $a_\bullet$ to denote the object $\{a_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$ and call the morphisms $\{a_{n+1} \to$

$r(a_n)\}_{n\geq 0}$ the *structure morphisms* of $a_\bullet$. The set of morphisms in $\mathrm{Tow}(\mathcal{T})$ between $a_\bullet$ and $b_\bullet$ consists of sequences of morphisms $\{f_n : a_n \to b_n\}_{n\geq 0}$ for which the following squares commute:

$$\begin{tikzcd} a_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & r(a_n) \arrow[d, "r(f_n)"] \\ b_{n+1} \arrow[r] & r(b_n) \end{tikzcd}$$

We write $f_\bullet : a_\bullet \to b_\bullet$ to denote the morphism $\{f_n : a_n \to b_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$.

For a morphism $f_\bullet : a_\bullet \to b_\bullet$ in $\mathrm{Tow}(\mathcal{T})$, define $p_0 := b_0$ and, for $n > 0$, define:

$$p_n := \lim\big(b_n \to r(b_{n-1}) \xleftarrow{r(f_{n-1})} r(a_{n-1})\big)$$

Set $\alpha_0 : a_0 \to p_0$ to be given by f_0 and $\beta_0 : p_0 \to b_0$ to be the identity. For $n > 0$, let $\beta_n : p_n \to b_n$ be the projection from the inverse limit onto the component b_n, $\overline{\alpha}_n : p_n \to r(a_{n-1})$ be the projection from the inverse limit onto the component $r(a_{n-1})$, and $\alpha_n : a_n \to p_n$ to be the unique morphism for which the following diagram commutes:

$$\begin{tikzcd} a_n \arrow[dr, "\alpha_n"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & p_n \arrow[r, "\overline{\alpha}_n"] \arrow[d, "\beta_n"] & r(a_{n-1}) \arrow[d, "r(f_{i-1})"] \\ & b_n \arrow[r] & r(b_{n-1}) \end{tikzcd}$$

The sequence $\{p_n\}_{n\geq 0}$ together with morphisms $\{p_{n+1} \xrightarrow{\overline{\alpha}_{n+1}} r(a_i) \xrightarrow{r(\alpha_n)} r(p_n)\}_{n\geq k}$ defines an object $p_\bullet$ in $\mathrm{Tow}(\mathcal{T})$. Moreover $\{\alpha_n : a_n \to p_n\}_{n\geq 0}$ and $\{\beta_n : p_n \to b_n\}_{n\geq 0}$ define morphisms $\alpha_\bullet : a_\bullet \to p_\bullet$ and $\beta_\bullet : p_\bullet \to b_\bullet$ whose composition is $f_\bullet$. For example, let $*_\bullet$ be given by the sequence consisting of the terminal objects $\{*\}_{n\geq 0}$ in $\mathcal{T}_n$ and $f_\bullet : a_\bullet \to *_\bullet$ be the unique morphism in $\mathrm{Tow}(\mathcal{T})$. Then $p_0 = *$, and, for $n > 0$, $p_n = r(a_{n-1})$. The morphism $\alpha_n : a_n \to p_n = r(a_{n-1})$ is given by the structure morphism of $a_\bullet$.

Definition 2.2. A morphism$\{f_n : a_n \to b_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$ is called a weak equivalence (cofibration) if for any n, f_n is a weak equivalence (cofibration) in $\mathcal{T}_n$. This morphism is called a fibration if, for any $n \geq 0$, $\alpha_n : a_n \to p_n$ is a fibration in $\mathcal{T}_n$.

For example the morphism $a_\bullet \to *_\bullet$ is a fibration if and only if a_0 is fibrant in $\mathcal{T}_0$ and, for $n > 0$, the structure morphism $a_n \to r(a_{n-1})$ is a fibration in $\mathcal{T}_n$.

Proposition 2.3. *The above choice of weak equivalences, cofibrations, and fibrations defines a model category structure on* $\mathrm{Tow}(\mathcal{T})$.

Proof. We start by observing first that the category $\mathrm{Tow}(\mathcal{T})$ is bicomplete. The limits and colimits are formed "degreewise". The structural morphisms of the limit are just the limits of the structural morphisms since the functors r, as right adjoints, commute with limits. For colimits, one considers the adjoints $l(a_{n+1}) \to b_n$ of the structural morphisms, takes the colimit $l(\mathrm{colim}(a_{n+1})) \cong \mathrm{colim}\, l(a_{n+1}) \to \mathrm{colim}(a_n)$, and its adjoint $\mathrm{colim}(a_{n+1}) \to r(\mathrm{colim}(a_n))$. These are precisely the structural morphisms of the colimit.

The 2 out of 3 property (MC2) for weak equivalences follows immediately from the same property for categories $\mathcal{T}_n$.

That retracts of weak equivalences (res. cofibrations) are weak equivalences (res. cofibrations) again follows from the same property in $\mathcal{T}_n$'s. For fibrations, notice that if $\{c_n \to d_n\}_{n\geq 0}$ is a retract of a fibration $\{a_n \to b_n\}_{n\geq 0}$, then $c_0 \to d_0$ is a fibration in $\mathcal{T}_0$. Next consider the following commutative diagram for $n > 0$:

$$\begin{tikzcd}
d_n \arrow[r] \arrow[d] & r(d_{n-1}) \arrow[d] & r(c_{n-1}) \arrow[l, two heads] \arrow[d] & p'_n \arrow[d] & c_n \arrow[l] \arrow[d] \\
b_n \arrow[r] \arrow[d] & r(b_{n-1}) \arrow[d] & r(a_{n-1}) \arrow[l, two heads] \arrow[d] \arrow[r, squiggly, "\lim"] & p_n \arrow[d] & a_n \arrow[l, two heads] \arrow[d] \\
d_n \arrow[r] & r(d_{n-1}) & r(c_{n-1}) \arrow[l, two heads] & p'_n & c_n \arrow[l]
\end{tikzcd}$$

By the retract property in $\mathcal{T}_n$ the morphism $c_n \to p'_n$ is fibration, for any $n \geq k$, and therefore so is $\{c_n \to d_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$. This proves axiom (MC3).

Let us prove now the right and left lifting properties (MC4). Consider a commutative diagram:

$$\begin{tikzcd}
a_\bullet \arrow[r] \arrow[d, hook, "\sim"] & c_\bullet \arrow[d, two heads] \\
b_\bullet \arrow[r] & d_\bullet
\end{tikzcd}$$

where the indicated arrows are respectively an acyclic cofibration and a fibration. In degree 0, a lift $b_0 \to c_0$ is provided by the model structure on $\mathcal{T}_0$. We construct the lift inductively. Take the solved lifting problem at level n and complete with the structural maps to get the following commutative cube:

$$
\begin{tikzcd}
 & r(a_n) \arrow[rr] \arrow[dd] & & r(c_n) \arrow[dd, two heads] \\
a_{n+1} \arrow[ur] \arrow[rr] \arrow[dd, hook, "\sim"'] & & c_{n+1} \arrow[ur] \arrow[dd] & \\
 & r(b_n) \arrow[rr] \arrow[uurr, bend right] & & r(d_n) \\
b_{n+1} \arrow[ur] \arrow[rr] & & d_{n+1} \arrow[ur] &
\end{tikzcd}
$$

Denote as usual by p_{d+1} the pull-back of $d_{n+1} \to r(d_n) \leftarrow r(c_n)$. By the universal property of the pull-back there is a morphism $b_{n+1} \to p_{n+1}$ that makes the resulting diagram commutative. Since by definition $c_{n+1} \to p_{n+1}$ is a fibration, the resulting lifting problem

$$
\begin{tikzcd}
a_{n+1} \arrow[r] \arrow[d, hook, "\sim"] & c_{n+1} \arrow[d, two heads] \\
b_{n+1} \arrow[r] & p_{n+1}
\end{tikzcd}
$$

has a solution and this is the desired morphism. The proof for the right lifting property for acyclic fibrations with respect to cofibrations is analogous.

We prove finally the factorization axiom (MC5). Consider a morphism $a_\bullet \to b_\bullet$. The morphism $a_k \to b_k$ can be factored as an acyclic cofibration followed by a fibration (respectively as a cofibration followed by an acyclic fibration) as (MC5) holds in $\mathcal{T}_0$. We construct by induction on the degree a factorization $a_{n+1} \hookrightarrow q_{n+1} \twoheadrightarrow b_{n+1}$. Consider the following commutative diagram:

$$
\begin{tikzcd}
a_{n+1} \arrow[rr] \arrow[dr] \arrow[dd] & & r(a_n) \arrow[d] \\
 & z_{n+1} \arrow[r] \arrow[dl, two heads] & r(q_n) \arrow[d, two heads] \\
b_{n+1} \arrow[rr] & & r(b_n)
\end{tikzcd}
$$

where the right column is obtained by applying the functor r to the factorization at level n and bottom right square is a pull-back. Since the functor r and cobase-change preserve (acyclic) fibrations, $z_{n+1} \to b_{n+1}$ is an (acyclic) fibration if so is $q_n \to b_n$. It is now enough to factor $a_{n+1} \to z_{n+1}$ in $\mathcal{T}_{n+1}$ in the desired way to obtain the factorization of $a_{n+1} \to b_{n+1}$. ∎

Example 2.4. Let $\mathcal{M}$ be a model category. The constant sequence $\{M\}_{n\geq 0}$ together with the the sequence of the identity functors $\{\mathrm{id} : \mathcal{M} \rightleftarrows \mathcal{M} : \mathrm{id}\}_{n\geq 0}$ form a tower of model categories. Its category of towers can be identified with the category of functors $\mathrm{Fun}(\mathbf{N}, \mathcal{M})$, where $\mathbf{N}$ is the poset whose objects are natural numbers, $\mathbf{N}(n,l)=\emptyset$ if $n<l$, and $\mathbf{N}(n,l)$ consists of one element if $n\geq l$. The model structure on $\mathrm{Fun}(\mathbf{N},\mathcal{M})$, given by Proposition 2.3, coincide with the standard model structure on the functor category $\mathrm{Fun}(\mathbf{N},\mathcal{M})$ (see []). For example, a functor F in $\mathrm{Fun}(\mathbf{N},\mathcal{M})$ is fibrant if the object $F(0)$ is fibrant in $\mathcal{M}$ and for any $n>0$, the morphism $F(n)\to F(n-1)$, induced by $n-1<n$, is a fibration in $\mathcal{M}$. A morphism $\alpha: F\to G$ is a cofibration in $\mathrm{Fun}(\mathbf{N},\mathcal{M})$ if, for any $n\geq k$, $\alpha_n: F(n)\to G(n)$ is a cofibration in $\mathcal{M}$.

3. An approximation for classical homological algebra

We have now explained our set up for doing homotopical algebra. In the rest of the paper we are going to illustrate how to use it to study unbounded chain complexes of modules over a commutative ring. We could as well do this in rather general abelian categories (satisfying the so-called axiom AB4*), but as soon as we come to the relative version which is the main subject of this article we will mainly focus on categories of modules anyway. The aim of this section is to provide a nice model category of towers which approximates the category of unbounded chain complexes. We are going to use the following notation:

3.1. We consider *cohomological* complexes (differentials raise the degree by one) in R-Mod, that is of the form $X=(\cdots\to X^i\xrightarrow{d^i}X^{i+1}\to\cdots)$. The category of such chain complexes is denoted by $\mathrm{Ch}(R)$. We identify R-Mod with the full subcategory of $\mathrm{Ch}(R)$ whose objects are chain complexes concentrated in degree 0.

For a chain complex $X\in\mathrm{Ch}(R)$, the cocycles $\mathrm{Ker}(d^i: X^i\to X^{i+1})$ are denoted by $Z^i(X)$, or simply by Z^i and the coboundaries $\mathrm{Im}(d^{i-1}: X^{i-1}\to X^i)$ are denoted by $B^i(X)$, or simply by B^i. The cohomology of X is as usual $H^i(X)=Z^i(X)/B^i(X)$. A morphism of chain complexes $f: X\to Y$ is a *quasi-isomorphism* if $H^i(f): H^i(X)\to H^i(Y)$ is an isomorphism for all $i\in\mathbb{Z}$. A chain complex is called *acyclic* if all its cohomology modules are trivial.

For an integer n, the symbol $\Sigma^n:\mathrm{Ch}(R)\to\mathrm{Ch}(R)$ denotes the shift functor that assigns to a complex X, the shifted complex given by $(\Sigma^nX)^i:=X^{i-n}$ with the differentials given by $(-1)^nd^{i-n}$. Similarly for a morphism $f: X\to Y$ in $\mathrm{Ch}(R)$, $(\Sigma^nf)^i:=f^{i-n}$. For example, if M is an R-module, then Σ^nM denotes a chain complex where $(\Sigma^nM)^n=M$ and $(\Sigma^nM)^i=0$ if $i\neq n$.

3.2. A morphism of chain complexes $f: X\to Y$ is a *homotopy equivalence* if there is a morphism $g: Y\to X$ such that fg and gf are homotopic to the identity morphisms. Homotopy equivalences are examples of quasi-isomorphisms. For any X, the morphism $h: X\to P(X)$ is an example of a homotopy equivalence.

A complex X is called *contractible* if $X\to 0$ is a homotopy equivalence. For an R-module M and an integer k, $D^k(M)$ denotes the chain complex where, for $i=k$ and $i=k+1$, $D^k(M)^i=M$ and otherwise $D^k(M)^i=0$, with the differential $d^k: D^k(M)^k\to D^k(M)^{k+1}$ given by id_M. Complexes of the form $D^k(M)$ are examples of contractible complexes.

3.3. Let n be an integer. The full subcategory of $\mathrm{Ch}(R)$ consisting of these chain complexes X such that $X^i = 0$, for $i < n$, is denoted by $\mathrm{Ch}(R)^{\geq n}$. The inclusion functor $\mathrm{in} : \mathrm{Ch}(R)^{\geq n} \subset \mathrm{Ch}(R)$ has both right and left adjoints. The left adjoint is denoted by $\tau_n : \mathrm{Ch}(R \to \mathrm{Ch}(R)^{\geq n}$. Explicitly, τ_n assigns to a complex X, the truncated complex:

$$\tau_n(X) := (\mathrm{coker}(d^{n-1}) \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \xrightarrow{d^{n+2}} \cdots)$$

where in degree n, $\tau_n(X)^n = \mathrm{coker}(d^{n-1})$, and for $i > n$, $\tau_n(X)^i = X^i$. For a morphism $f : X \to Y$ in $\mathrm{Ch}(R)$, $\tau_n(f)^n$ is induced by f^n and, for $i > n$, $\tau_n(f)^i = f^i$.

For any $X \in \mathrm{Ch}(R)$, the *canonical morphism* $X \to \tau_n(X)$ is defined to be the morphism in $\mathrm{Ch}(R)$ which is the adjoint to the identity morphism $\mathrm{id} : \tau_n(X) \to \tau_n(X)$ in $\mathrm{Ch}(R)^{\geq n}$. Explicitly this morphism is given by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d] & (\cdots \arrow[r, "d^{n-2}"] & X^{n-1} \arrow[r, "d^{n-1}"] \arrow[d] & X^n \arrow[r, "d^n"] \arrow[d, "q"] & X^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "\mathrm{id}"] & \cdots) \\
\tau_n(X) \arrow[r, equal] & (\cdots \arrow[r] & 0 \arrow[r] & \mathrm{coker}(d^{n-1}) \arrow[r, "d^n"] & X^{n+1} \arrow[r, "d^{n+1}"] & \cdots)
\end{tikzcd}$$

where q denotes the quotient morphism.

Whereas unbounded complexes are more difficult to understand, bounded ones are relatively simple and it was already known to Quillen, [9], that they form a model category.

Theorem 3.4. *The category of bounded chain complexes* $\mathrm{Ch}(R)^{\geq n}$ *is equipped with a model category structure where weak equivalences are quasi-isomorphisms, cofibrations are degreewise monomorphisms in degrees* $> n$*, and fibrations are degreewise split epimorphisms with injective kernel. In particular* X *is fibrant if* X^i *is an injective module for all* $i \geq n$*.*

As announced we are going to use the model categories $\mathrm{Ch}(R)^{\geq n}$ to approximate the category $\mathrm{Ch}(R)$ of unbounded chain complexes.

For $n \geq k$, the restriction of $\tau_k : \mathrm{Ch}(R) \to \mathrm{Ch}(R)^{\geq k}$ to the subcategory $\mathrm{Ch}(R)^{\geq n} \subset \mathrm{Ch}(R)$ is denoted by the same symbol $\tau_k : \mathrm{Ch}(R)^{\geq n} \to \mathrm{Ch}(R)^{\geq k}$. Note that this restriction is left adjoint to the inclusion $\mathrm{in} : \mathrm{Ch}(R)^{\geq k} \subset \mathrm{Ch}(R)^{\geq n}$. Moreover the canonical morphism $X \to \tau_k(X)$ can be expressed uniquely as the composition $X \to \tau_n(X) \to \tau_k(X)$, of the canonical morphism $X \to \tau_n(X)$ for X and n, and the canonical morphism $\tau_n(X) \to \tau_k(X) = \tau_k(\tau_k(X))$ for $\tau_k(X)$ and k.

Consider now the sequence of model categories $\{\mathrm{Ch}(R)^{\geq n}\}_{n\geq 0}$, with the model structures given by Theorem 3.4. The functor $\mathrm{in} : \mathrm{Ch}(R)^{\geq n} \subset \mathrm{Ch}(R)^{\geq n+1}$ takes (acyclic) fibrations to (acyclic) fibrations and hence the following is a sequence of Quillen functors:

$$\{\tau_n : \mathrm{Ch}(R)^{\geq n+1} \rightleftarrows \mathrm{Ch}(R)^{\geq n} : \mathrm{in}\}_{n\geq 0}$$

We will use the symbol $\mathrm{Tow}(R)$ to denote the associated category of towers (see Section 2). Let $X_\bullet$ be an object in $\mathrm{Tow}(R)$. We can think about this object as a tower of morphisms:

$$\cdots \xrightarrow{t_3} X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} X_0$$

in $\mathrm{Ch}(R)$ given by the structure morphisms of $X_\bullet$. Conversely, for any such tower where X_n is a chain complex that belongs to $\mathrm{Ch}(R)^{\geq n}$, we can define an object $X_\bullet$ in $\mathrm{Tow}(R)$ given by the sequence $\{X_n\}_{n\geq 0}$ with the morphisms $\{t_{n+1}\}_{n\geq 0}$ as its structure morphisms. In this way we can think about $\mathrm{Tow}(R)$ as a full subcategory of the functor category $\mathrm{Fun}(\mathbf{N}, \mathrm{Ch}(R))$ consisting of these functors $X : \mathbf{N} \to \mathrm{Ch}(R)$ for which $X(n) \in \mathrm{Ch}(R)^{\geq n}$.

To be very explicit, $\mathrm{Tow}(R)$ is the category of the following commutative diagrams of R-modules:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_2^{-2} \arrow[r, "d_2^{-2}"] \arrow[d, "t_2^{-2}"] & X_2^{-1} \arrow[r, "d_2^{-1}"] \arrow[d, "t_2^{-1}"] & X_2^0 \arrow[r, "d_2^0"] \arrow[d, "t_2^0"] & X_2^1 \arrow[r, "d_2^1"] \arrow[d, "t_2^1"] & X_2^2 \arrow[r, "d_2^2"] \arrow[d, "t_2^2"] & \cdots \\
 & 0 \arrow[r] & X_1^{-1} \arrow[r, "d_1^{-1}"] \arrow[d, "t_1^{-1}"] & X_1^0 \arrow[r, "d_1^0"] \arrow[d, "{}^0_1"] & X_1^1 \arrow[r, "d_1^1"] \arrow[d, "t_1^1"] & X_1^2 \arrow[r, "d_1^2"] \arrow[d, "t_1^2"] & \cdots \\
 & & 0 \arrow[r] & X_0^0 \arrow[r, "d_0^0"] & X_0^1 \arrow[r, "d_0^1"] & X_0^2 \arrow[r, "d_0^2"] & \cdots
\end{tikzcd}$$

where, for any $n \geq 0$ and $i \leq n$, $d_{n,i-1}d_{n,i} = 0$, i.e., horizontal lines are chain complexes.

We will always think about $\mathrm{Tow}(R)$ as a model category, with the model structure given by Proposition 2.3. For example, if we think about $X_\bullet$ as a tower $(\cdots \xrightarrow{t_3} X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} X_0)$, then $X_\bullet$ is fibrant if and only if X_0 is fibrant in $\mathrm{Ch}(R)_{\geq 0}$ and, for any $n \geq 0$, $t_{n+1} : X_{n+1} \to X_n$ is a fibration in $\mathrm{Ch}(R)_{\geq n+1}$. If we think about $X_\bullet$ as a commutative diagram above, then $X_\bullet$ is fibrant if, for any $i \geq 0$, the objects $X_{0,i}$ are injective, and, for any $n > 0$ and $i \geq n$, $t_{n,i}$ has a section and its kernel is injective. Note also that since all objects in $\mathrm{Ch}(R)_{\geq n}$ are cofibrant, then so are all objects in $\mathrm{Tow}(R)$

Here is another way of describing the category $\mathrm{Tow}(R)$. Consider the constant sequence of model categories $\{\mathrm{Ch}(R)_{\geq 0}\}_{n\geq 0}$ with the model structure given by Theorem 3.4 and the sequence of functors $\{\tau : \mathrm{Ch}(R)_{\geq 0} \rightleftarrows \mathrm{Ch}(R)_{\geq 0} : \Sigma\}_{n\geq 0}$, where Σ is the shift functor defined in 3.1 that assigns to $X = (X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots)$ the shifted complex $\Sigma X := (0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots)$. The functor τ introduced in 3.3 is left adjoint to Σ. It is clear that Σ takes (acyclic) fibrations in $\mathrm{Ch}(R)^{\geq 0}$ into (acyclic) fibrations in $\mathrm{Ch}(R)^{\geq 0}$. Let us denote this tower of model categories by $\mathcal{T}$.

Let $X_\bullet$ be an object in $\mathrm{Tow}(\mathcal{T})$. The structure morphisms of $X_\bullet$ and the differentials of the chain complexes X_i can be assembled to form a diagram similar to the one we have seen above, in which the horizontal lines are chain complexes. It then follows that $\mathrm{Tow}(\mathcal{T})$ is isomorphic to $\mathrm{Tow}(R)$.

Our aim is to approximate the category of unbounded chain complexes of R-modules by towers of bounded complexes. For this purpose we first need to construct a pair of adjoint functors $\mathrm{tow} : \mathrm{Ch}(R) \rightleftarrows \mathrm{Tow}(R) : \lim$.

Definition 3.5. Let X be an object in $\mathrm{Ch}(R)$. The *tower* $\mathrm{tow}(X)$ is the object in $\mathrm{Tow}(R)$ given by the sequence $\{\tau_n(X)\}_{n\geq 0}$ with the structural morphisms given by the the canonical morphisms $\{t_{n+1} : \tau_{n+1}(X) \to \tau_n(X)\}_{n\geq 0}$.

Explicitly, $\mathrm{tow}(X)$ is represented by the following commutative diagram in R:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\tau_2(X) \arrow[d] & 0 \arrow[r] & \mathrm{coker}(d^{-3}) \arrow[r, "d^{-2}"] \arrow[d] & X^{-1} \arrow[r, "d^{-1}"] \arrow[d, "q"] & X^0 \arrow[r, "d^0"] \arrow[d, "\mathrm{id}"] & X^1 \arrow[r, "d^1"] \arrow[d, "\mathrm{id}"] & \cdots \\
\tau_1(X) \arrow[d] & & 0 \arrow[r] & \mathrm{coker}(d^{-2}) \arrow[r, "d^{-1}"] \arrow[d] & X^0 \arrow[r, "d^0"] \arrow[d, "q"] & X^1 \arrow[r, "d^1"] \arrow[d, "\mathrm{id}"] & \cdots \\
\tau_0(X) & & & 0 \arrow[r] & \mathrm{coker}(d^{-1}) \arrow[r, "d^0"] & X^1 \arrow[r, "d^1"] & \cdots
\end{tikzcd}
$$

where q's denote the quotient morphisms. For a chain map $f : X \to Y$, the morphism $\mathrm{tow}(f)$ is defined to be given by the sequence of morphisms $\{\tau_n(f)\}_{n\geq 0}$.

Recall that the category of towers $\mathrm{Tow}(R)$ can be identified with a full subcategory of the functor category $\mathrm{Fun}(\mathbf{N}, \mathrm{Ch}(R))$.

Definition 3.6. The *limit* functor $\lim : \mathrm{Tow}(R, \mathcal{I}) \to \mathrm{Ch}(R)$ is the restriction of the standard limit functor $\lim : \mathrm{Fun}(\mathbf{N}, \mathrm{Ch}(R)) \to \mathrm{Ch}(R)$.

Explicitly, let $X_\bullet$ be an object in $\mathrm{Tow}(R, \mathcal{I})$ described by the following commutative diagram in R with horizontal lines being chain complexes:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_2^{-2} \arrow[r, "d_2^{-2}"] \arrow[d, "t_2^{-2}"] & X_2^{-1} \arrow[r, "d_2^{-1}"] \arrow[d, "t_2^{-1}"] & X_2^0 \arrow[r, "d_2^0"] \arrow[d, "t_2^0"] & X_2^1 \arrow[r, "d_2^1"] \arrow[d, "t_2^1"] & X_2^2 \arrow[r, "d_2^2"] \arrow[d, "t_2^2"] & \cdots \\
 & 0 \arrow[r] & X_1^{-1} \arrow[r, "d_1^{-1}"] \arrow[d, "t_1^{-1}"] & X_1^0 \arrow[r, "d_1^0"] \arrow[d, "t_1^0"] & X_1^1 \arrow[r, "d_1^1"] \arrow[d, "t_1^1"] & X_1^2 \arrow[r, "d_1^2"] \arrow[d, "t_1^2"] & \cdots \\
 & & 0 \arrow[r] & X_0^0 \arrow[r, "d_0^0"] & X_0^1 \arrow[r, "d_0^1"] & X_0^2 \arrow[r, "d_0^2"] & \cdots
\end{tikzcd}$$

Then $\lim(X_\bullet)$ is the chain complex obtained by taking the inverse limit of the above diagram in the vertical direction:

$$\lim(X_\bullet)^i := \lim(\cdots \xrightarrow{t_3^i} X_2^i \xrightarrow{t_2^i} X_1^i \xrightarrow{t_1^i} X_0^i)$$

with the differential $d_i : \lim(X_\bullet)^i \to \lim(X_\bullet)^{i+1}$ given by $\lim_n(d_n^i)$. On morphisms, the functor $\lim : \mathrm{Tow}(R) \to \mathrm{Ch}(R)$ is defined in the analogous way by taking the inverse limits in the vertical direction.

Lemma 3.7. *The functor* $\mathrm{tow} : \mathrm{Ch}(R) \to \mathrm{Tow}(R)$ *is left adjoint to* $\lim : \mathrm{Tow}(R) \to \mathrm{Ch}(R)$*.*

Proof. Let Y be a chain complex in $\mathrm{Ch}(R)$ and $X_\bullet$ be an object in $\mathrm{Tow}(R)$ given by the tower $(\cdots X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} X_0)$ of morphisms in $\mathrm{Ch}(R)$ with $X_n \in \mathrm{Ch}(R)^{\geq n}$. Consider a morphism of chain complexes $f : Y \to \lim(X_\bullet)$. Since $\lim(X_\bullet)$ is the inverse limit of the tower $X_\bullet$, the morphism f corresponds to a sequence of morphisms $\{f_n : Y \to X_n)\}_{n\geq 0}$ for which the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & Y \arrow[r, "\mathrm{id}"] \arrow[d, "f_2"] & Y \arrow[r, "\mathrm{id}"] \arrow[d, "f_1"] & Y \arrow[d, "f_0"] \\
\cdots \arrow[r, "t_3"] & X_2 \arrow[r, "t_2"] & X_1 \arrow[r, "t_1"] & X_0
\end{tikzcd}$$

Since the chain complex X_n belongs to $\mathrm{Ch}(R)^{\geq n}$, the morphism $f_n : Y \to X_n$ can be expressed in a unique way as a composition $Y \to \tau_n(Y) \to X_n$ where $Y \to \tau_n(Y)$ is the canonical morphism. The sequence $\{\tau_n(Y) \to X_n\}_{n\geq 0}$ describes a morphism $\mathrm{tow}(Y) \to X_\bullet$ in $\mathrm{Tow}(R)$. It is straightforward to check that this procedure defines a natural bijection from the set of morphisms between Y and $\lim(X_\bullet)$ in $\mathrm{Ch}(R)$ onto the set of morphisms between $\mathrm{tow}(Y)$ and $X_\bullet$ in $\mathrm{Tow}(R)$. ∎

The next proposition is the reinterpretation of the work of Spaltenstein [10] or Bökstedt and Neeman [2].

Theorem 3.8. *The pair of functors* $tow : \mathrm{Ch}(R) \leftrightarrows Tow(R) : \lim$ *is a right Quillen pair. It forms moreover a model approximation.*

Proof. Properties (1), (2), and (3) are easy to verify, so that the pair of functors forms a Quillen pair. The content of the proposition is thus in property (4). It says that if $X_\bullet$ is a fibrant replacement for the truncation tower of some unbounded chain complexe A, then the canonical morphism $A \to \mathrm{in}(X_\bullet)$ is a quasi-isomorphism. This is basically a cohomological computation. Notice that the structure map $I_{m+1} \to I_m$ induces an isomorphism $H^i(A) \cong H^i(I_{m+1}) \to H^i(I_m)$ for $i > n - m$. This observation is [2, Application 2.4] and proves the claim. ∎

Let us be very explicit about how one constructs then an injective resolution for an unbounded chain complex A. First we construct its truncation tower $\{\tau_n A\}$, then we form a fibrant replacement in the category of towers of bounded chain complexes. It can be obtained in a simple inductive process as follows. Construct an injective resolution I_0 for $\tau_0 A$, and if I_n has been constructed choose I_{n+1} so that the structure map $I_{n+1} \to I_n$ be a fibration, i.e. a degreewise split epimorphism with injective kernel. The tower $I_\bullet$ is a *special tower* in Spaltenstein's terminology. Its inverse limit is an unbounded chain complex of injective modules and the canonical map $A \to \lim(I_\bullet)$ is a quasi-isomorphism as we have proven in the previous proposition.

This concludes the part of the article devoted to classical resolutions.

References

1. I. T. Adamson, *Cohomology theory for non-normal subgroups and non-normal fields*, Proc. Glasgow Math. Assoc. **2** (1954), 66–76.
2. M. Bökstedt and A. Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234.
3. W. Chachólski, A. Neeman, W. Pitsch, and J. Scherer, *Relative injective resolutions via truncations, part II*, Preprint (2009), 11 pages.
4. W. Chachólski and J. Scherer, *Homotopy theory of diagrams*, Mem. Amer. Math. Soc. **155** (2002), no. 736, x+90.
5. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124. MR MR0024908 (9,567a)
6. S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. No. **55** (1965), 39.
7. G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc. **82** (1956), 246–269. MR MR0080654 (18,278a)
8. M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.
9. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.
10. N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.