RELATIVE INJECTIVE RESOLUTIONS VIA TRUNCATIONS, PART II

WOJCIECH CHACHÓLSKI, AMNON NEEMAN, WOLFGANG PITSCH, AND JÉRÔME SCHERER

Abstract. In this first part we reinterpreted the by now classical result of Spaltenstein or Bökstedt and Neeman of the construction of injective resolutions for unbounded chain complexes: There is a model category of towers of bounded chain complexes which forms a model approximation for unbounded chain complexes. We then wonder in this part whether this point of view can be relativized, i.e. if the same category of towers forms a model approximation for unbounded complexes, but where one changes the weak equivalences (formerly quasi-isomorphisms) to $\mathcal{I}$-equivalences, where $\mathcal{I}$ is a class of injective modules. We prove that this approach is valid if we work over a Noetherian ring of finite Krull dimension, but it fails in general when the ring is Nagata's "bad ring" of infinite Krull dimension.

Introduction

The idea to alter the choice of projective or injective objects to construct resolutions, and hence to do "relative" homological algebra, goes back at least to Adamson [1] for group cohomology and Chevalley-Eilenberg [5] for Lie algebra homology. This was subsumed in a general theory by Hochschild, [9], but the most complete reference for the classical point of view is probably the work of Eilenberg and Moore, [8]. Christensen and Hovey, [6], studied the projective case from the point of view of Quillen's homotopical algebra. They show that, in many cases, one can equip the category of unbounded chain complexes with a model category structure where the weak equivalences reflect the choice of the projective objects. However this approach is not suitable in the injective case, because of the lack of a dual small object argument.

We decided therefore to see how far the point of view exposed in the first part [4] of this work can be pushed and will thus try to do relative homological algebra for unbounded chain complexes by constructing a suitable *model approximation.* Recall that in the absolute case the approximation is given by a category of towers of bounded chain complexes. In the relative setting it is easy to provide the category of bounded chain complexes with a model category structure where the weak equivalences are now determined by a class $\mathcal{I}$ of modules. The question is whether the category of towers still approximates the category of unbounded chain complexes, that is whether the pair of adjoint functors between unbounded chain complexes and towers of bounded chain complexes allows us to do homotopical algebra.

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What kind of classes of modules are suitable to do relative homological algebra? This is the subject of the companion paper [3] where we mostly focus on the nicest classes, i.e. those consisting of injective modules. We show that these classes form a lattice and prove they are determined by certain sets of ideals. Equipped with this knowledge about how injective classes look like we come back to our objective and prove that our approach does work for any injective class $\mathcal{I}$ of injective modules over a Noetherian ring of finite Krull dimension.

Theorem 3.1. *Let R be a Noetherian ring with finite Krull dimension d and $\mathcal{I}$ an injective class of injective modules. The category of towers forms then a model approximation for $Ch(R)$ equipped with $\mathcal{I}$-equivalences.*

The key to this result is local cohomology. It allows to study whether or not the relative homology of the inverse limit of certain towers stabilizes. However, when the Krull dimension is infinite, this approach fails and we provide an example of an injective class $\mathcal{I}$ and a ring (Nagata's well known "bad ring", [12]) where the limit of an $\mathcal{I}$-injective resolution of the tower of truncations of certain unbounded chain complexes does not give an $\mathcal{I}$-injective resolution of the original complex, see Theorem 4.4.

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1. Relative equivalences for chain complexes

Let us briefly recall from [3] the basic definitions about how we will do relative homological algebra.

Definition 1.1. Let $\mathcal{I}$ be a collection of R-modules. A homomorphism $f : M \to N$ is an $\mathcal{I}$-*monomorphism* if, for any $W \in \mathcal{I}$, $f^* : \mathrm{Hom}_R(N, W) \to \mathrm{Hom}_R(M, W)$ is a surjection of sets. We say that R-Mod has enough $\mathcal{I}$-injectives if, for any object M, there is an $\mathcal{I}$-monomorphism $M \to W$ with $W \in \mathcal{I}$.

Definition 1.2. A collection $\mathcal{I}$ of R-modules is an *injective class* if R-Mod has enough $\mathcal{I}$-injectives and $\mathcal{I}$ is closed under retracts and products.

In [3] we proved that injective classes consisting of injective modules are in bijection with certain sets of ideals I in R, which we called saturated. The precise definition is given in [3, Definition 2.1] and it will be sufficient for us to know that it is designed in such a way that the class of retracts of products of injective envelopes of quotient modules R/I forms an injective class. We start now by explaining how a choice of an injective class leads to relative weak equivalences between chain complexes. Throughout the rest of the paper $\mathcal{I}$ denotes an injective class of R-modules.

Definition 1.3. A morphism $f : X \to Y$ in $\mathrm{Ch}(R)$ is called an *$\mathcal{I}$-weak equivalence* if, for any $W \in \mathcal{I}$, $\mathrm{Hom}_R(f, W) : \mathrm{Hom}_R(Y, W) \to \mathrm{Hom}_R(X, W)$ is a quasi-isomorphism of complexes of abelian groups. An object X in $\mathrm{Ch}(R)$ is called *$\mathcal{I}$-trivial* if $X \to 0$ is an $\mathcal{I}$-weak equivalence, i.e. when $\mathrm{Hom}_R(X, W)$ is an acyclic complex of abelian groups for all $W \in \mathcal{I}$.

It is clear that $\mathcal{I}$-weak equivalences satisfy the "2 out of 3" property and that isomorphisms are $\mathcal{I}$-weak equivalences. Further:

Proposition 1.4. (1) *A homotopy equivalence in $Ch(R)$ is an $\mathcal{I}$-weak equivalence.*

(2) *A morphism $f : X \to Y$ in $Ch(R)$ is an $\mathcal{I}$-weak equivalence if and only if the cone* $\mathrm{Cone}(f)$ *is $\mathcal{I}$-trivial.*

(3) *Coproducts of $\mathcal{I}$-weak equivalences are $\mathcal{I}$-weak equivalences.*

(4) *A contractible chain complex in $Ch(R)$ is $\mathcal{I}$-trivial.*

(5) *Coproducts of $\mathcal{I}$-trivial complexes are $\mathcal{I}$-trivial.*

(6) *X is $\mathcal{I}$-trivial if and only if, for any i, $d^i : \mathrm{coker}(d^{i-1}) \to X^{i+1}$ is an $\mathcal{I}$-monomorphism.*

(7) *Let X be a complex such that, for all i, $\mathrm{coker}(d^{i+1}) \in \mathcal{I}$. Then X is $\mathcal{I}$-trivial if and only if X is isomorphic to $\bigoplus D^i(W_i)$.*

Proof. (1): This is a consequence of the fact that $\mathrm{Hom}_R(-, W)$ is an additive functor.

(2): Note that the cone of $\mathrm{Hom}_R(f, W) : \mathrm{Hom}_R(Y, W) \to \mathrm{Hom}_R(X, W)$ is isomorphic to the shift of the complex $\mathrm{Hom}_R(\mathrm{Cone}(f), W)$, for any $W \in \mathcal{I}$. Thus $\mathrm{Hom}_R(f, W)$ is a quasi-isomorphism if and only if $\mathrm{Hom}_R(\mathrm{Cone}(f), W)$ is acyclic. Statement (2) follows.

(3): This is a consequence of two facts. First is that $\mathrm{Hom}_R(-, W)$ takes coproducts in R-Mod into products of abelian groups. Second, products of quasi-isomorphisms of chain complexes of abelian groups are quasi-isomorphisms.

(4): This is a consequence of (1).

(5): This is a consequence of (3).

(6): The kernel of $\mathrm{Hom}_R(d^{i-1}, W)$ is given by $\mathrm{Hom}_R(\mathrm{coker}(d^{i-1}), W)$. Thus the homology of $\mathrm{Hom}_R(X, W)$ is trivial if and only if the morphism $\mathrm{Hom}_R(X^{i+1}, W) \to \mathrm{Hom}_R(\mathrm{coker}(d^{i-1}), W)$ induced by d^i is an epimorphism for all i. By definition this happens if and only if the morphism $d^i : \mathrm{coker}(d^{i-1}) \to X^{i+1}$ is an $\mathcal{I}$-monomorphism.

(7): If X can be expressed as $\bigoplus D^i(W_i)$, then X is contractible and according to (4) it is $\mathcal{I}$-trivial. Assume now that X is $\mathcal{I}$-trivial. Define $W_i := \mathrm{coker}(d^{i-1})$. According to (6), the morphism $d^i : \mathrm{coker}(d^{i-1}) \to X^{i+1}$ is an I-monomorphism. As $\mathrm{coker}(d^{i-1})$ is assumed to belong to $\mathcal{I}$, it follows that the morphism $d^i : \mathrm{coker}(d^{i-1}) \to X^{i+1}$ has a retraction. This retraction can be used to define a morphism of chain complexes $X \to D^i(W_i)$. By assembling these morphisms together we get the desired isomorphism $X \to \bigoplus D^i(W_i)$. ∎

Example 1.5. If $\mathcal{I}$ consists of all R-modules, then $f : X \to Y$ is an $\mathcal{I}$-weak equivalences if and only if it is a homotopy equivalence. A chain complex is $\mathcal{I}$-trivial if and only if it is isomorphic to $\bigoplus_i D(M_i)$ for some sequence of modules M_i.

The collection $\mathcal{I}$ of all injective R-modules is an injective class. A morphism $f : X \to Y$ in $\mathrm{Ch}(R)$ is an $\mathcal{I}$-weak equivalence if and only if it is a quasi-isomorphism. A chain complex is $\mathcal{I}$-trivial if and only if it has trivial homology.

Let $l : S\text{-Mod} \leftrightarrows R\text{-Mod} : r$ be a pair of adjoint functors and $\mathcal{I}$ be an injective class of R-modules. According to [3, Proposition **??** (5)], the collection $\mathcal{J}$ of retracts of objects of the form $r(W)$, for $W \in \mathcal{I}$, is an injective class in S-Mod. By applying l and r degree-wise, we get an induced pair of adjoint functors, denoted by the same symbols: $l : \mathrm{Ch}(S) \leftrightarrows \mathrm{Ch}(R) : r$. A morphism $f : X \to Y$ in $\mathrm{Ch}(S)$ is a $\mathcal{J}$ -weak equivalence if and only if $l(f) : l(X) \to l(Y)$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(R)$.

Example 1.6. Let R be a commutative ring and $\mathcal{L}$ be a saturated set of ideals in R. Consider the injective class $\mathcal{E}(\mathcal{L})$ that consists of retracts of products of injective envelopes $E(R/I)$ for $I \in \mathcal{L}$. A morphism $f : X \to Y$ in $\mathrm{Ch}(R)$ is an $\mathcal{E}(\mathcal{L})$-weak equivalence if and only if $\mathrm{Hom}(H_n(f), E(R/I))$ is a bijection for any n and $I \in \mathcal{L}$. This happens if and only if the annihilator of any element in $\ker(\mathrm{H}_n(f))$ or $\mathrm{coker}(\mathrm{H}_n(f))$ is not included in any ideal that belongs to $\mathcal{L}$.

Our starting point for studying unbounded chain complexes is the following fundamental result about bounded chain complexes. It is the relative analogue of the classical result of Quillen, [13], which was the starting point for our reinterpretation of the classical case in [4]. One could prove both of these results by constructing explicitly factorizations, but we prefer to simply refer to the general work of Bousfield [2, Section 4.4].

Theorem 1.7. *The following choice of weak equivalence, cofibrations and fibrations in* $\mathrm{Ch}(R)^{\geq n}$ *satisfies the axioms of a model category:*

- $f : X \to Y$ *is called an $\mathcal{I}$-weak equivalence if* $f^* : \mathrm{Hom}_R(Y, W) \to \mathrm{Hom}_R(X, W)$ *is a quasi-isomorphism of complexes of abelian groups for any* $W \in \mathcal{I}$.
- $f : X \to Y$ *is called an $\mathcal{I}$-cofibration if* $f^i : X^i \to Y^i$ *is an $\mathcal{I}$-monomorphism for all* $i > n$.
- $f : X \to Y$ *is called an $\mathcal{I}$-fibration if* $f^i : X^i \to Y^i$ *has a section and its kernel belongs to* $\mathcal{I}$ *for all* $i \geq n$. *In particular X is $\mathcal{I}$-fibrant if* $X^i \in \mathcal{I}$ *for all* $i \geq n$.

We will often write $X \to I(X)$ for a fibrant replacement in this model category, to emphasize the fact that the modules $I(X)^k$ all belong to the class $\mathcal{I}$. Here are some basic properties of this model structure on $\mathrm{Ch}(R)^{\geq n}$:

Proposition 1.8. (1) *All objects in* $\mathrm{Ch}(R)^{\geq n}$ *are* $\mathcal{I}$*-cofibrant.*

(2) *Let* $f : X \to Y$ *be an* $\mathcal{I}$*-fibration. Then* $\mathrm{Ker}(f)$ *is fibrant.*

(3) *An* $\mathcal{I}$*-fibration* $f : X \to Y$ *is an* $\mathcal{I}$*-weak equivalence if and only if* $\mathrm{Ker}(f)$ *is* $\mathcal{I}$*-trivial. Moreover, if* f *is an acyclic* $\mathcal{I}$*-fibration, then there is an isomorphism* $\alpha : Y \oplus \mathrm{Ker}(f) \to X$ *for which the following diagram commutes:*

$$\begin{tikzcd} Y \oplus \mathrm{Ker}(f) \arrow[rr, "\alpha"] \arrow[dr, "pr"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

(4) *An* $\mathcal{I}$*-weak equivalence between* $\mathcal{I}$*-fibrant chain complexes in* $\mathrm{Ch}(R)^{\geq n}$ *is a homotopy equivalence.*

(5) *An* $\mathcal{I}$*-fibrant object in* $\mathrm{Ch}(R)^{\geq n}$ *is* $\mathcal{I}$*-trivial if and only if it is isomorphic to a complex of the form* $\bigoplus_{i\leq n} D^i(W_i)$*.*

(6) *Products of* $\mathcal{I}$*-fibrant and* $\mathcal{I}$*-trivial complexes are* $\mathcal{I}$*-trivial.*

(7) *Assume that the following is a sequence of* $\mathcal{I}$*-fibrations and* $\mathcal{I}$*-weak equivalences in* $\mathrm{Ch}(R)^{\geq n}$*:*

$$(\cdots X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0)$$

Then, for any $k \geq 0$*, the projection morphism* $\lim_{i\geq 0} X_i \to X_k$ *is an* $\mathcal{I}$*-fibration and is an* $\mathcal{I}$*-weak equivalence.*

Proof. (1): This follows from the fact that, for any R-module W, the morphism $0 \to W$ is an $\mathcal{I}$-monomorphism.

(2): This follows from the fact that, for any W, the following is an exact sequence of chain complexes of abelian groups:

$$0 \to \mathrm{Hom}_R(Y, W) \xrightarrow{\mathrm{Hom}_R(f,W)} \mathrm{Hom}_R(X, W) \to \mathrm{Hom}_R(\mathrm{Ker}(f), W) \to 0$$

(3): The first part follows from (2). If $f : X \to Y$ is an acyclic $\mathcal{I}$-fibration, then because all objects in $\mathrm{Ch}(R)^{\geq n}$ are $\mathcal{I}$-cofibrant, there is a morphism $s : Y \to X$ for which $fs = \mathrm{id}_Y$. This implies the second part of (3).

(4): This statement is a consequence of two facts: all objects in $\mathrm{Ch}(R)^{\geq n}$ are $\mathcal{I}$-cofibrant and, for any $\mathcal{I}$-fibrant chain complex $Z \in \mathrm{Ch}(R)^{\geq n}$, the standard path object $Z \subset P(Z) \xrightarrow{\pi} Z \oplus Z$ is a very good path object (THIS DISAPPEARED FROM THIS VERSION, WE CAN'T REFER To THIS) for Z in the $\mathcal{I}$-model structure on $\mathrm{Ch}(R)_{\leq n}$.

(5): According to Proposition 1.4.(8) we need to show that the assumptions X is $\mathcal{I}$-fibrant and $\mathcal{I}$-trivial imply that, for all i, $W_i := \mathrm{coker}(d_{i+1})$ belongs to $\mathcal{I}$. We do it by induction on i. For $i = n$, $\mathrm{coker}(d_{n+1}) = X_n$, which belongs to $\mathcal{I}$ as X is $\mathcal{I}$-fibrant. Assume now that $W_{i+1} \in \mathcal{I}$. This with the fact that $d_{i+1} : W_{i+1} \to X_i$ is an $\mathcal{I}$-monomorphism, implies that d_{i+1} has a retraction. It follows that $X_i = W_{i+1} \oplus W_i$. Consequently W_i, as a retract of a membre of $\mathcal{I}$, also belongs to $\mathcal{I}$.

(6): This is a consequence of (5).

(7): This is a consequence of (3) and (6). ∎

Just like in the classical case (where weak equivalences are quasi-isomorphisms), we have a model category of towers with relative equivalences. Consider as in [4] the sequence of model categories $\{\mathrm{Ch}(R)^{\geq n}\}_{n\geq 0}$, but this time with the relative model structures given by Theorem 1.7. The functor $\mathrm{in} : \mathrm{Ch}(R)^{\geq n} \subset \mathrm{Ch}(R)^{\geq n+1}$ takes (acyclic) fibrations to (acyclic) fibrations and hence the following is a sequence of Quillen pairs, where τ_n is truncation:

$$\{\tau_n : \mathrm{Ch}(R)^{\geq n+1} \rightleftarrows \mathrm{Ch}(R)^{\geq n} : \mathrm{in}\}_{n\geq 0}$$

We will denote this tower of model categories by $\mathcal{T}(R,\mathcal{I})$ and use the symbol $\mathrm{Tow}(R,\mathcal{I})$ to denote the category of towers in $\mathcal{T}(R,\mathcal{I})$ (see [4, Section **??**]). The objects are sequences of bounded chain complexes X_n together with structure morphisms $X_{n+1} \to \mathrm{in}(X_n)$. This category of towers forms a model category where weak equivalences are morphisms between towers which are levelwise $\mathcal{I}$-weak equivalences, [4, Proposition 2.3].

2. A right Quillen pair for Ch(R)

In this section we define a right Quillen pair for $\mathrm{Ch}(R)$ that has the potential to form a model approximation. We are going to use the model category $\mathrm{Tow}(R,\mathcal{I})$ described in the previous section. Recall also from [4, Lemma **??**] that there is a "tower" functor $\mathrm{tow} : \mathrm{Ch}(R) \to \mathrm{Tow}(R,\mathcal{I})$ which takes an unbounded chain complex to the tower of its truncations. It is left adjoint to the limit functor which takes an arbitrary tower to the chain complex defined as the degreewise inverse limit. We are almost ready to prove that this pair of adjoint functors forms a Quillen pair.

Lemma 2.1. *Let $K_\bullet \in \mathrm{Tow}(R,\mathcal{I})$ be a fibrant object such that, for any $n \geq 0$, K_n is $\mathcal{I}$-trivial in $\mathrm{Ch}(R)^{\geq n}$. Then $\lim(K_\bullet)$ is $\mathcal{I}$-trivial in $\mathrm{Ch}(R)$.*

Proof. Since $K_\bullet$ is fibrant in $\mathrm{Tow}(R,\mathcal{I})$, K_0 is $\mathcal{I}$-fibrant in $\mathrm{Ch}(R)^{\geq 0}$ and, for $n > 0$, the structure morphism $t_n : K_n \to K_{n-1}$ is an $\mathcal{I}$-fibration in $\mathrm{Ch}(R)^{\geq n}$. As the bounded chain complexes K_n are assumed to be $\mathcal{I}$-trivial, the $\mathcal{I}$-fibrations t_n are also $\mathcal{I}$-weak equivalences. It then follows from Proposition 1.8.(2) that $K_\bullet$ is isomorphic to the following tower of chain complexes:

$$\cdots \to M_0 \oplus M_1 \oplus M_2 \oplus M_3 \xrightarrow{\mathrm{pr}} M_0 \oplus M_1 \oplus M_2 \xrightarrow{\mathrm{pr}} M_0 \oplus M_1 \xrightarrow{\mathrm{pr}} M_0$$

where $M_0 := K_0$ and, for $n > 0$, $M_n := \mathrm{Ker}(t_n : K_n \to K_{n-1})$. It then follows that $\lim(K_\bullet) = \prod_{n\geq 0} M_n$.

The fact that M_n is $\mathcal{I}$-trivial and $\mathcal{I}$-fibrant in $\mathrm{Ch}(R)_{\leq n}$ implies that M_n is isomorphic to $\bigoplus_{i\geq n} D^i(W_{n,i})$ for some sequence $\{W_{n,i}\}_{i\leq n}$ of objects in $\mathcal{I}$ (see Proposition 1.8.(4)). Substituting this to the above product describing $\lim(K_\bullet)$ we get the following isomorphisms:

$$\lim(K_\bullet) \cong \prod_{n\geq 0} M_n = \prod_{n\geq 0} \bigoplus_{i\geq n} D^i(W_{n,i}) \cong \bigoplus_i D^i(\prod_{i\geq n} W_{n,i})$$

It is now clear that $\lim(K_\bullet)$ is $\mathcal{I}$-trivial. In fact $\lim(K_\bullet)$ is even homotopy equivalent to the trivial chain complex 0. ∎

Proposition 2.2. *The functors* $\mathrm{tow} : \mathrm{Ch}(R) \rightleftarrows \mathrm{Tow}(R, \mathcal{I}) : \lim$ *form a right Quillen pair for* $\mathrm{Ch}(R)$ *with* $\mathcal{I}$*-weak equivalences as weak equivalences.*

This right Quillen pair $\mathrm{tow} : \mathrm{Ch}(R) \rightleftarrows \mathrm{Tow}(R, \mathcal{I}) : \lim$ is going to be called the *standard Quillen pair* for $\mathrm{Ch}(R)$.

Proof. If $f : X \to Y$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(R)$, then the tower of truncations $\mathrm{tow}(f)$ is a weak equivalence in $\mathrm{Tow}(R, \mathcal{I})$ since our weak equivalences are defined by homing into certain objects W and truncation is a left adjoint. We have only to show if $f_\bullet : X_\bullet \to Y_\bullet$ is a weak equivalence in $\mathrm{Tow}(R, \mathcal{I})$ between fibrant objects, then $\lim(f_\bullet)$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(R)$.

By Ken Brown Lemma (see for example [7, Lemma 9.9]), it is enough to show the statement under the additional assumption that $f_\bullet : X_\bullet \to Y_\bullet$ is an $\mathcal{I}$-fibration. Let us define $K_\bullet$ to be an object in $\mathrm{Tow}(R, \mathcal{I})$ given by the sequence $\{\mathrm{Ker}(f_n)\}_{n\geq 0}$ with the structure morphisms being the restrictions of the structure morphisms of $X_\bullet$. Since all objects in $\mathrm{Ch}(R)_{\leq n}$ are $\mathcal{I}$-cofibrant, then so are all objects in $\mathrm{Tow}(R, \mathcal{I})$. It follows that there is $s_\bullet : Y_\bullet \to X_\bullet$ for which $f_\bullet s_\bullet = \mathrm{id}$. By applying the functor lim, we then get the following split exact sequence in $\mathrm{Ch}(R)$:

$$0 \to \lim(K_\bullet) \to \lim(X_\bullet) \xrightarrow{\lim(f_\bullet)} \lim(Y_\bullet) \to 0$$

Since $X_\bullet$ is isomorphic to $K_\bullet \oplus Y_\bullet$, as a retract of a fibrant object $X_\bullet$, the object $K_\bullet$ is then also fibrant. Moreover, for any $n \geq 0$, K_n is $\mathcal{I}$-trivial in $\mathrm{Ch}(R)^{\geq n}$. This is a consequence of the fact that f_n is an $\mathcal{I}$-equivalence in $\mathrm{Ch}(R)^{\geq n}$. We can then apply Lemma 2.1 to conclude that $\lim(K_\bullet)$ is an $\mathcal{I}$-trivial chain complex in $\mathrm{Ch}(R)$. The morphism $\lim(f_\bullet) : \lim(X_\bullet) \to \lim(Y_\bullet)$ must be then an $\mathcal{I}$-weak equivalence. It is in fact a homotopy equivalence. ∎

The question which remains now to be answered is when does this right Quillen pair form a model approximation, or in other words when is it possible to form relative injective resolutions by the procedure encoded in our pair of adjoitn functors. This is the subject of the final part of the article.

3. Noetherian rings with finite Krull dimension

In this section R is a Noetherian ring with finite Krull dimension d. Our main theorem is that the model category of towers always provides a model approximation for the category of unbounded chain complexes with W-equivalences for all choices of injective classes $\mathcal{I}$ consisting of injective modules. In this section and the next one we will rely on certains local cohomology computations which we postpone to Appendix A at the end of the article.

Theorem 3.1. *Let R be a Noetherian ring with finite Krull dimension d and $\mathcal{I}$ an injective class of injective modules. The category of towers forms then a model approximation for $Ch(R)$ equipped with $\mathcal{I}$-equivalences.*

We need some preparation before proving this theorem. The key ingredient is the vanishing of the homology of a W-relative resolution above the Krull dimension of the ring.

Lemma 3.2. *Let R be a local Noetherian ring with finite Krull dimension d, p be any prime ideal, and $I \in Ch(R)^{\geq 0}$ be an injective resolution of a module M. The complex $I(p)$ obtained from I by keeping only the copies of $E(R/p)$ has no homology in degrees $> d$.*

Proof. To see why this is a complex one can localize I at p so as to obtain a complex made of copies of $E(R/q)$ for $q \subset p$ by Lemma A.2. The subcomplex $\Gamma_p(I \otimes R_p)$ is then precisely $I(p)$ by Lemma A.5. The ring R_p is flat so the cohomology of this complex computes the local cohomology of the module M_p with support in p. We conclude then by Proposition A.7 and Remark A.8 because the Krull dimension of R_p is at most $d-1$. ∎

In [3] we have seen that an injective class of R-modules is completely determined by the *saturated* set of ideals of the form $\mathrm{ann}(x) = \{r \in R \mid rx = 0\}$ for some non-zero element $x \in W \in \mathcal{I}$. When R is Noetherian it is even sufficient to consider the prime ideals in this set. This subset is generization closed.

Proposition 3.3. *Let R be a local Noetherian ring with finite Krull dimension d and $\mathcal{I}$ an injective class of injective modules. For any module M and a $\mathcal{I}$-relative resolution $I \in Ch(R)^{\geq 0}$, $H^k(I) = 0$ if $k > d+1$.*

Proof. Let S be the generization closed subset of $Spec(R)$ corresponding to the injective class $\mathcal{I}$. Let a be the length of the maximal chain of prime ideals in the complement of S. If $a = 0$, $\mathcal{I}$ consists of all injective modules, so that $H^k(I) = 0$ for all $k > 0$. If $a = 1$, S consists of all prime ideals except the maximal ideal m. In this case it follows from Remark A.6 that the higher cohomology modules coincide up to a shift with the higher local cohomology modules $H^k_m(M)$. As m can be generated up to nilpotency by d elements and since local cohomology only depends on the radical of the ideal, the local cohomology modules vanish in degrees $> d$.

Let us thus assume that $a > 1$. We prove the proposition by induction on a. Consider the minimal ideals p_i in the complement of S so that we know the result is true for the injective class corresponding to the set $S' = S \cup \{p_i\}$. We denote by I' the $\mathcal{I}'$-relative resolution of M made of copies of $E(R/p)$ with $p \in S'$. From Lemma A.3 we see that I can be constructed from I' by taking the quotient by the subcomplex I'' made of copies of modules $E(R/p_i)$. The long exact sequence in homology reduces the question of the vanishing of the homology modules of I in degrees $> d+1$ to the vanishing of the homology modules I'' in degrees $> d$. But we infer from Lemma A.3 again that the complex I'' splits into a direct sum of the complexes $I(p_i)$ that we have introduced in Lemma 3.2. We conclude by this lemma. ∎

Here is a last proposition we will apply in the proof of our main theorem to measure the difference between the resolutions of a bounded complex and a truncation. Recall that $I(X)$ denotes the fibrant replacement of the bounded complex X in the $\mathcal{I}$-relative model structure described in Theorem 1.7, i.e. an $\mathcal{I}$-relative injective resolution of X.

Proposition 3.4. *Let R be a local Noetherian ring with finite Krull dimension d and $\mathcal{I}$ an injective class of injective modules. Let $X \in Ch(R)^{\geq 0}$ be a bounded complex and $\tau_1 X$ its first truncation. Then the canonical morphism $X \to \tau_1 X$ induces isomorphisms in cohomology $H^k(I(X)) \to H^k(I(\tau_1 X))$ for any $k > d+1$.*

Proof. Let us replace $X \to \tau_1 X$ by an $\mathcal{I}$-fibration $I(X) \to I(\tau_1 X)$ between $\mathcal{I}$-fibrant objects. The kernel K is a chain complex made of injective modules in $\mathcal{I}$, and forms therefore an $\mathcal{I}$-fibrant replacement for $H^0(X)$, the kernel of the canonical morphism.

From the previous proposition we know that $H^k(K) = 0$ if $k > d+1$ (since R_p is flat and by Lemma A.1 it is sufficient to prove that the same holds for the localized complex at p for all prime ideals p). The long exact sequence in cohomology allows us to conclude. ∎

Proof of Theorem 3.1. To show that the Quillen pair is in fact a model approximation, we must check that Condition (4) of [4, Definition **??**] holds, or equivalently that the canonical morphism $\lim I(\mathrm{tow} X) \to X$ is an $\mathcal{I}$-equivalence for any unbounded chain complex X. We have learned from Proposition 3.4 that the cohomology of $I(\tau_n X)$ and $I(\tau_{n-1} X)$ only differ in low degrees (in degrees $< n + d + 1$). This means that the cohomology of the $\mathcal{I}$-fibrant replacement of the tower $\mathrm{tow}(X)$ stabilizes. Therefore $H^k(\lim I(\mathrm{tow} X)) \cong H^k(I(\tau_{k+d+1} X)$. ∎

4. A bad Noetherian ring

The objective of this section is to show that even under the Noetherian assumption towers do not always approximate unbounded chain complexes. We have seen in the previous section that no problems arise when the Krull dimension is finite. However when the Krull dimension is infinite it is always possible to find a problematic injective class. Let us first briefly recall the typical example of Noetherian ring with infinite Krull dimension, as constructed by Nagata in the appendix of [12].

Example 4.1. Let k be a field and consider the polynomial ring on countably many variables $A = k[x_1, x_2, \dots]$. Choose prime ideals $p_1 = (x_1)$, $p_2 = (x_2, x_3)$, $p_3 = (x_4, x_5, x_6)$ such that the depth of p_i is precisely i. Take S to be the multiplicative set consisting of elements of A which are not in any of the p_i's. The localized ring $R = S^{-1}A$ is Noetherian, but of infinite Krull dimension. In fact its maximal ideals are $m_i = S^{-1}p_i$, a sequence of ideals of strictly increasing height.

We choose now C to be the specialization closed subset of $\mathrm{Spec}(R)$ consisting of the m_i's. For us this means that we will do relative homological algebra with respect to the injective class $\mathcal{I}$ of injective R-modules generated by the injective envelopes $E(A/p)$ for all prime ideals $p \neq m_i$ for all $i \geq 1$. We noticed earlier that the class of $\mathcal{I}$-acyclic chain complexes is a localizing subcategory of $D(R)$. As it contains R/m_i but not any other R/p, we know from Neeman's classification that this localizing subcategory is precisely generated by $\oplus R/m_i$.

Lemma 4.2. *Let R be Nagata's ring described in Example 4.1 and denote by $I(R)$ an $\mathcal{I}$-injective resolution of R. For any $i > 0$ we have an isomorphism $H^{i-1}(I(R)) \cong E(R/m_i)$.*

Proof. Let us consider a minimal injective resolution I of the ring R considered as a chain complex concentrated in degree 0. By the description Matlis gave of injective modules, each I^n is a direct sum of modules of the form $E(R/p)$ where p runs over all prime ideals of R.

By Lemma A.3 we see that there is a subcomplex K of I made of copies of $E(R/m_i)$ and we take $I(R) = I/K$. This is a fibrant replacement for R in the relative model structure described in Theorem 1.7. Since the cohomology of I is concentrated in degree 0, we see from the long exact sequence in cohomology for the short exact sequence of complexes $K \to I \to I(R)$ that the cohomology modules of $I(R)$ are isomorphic to those of K up to a shift. But K splits as a direct sum $\oplus_i \Gamma_{m_i}(I)$ by Lemma A.3 again. Therefore $H^*(K) \cong \oplus H^*_{m_i}(R) \cong \oplus_i H^*_{m_i}(R_{m_i})$, where the second isomorphism comes from Lemma 3.2. The local ring R_{m_i} is regular, hence Gorenstein, of dimension i. Therefore the computation done in [10, Theorem 11.29] yields that $H^{i-1}(K) \cong E(R/m_i)$. ∎

We consider now the unbounded chain complex X with $X^n = R$ for all n and zero differential. The zeroth truncation of X is the non-positively graded complex with zero differential and where every module is R, in other words this complex is $\oplus_{i\geq 0}\Sigma^i R$. We know how to construct explicitly an $\mathcal{I}$-relative resolution for this bounded complex by the previous lemma: it is a direct sum $\oplus_{i\geq 0}\Sigma^i I(R)$.

Lemma 4.3. *Let X be the unbounded complex $\oplus_i \Sigma^i R$, $\tau_0 X$ be its zeroth truncation, and $I(\tau_0 X)$ denote the $\mathcal{I}$-relative resolution of the latter. Then $H^{i-1}(I(\tau_0 X)) \cong R \oplus \oplus_{j\leq i} E(A/m_j)$ for any $i \geq 1$.*

Proof. This is a direct consequence of the previous lemma. ∎

Let us now consider the tower approximation of the complex X. By definition it is the limit of the tower given by the $\mathcal{I}$-relative resolution of the truncations of X. From the previous lemma the nth level of this tower is $\oplus_{i\geq -n}\Sigma^i I(R)$ and the structure maps are the projections. Therefore the limit is the product $\prod_i \Sigma^i I(R)$. In particular the $(i-1)$-st cohomology module is isomorphic to the product $R \times \prod_{j\geq 1} E(A/m_j)$.

Theorem 4.4. *For the choice of the Noetherian ring R and the injective class $\mathcal{I}$, the category of towers $Tow(R, \mathcal{I})$ does not form a model approximation for the unbounded chain complexes. More precisely there exists a complex X which is not $\mathcal{I}$-equivalent to the limit of the fibrant replacement of its truncation tower.*

Proof. The complex X is the one we have constructed above, namely $\oplus_{i=-\infty}^{\infty}\Sigma^i R$. We have just computed the homology of Y the limit of the fibrant replacement of its truncation tower. The homotopy fiber of the natural map $X \to Y$ is an unbounded complex whose homology is $\prod_{j\geq 1} E(A/m_j)$ in each degree. This complex cannot be $\mathcal{I}$-acyclic since the annihilator of the image of 1 via the composite map $A \to \prod_i A \to \prod_i A/m_i \to \prod_i E(A/m_i)$ is zero. ∎

Appendix A. Some facts about local cohomology

In this section R is a Noetherian ring. We will recall a few elementary (and well-known to the algebraists) facts about localization, injective envelopes, and local cohomology. For a prime ideal p, we denote by M_p the localization of an R-module M at p. The first lemma will allow us to reduce certain problems to the case of a local ring, namely R_p.

Lemma A.1. *An R-module M is zero if and only if M_p is zero for all prime ideals p.*

Proof. Let us assume that M is non-zero, but $M_p = 0$ for any prime ideal p. We choose a non-zero element $x \in M$ and consider its annihilator. This ideal is contained in a maximal ideal m and since $M_m = 0$, there must exist an element $r \in R \setminus m$ such that $rx = 0$, a contradiction. ∎

A theorem of Matlis, [11], describes the injective modules as direct sums of injective hulls $E(R/p)$ of quotients of the ring by prime ideals. The following two lemmas give some properties of these indecomposable injective modules.

Lemma A.2. *If $q \subset p$, the module $E(R/q)$ is p-local, and otherwise $E(R/q)_p = 0$.*

Proof. Assume $q \subset p$ and fix $r \notin p$. The multiplication by r on $E(R/q)$ is an ismorphism, so $E(R/q)$ is p-local. Assume now that $q \not\subset p$. Then $q^m \not\subset p$ for any $m \geq 1$. If x is any element of $E(R/q)$, its annihilator is q^m for some positive integer m since $E(R/q)$ is q-torsion. There exists thus an element $s \in q^m$ which does not belong to p and such that $sx = 0$. Hence $x_p = 0$. This shows that $E(R/q)_p = 0$. ∎

Lemma A.3. *The R-module of homomorphisms* $\mathrm{Hom}_R(E(R/p), E(R/q))$ *is non-zero if and only if* $p \subset q$.

Proof. Since $E(R/q)$ is q-local by the previous lemma, any homomorphism factors through the q-localization of $E(R/p)$, which is zero unless $p \subset q$. ∎

Let us now introduce local cohomology, a subject which at least one of the authors enjoyed learning in [10].

Definition A.4. Given an ideal p in R, the *p-torsion* of an R-module M is the submodule $\Gamma_p(M)$ of elements with annihilator p^m for some positive integer m. The *local cohomology* modules $H_p^*(-)$ with support in p are the right derived functors of Γ_p.

Explicitly, to compute the local cohomology of a module M, we construct an injective resolution $I_\bullet$ of M and compute $H_p^j(M) = H^j(\Gamma_p(I_\bullet))$. Our last lemma helps us to understand how this p-torsion injective complex look like.

Lemma A.5. *The p-torsion module* $\Gamma_p(E(R/q)) = E(R/q)$ *if* $p \subset q$ *and is zero otherwise.*

Proof. Again this follows from the fact that $E(R/q)$ is q-torsion. ∎

Remark A.6. Let R be a local ring with maximal ideal m and let us consider the generization closed subset of *Spec*(R) given by $S = \{q \,|\, q \neq m\}$. It yields the injective class W generated by all injective envelopes $E(R/q)$ with $q \neq m$. Given a module M and an injective resolution $I_\bullet$, we have a triangle in the derived category $\Gamma_m(I_\bullet) \to I_\bullet \to W_\bullet$, where $W_\bullet$ is a W-relative injective resolution of M. In particular $H_k(W_\bullet) \cong H_m^{k+1}(M)$ for $k \geq 2$.

Proposition A.7. *Let R be a Noetherian ring and p be the radical of* $(x_1, \dots, x_n)$. *Then* $H_p^k(M) = 0$ *for any* $k > n$ *and any module* M.

Proof. Since the torsion functor does not see the difference between an ideal and its radical, we can assume that $p = (x_1, \dots, x_n)$. Then the local cohomology can be computed by means of the Čech complex $\otimes_i \check{C}(x_i, R) \otimes M$, [10, Theorem 7.13]. Here $\check{C}(x, R)$ is the complex $0 \to R \to R_x \to 0$ concentrated in degrees 0 and 1. The Čech complex is thus concentrated in degrees $\leq n$. ∎

Remark A.8. If R is a Noetherian local ring of dimension d, then the maximal ideal can always be expressed as the radical of an ideal generated by n elements, see [10, Theorem 1.17].

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