New approaches to geometric structures: generalized and complex Dirac geometry

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Universitat Autònoma
de Barcelona

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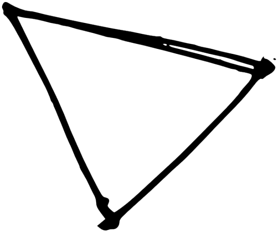
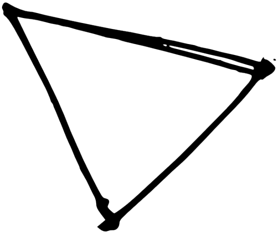
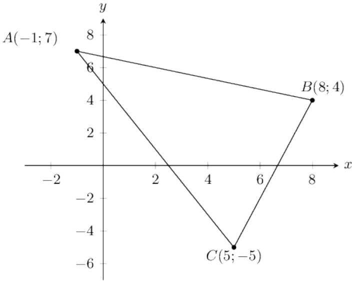
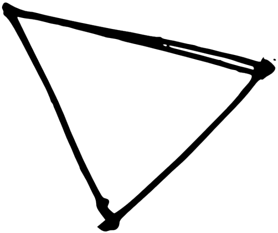
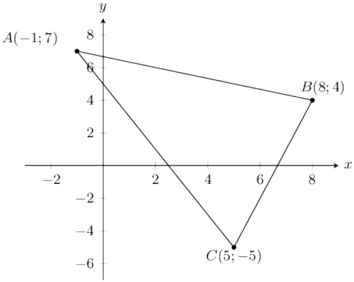
Barcelona, 5 September 2022






y
A(−1; 7)
8
6
B(8; 4)
4
2
x
−2
2
4
6
8
−2
−4
−6
C(5; −5)

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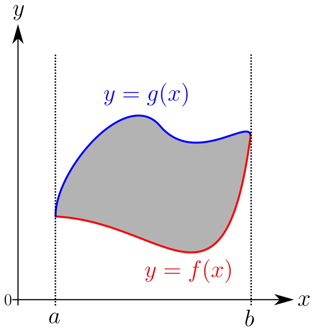
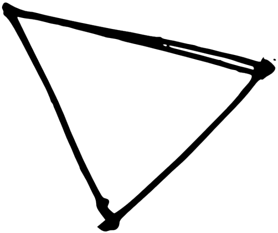
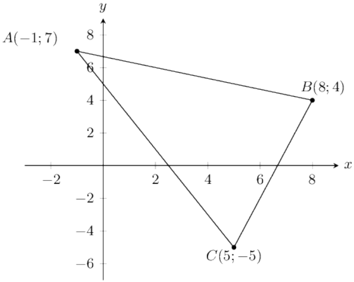
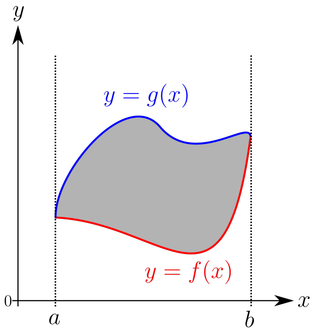
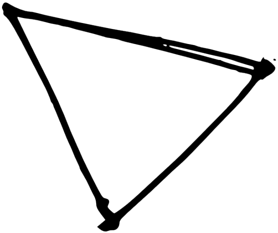

y
y = g(x)
y = f(x)
0
a
b
x

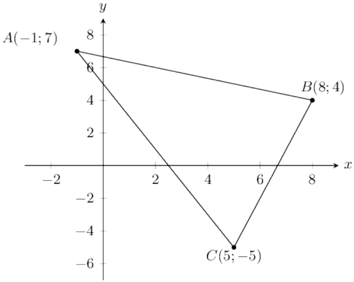


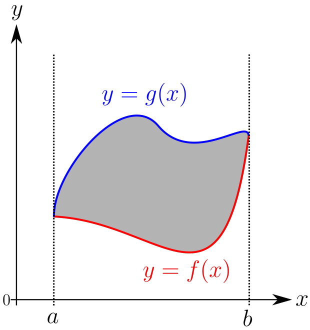




$$\mathbb{R}^2$$





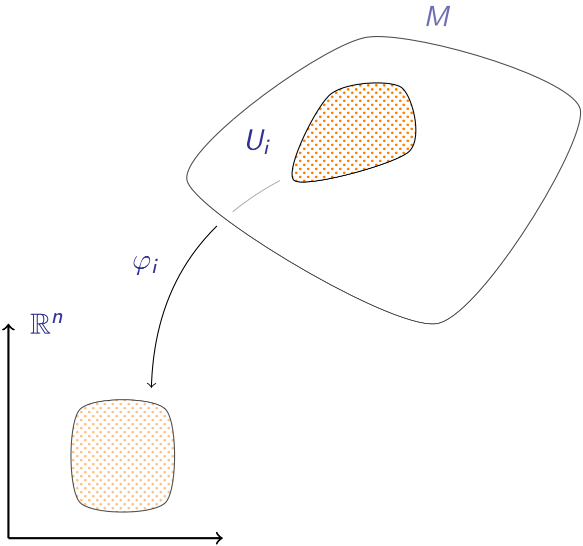


$$\mathbb{R}^2$$

$$\mathbb{R}^n$$

How to do geometry/analysis beyond $\mathbb{R}^n$.

On a set M:



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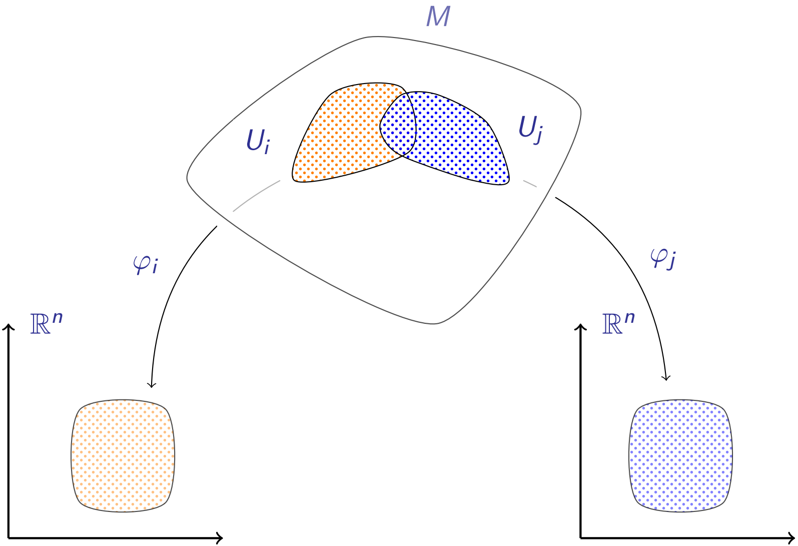

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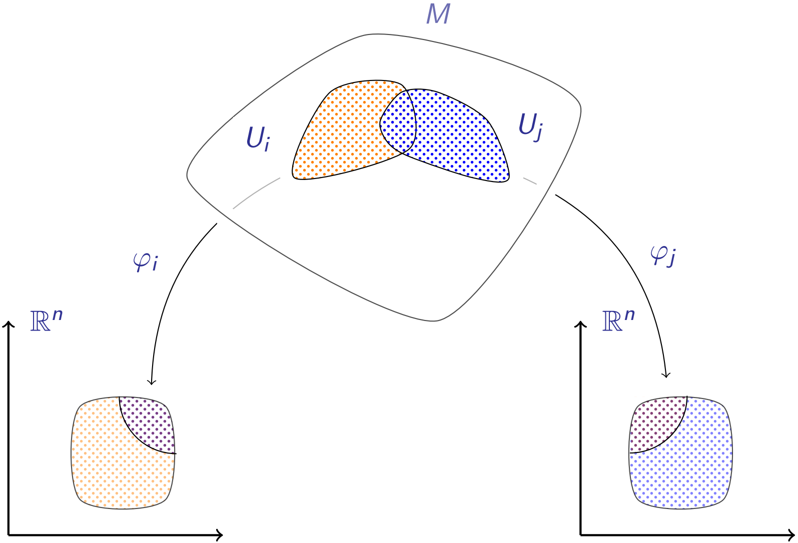

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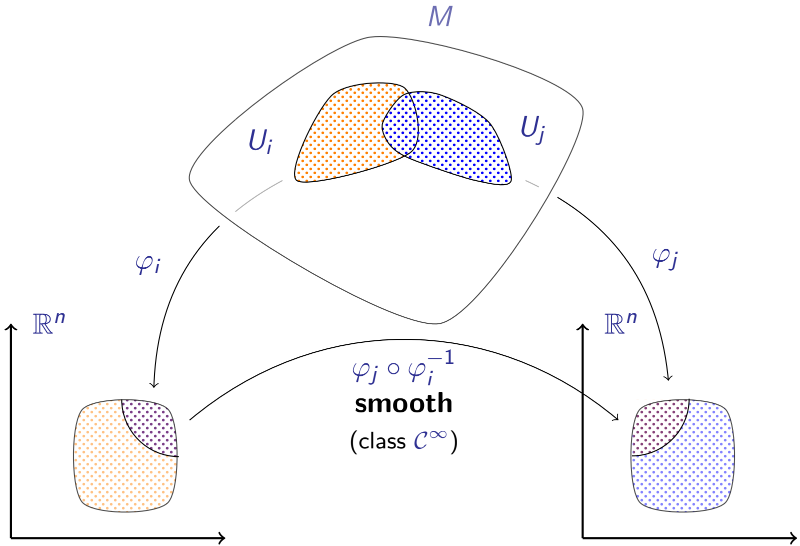
M
U_i
U_j
φ_i
φ_j
$\mathbb{R}^n$
$\mathbb{R}^n$
$\varphi_j \circ \varphi_i^{-1}$
smooth
(class $\mathcal{C}^\infty$)

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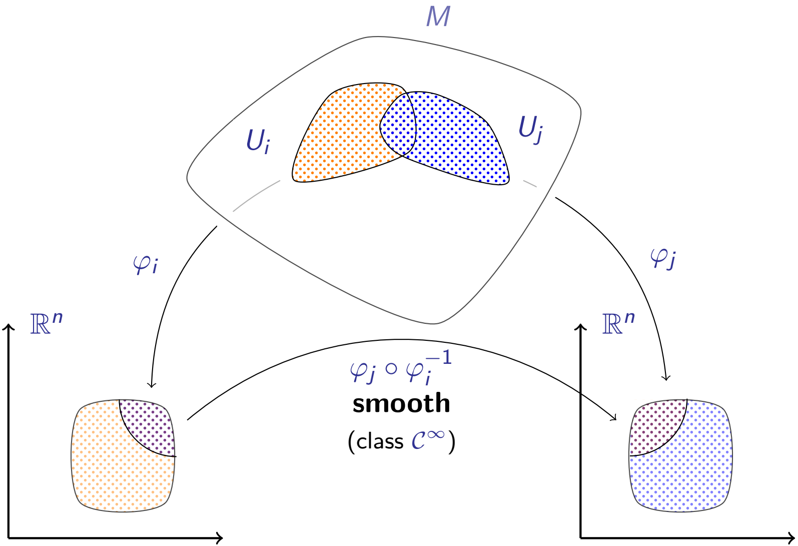


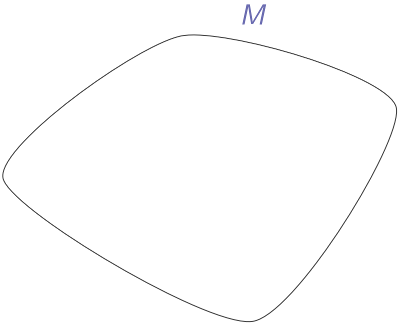
smooth

(class $\mathcal{C}^\infty$)

M gets a topology

(usually required to be Hausdorff + countable basis)

M is called a (smooth) **manifold**:



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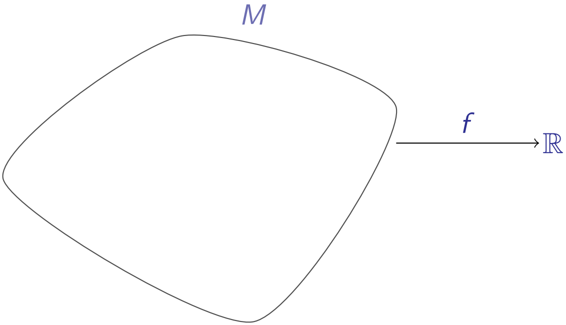
We say $f : M \to \mathbb{R}$ is smooth...



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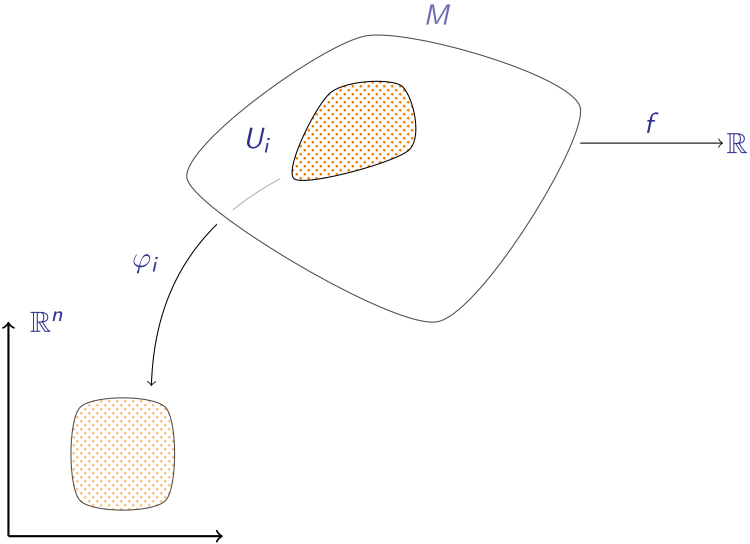

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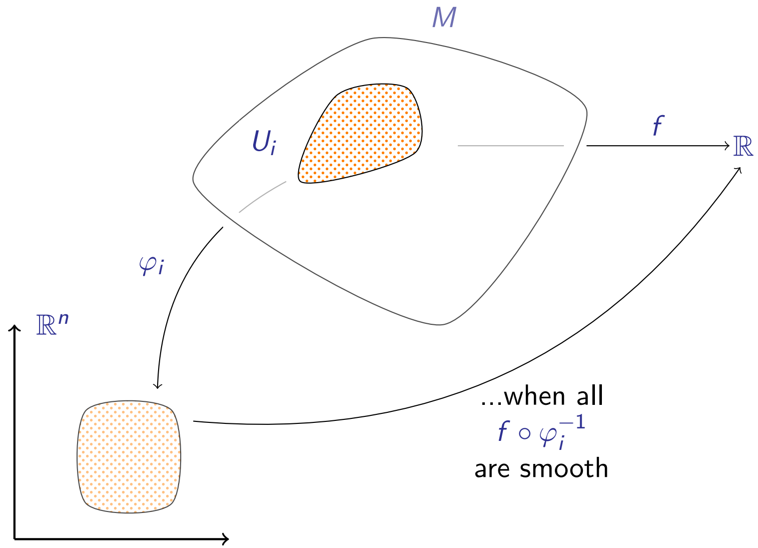
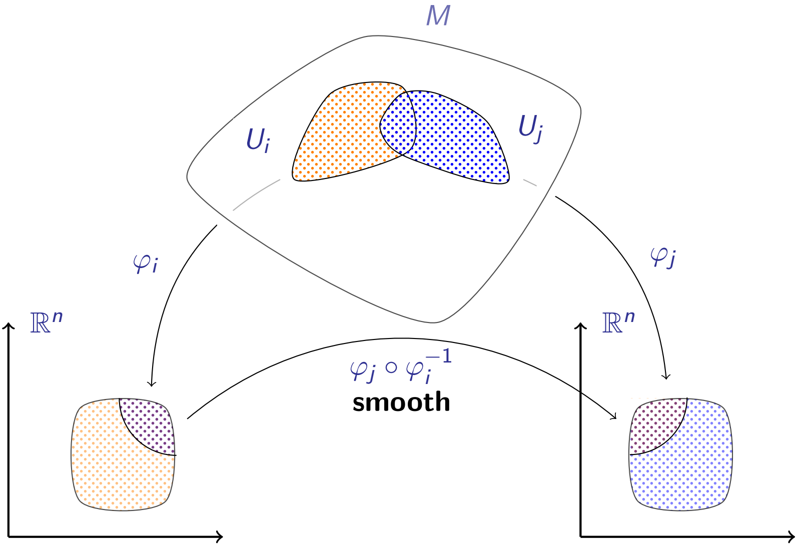


...when all

$f \circ \varphi_i^{-1}$

are smooth

A geometric structure is
an enrichment of the local
model and changes of chart of M.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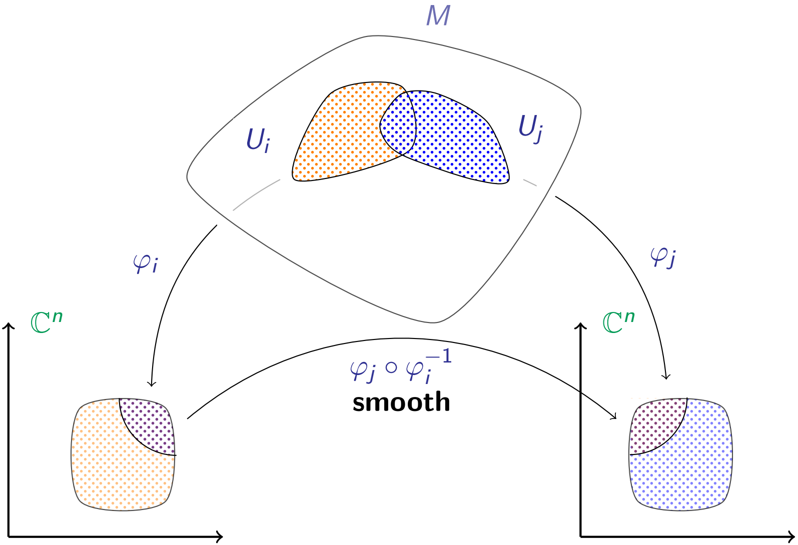
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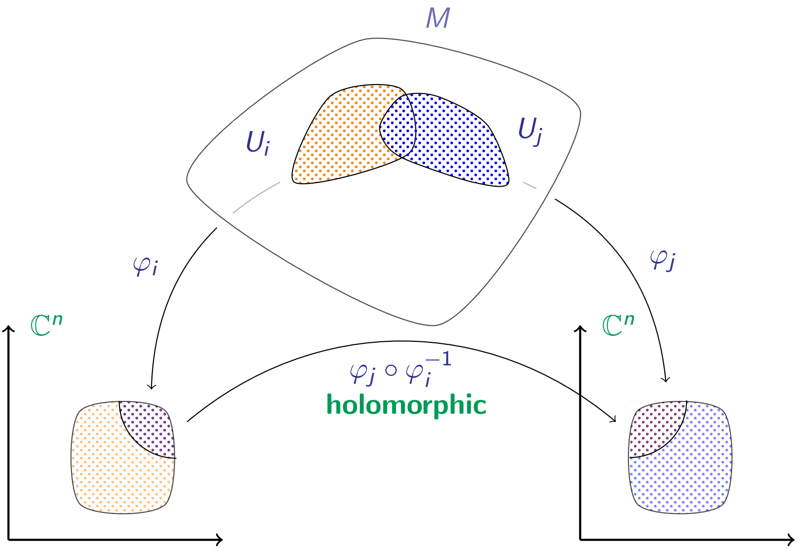
smooth



M
U_i
U_j
φ_i
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$\mathbb{C}^n$
$\mathbb{C}^n$
$\varphi_j \circ \varphi_i^{-1}$
holomorphic

Charts are complicated...

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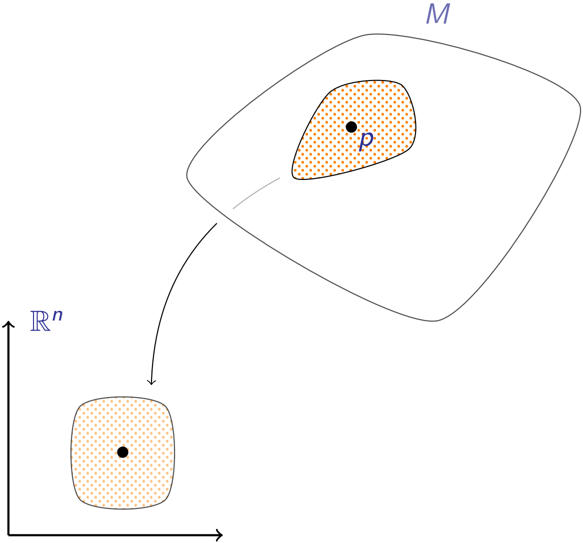

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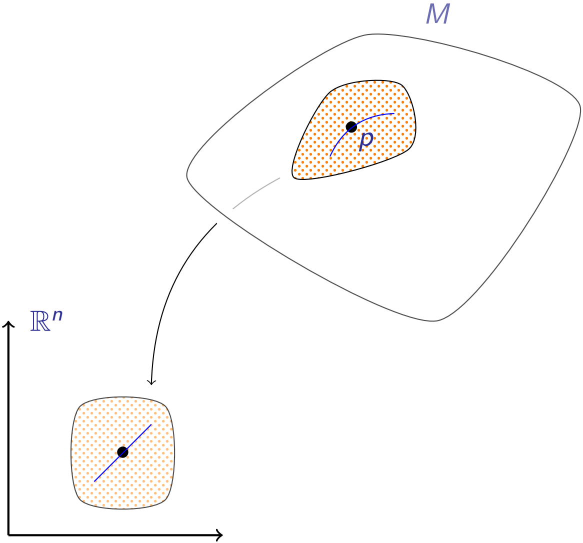

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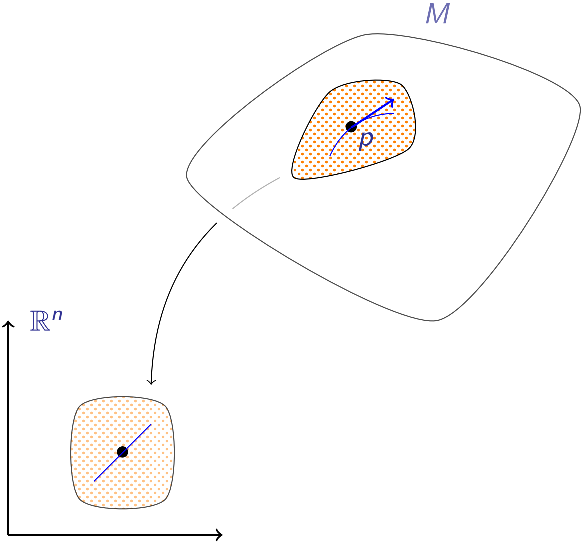

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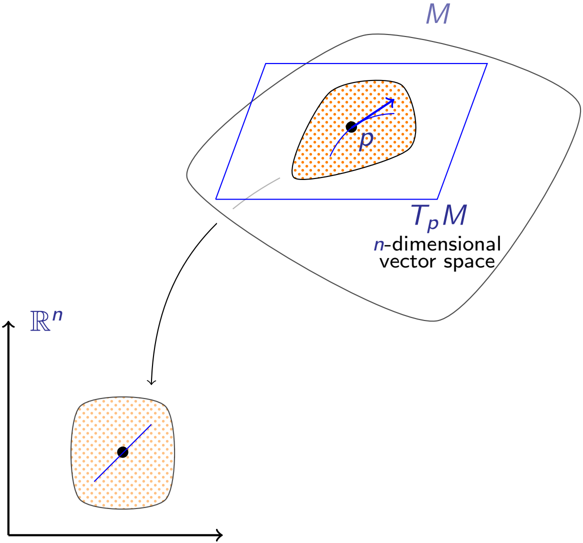

…vector spaces are easier to handle.

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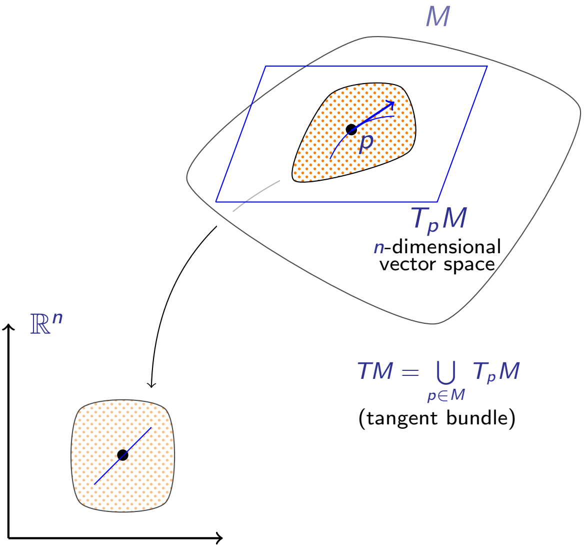

$$TM = \bigcup_{p \in M} T_pM$$

(tangent bundle)

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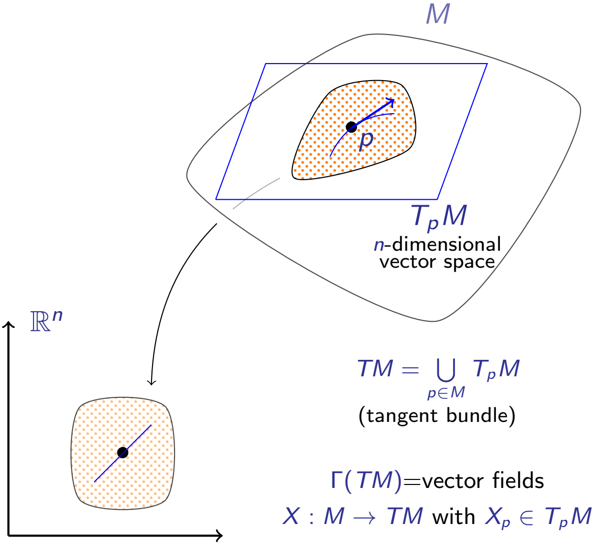

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(tangent bundle)

$\Gamma(TM)$=vector fields

$X : M \to TM$ with $X_p \in T_pM$

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We bring the geometric structure to the tangent bundle:

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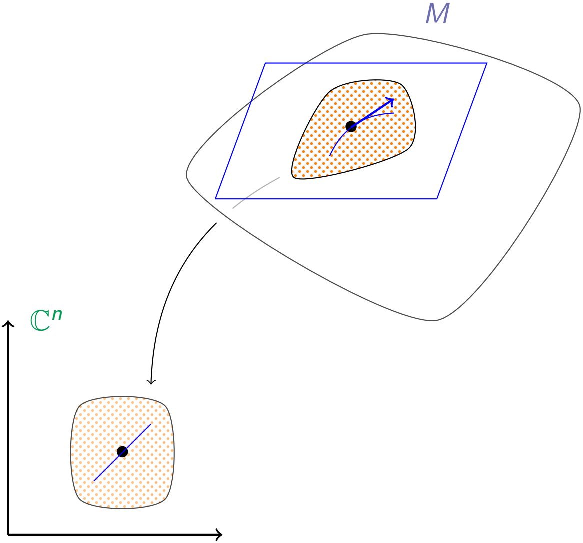

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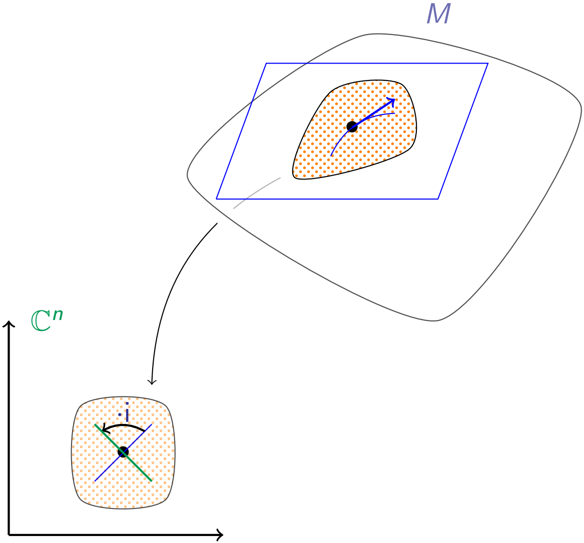

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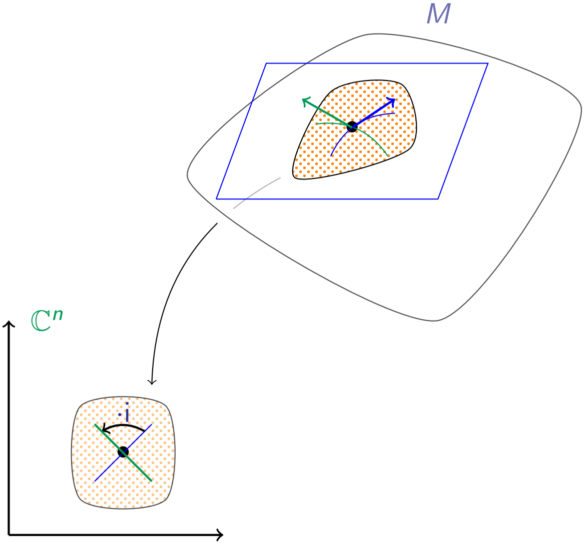

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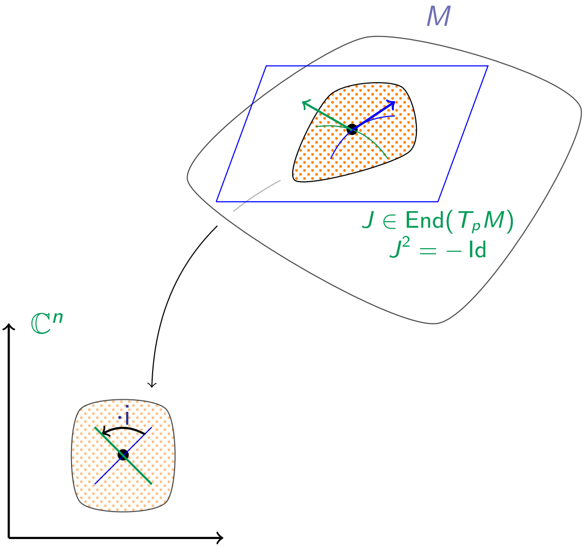

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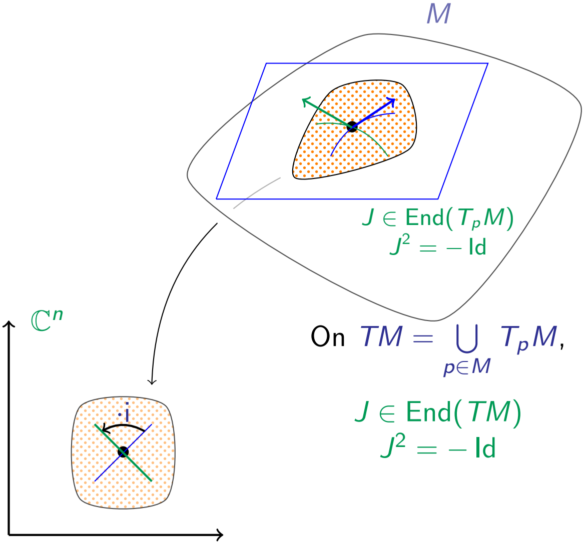

On $TM = \bigcup_{p \in M} T_pM$,

$$J \in \mathrm{End}(TM)$$
$$J^2 = -\mathrm{Id}$$

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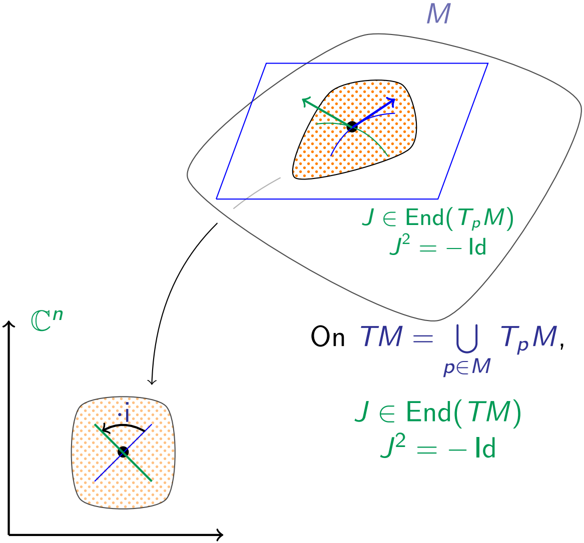

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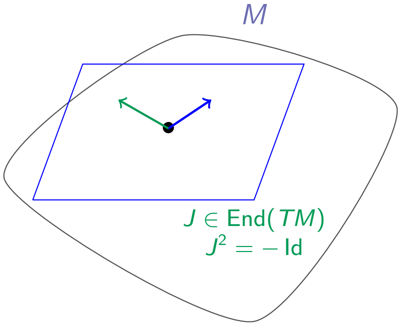
...but we loose information.

Starting with J (almost complex manifold):



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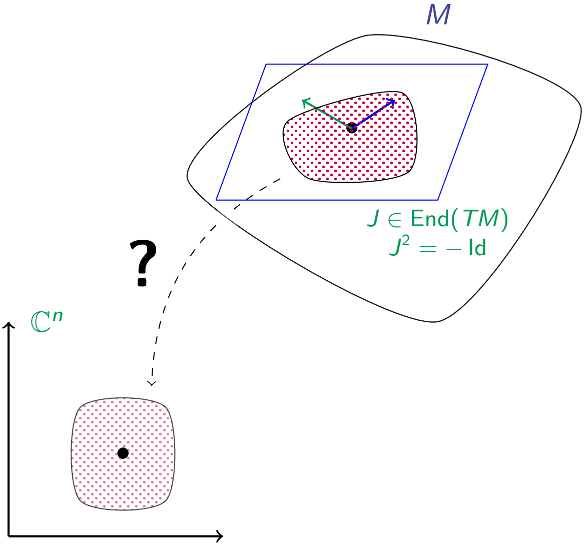

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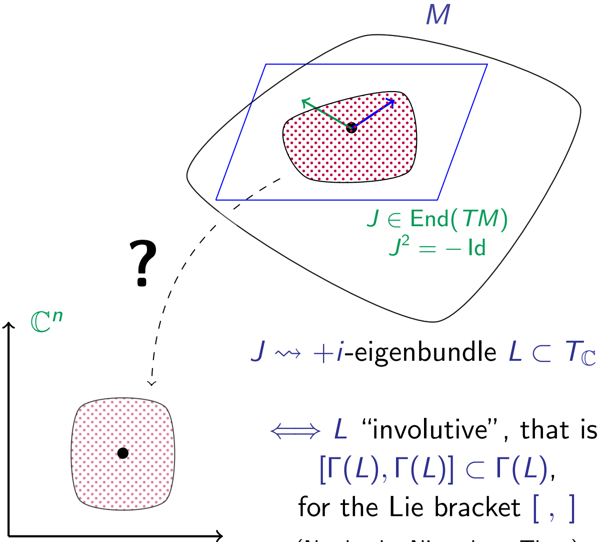

$J \rightsquigarrow +i$-eigenbundle $L \subset T_{\mathbb{C}}M$

$\Longleftrightarrow$ L "involutive", that is,
$[\Gamma(L), \Gamma(L)] \subset \Gamma(L)$,
for the Lie bracket $[\ ,\ ]$

(Newlander-Nierenberg Thm.)

High school Physics revisited:

$$F = m \cdot a$$

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Position $x \in \mathbb{R}^n$

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$$\nabla_p (p^2/2m) = \quad \frac{dx}{dt}$$

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Define $H = \varphi + p^2/2m$. Notation $dH = \nabla_x H \ \cdot \ dx + \nabla_p H \ \cdot \ dp$.

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A solution/trajectory $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n}$ (phase space) satisfies

$$dH = (\textstyle\sum_j dx_j \wedge dp_j)(\frac{d(\mathbf{x},\mathbf{p})}{dt})$$

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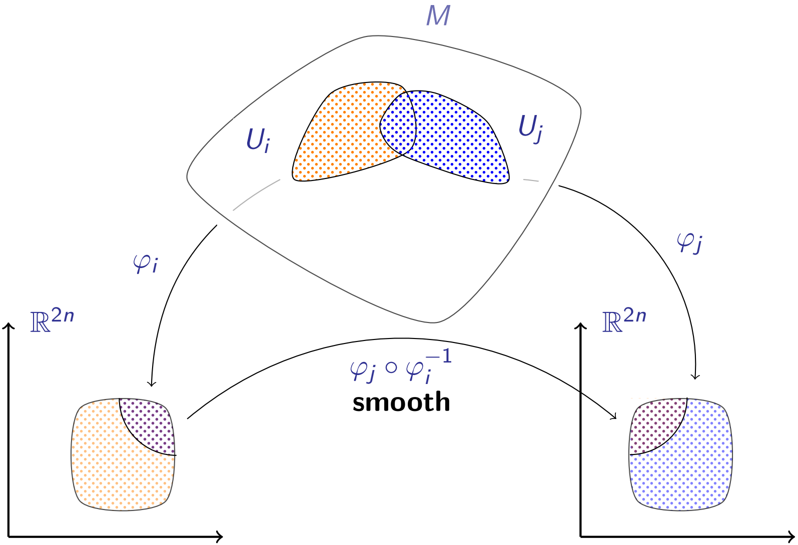
$\omega_0 := \sum_j dx_j \wedge dp_j$ gives a correspondence between vectors and 1-forms


M
U_i
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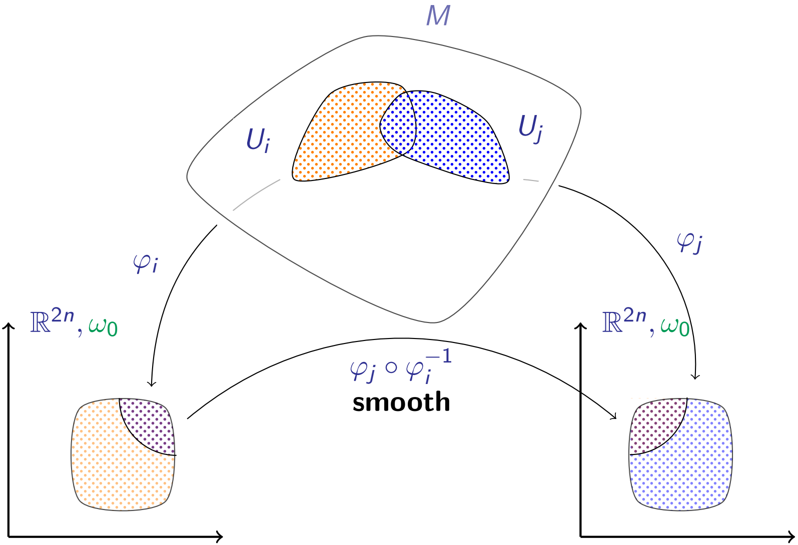
$\varphi_j \circ \varphi_i^{-1}$
smooth


M
U_i
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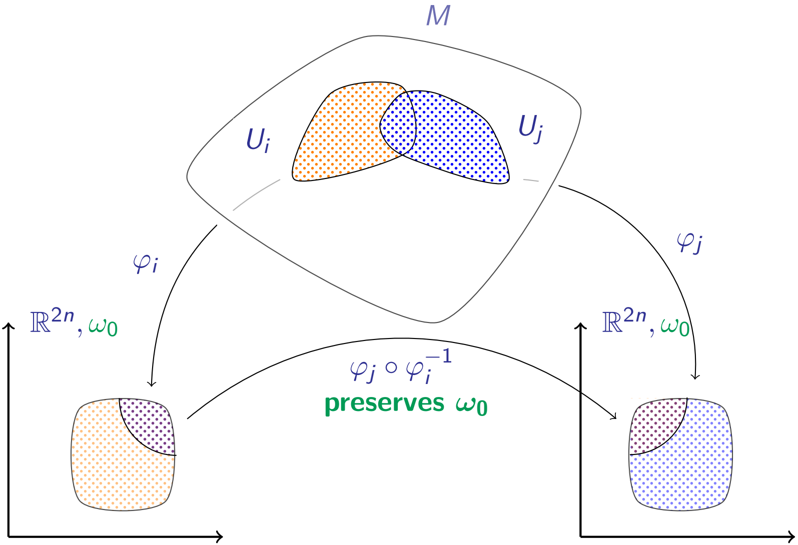
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preserves ω_0

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  (such an ω is called a 2-form and denoted by $\omega \in \Gamma(\wedge^2 T^*M)$ or $\omega \in \Omega^2(M)$)

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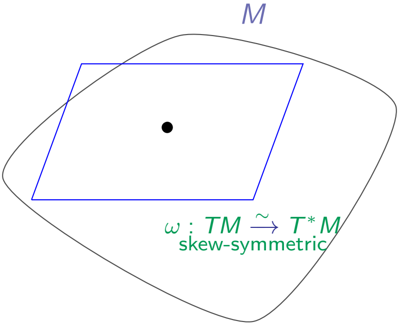
When does ω come from local charts $(\mathbb{R}^{2n}, \sum_j dx_j \wedge dp_j)$?

Starting with ω (non-degenerate 2-form):



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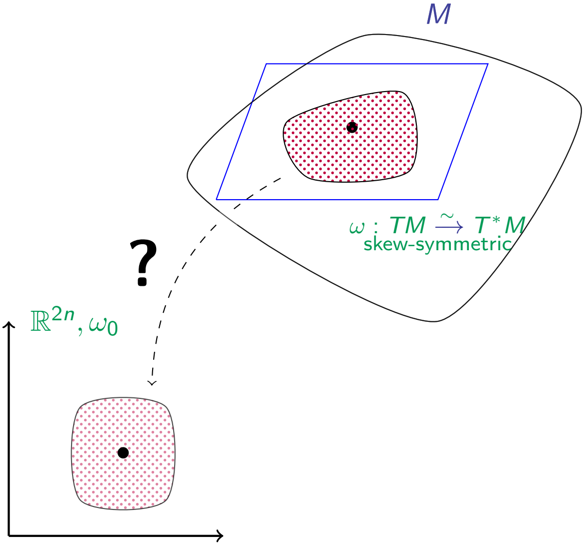

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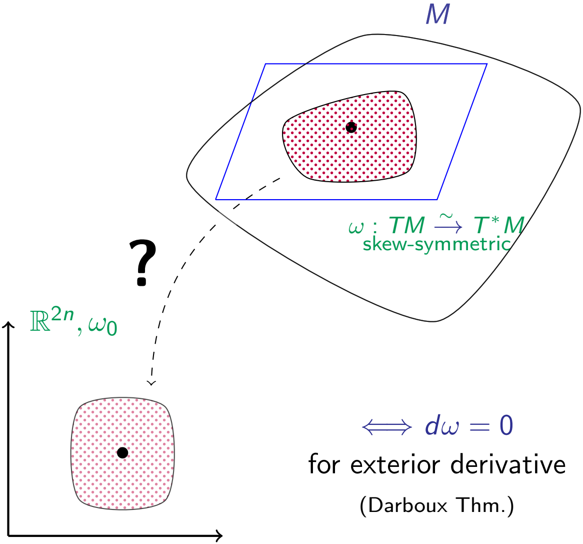

$\iff d\omega = 0$

for exterior derivative

(Darboux Thm.)

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- non-degenerate: $\omega : TM \xrightarrow{\sim} T^*M$ (trajectories are unique)

When does ω come from local charts $(\mathbb{R}^{2n}, \sum_j dx_j \wedge dp_j)$?

- closed: $\mathrm{d}\omega = 0$ (time-independent)

  (Darboux Thm.)

Again, charts are complicated...

Before we built TM, we considered $T_{\mathbb{C}}M$, $L \subset T_{\mathbb{C}}M$...

We can also consider T^*M.

Then, $\sum_j dx_j \wedge dp_j$ brought to TM is

$$\omega : TM \to T^*M$$

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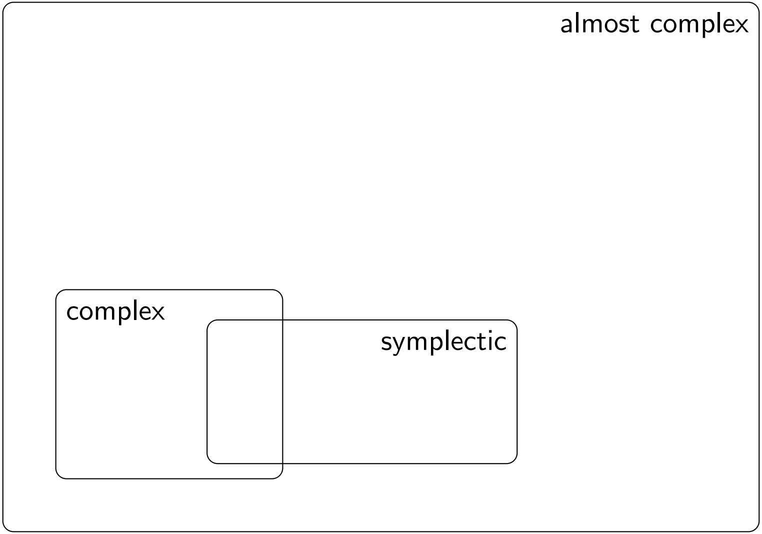
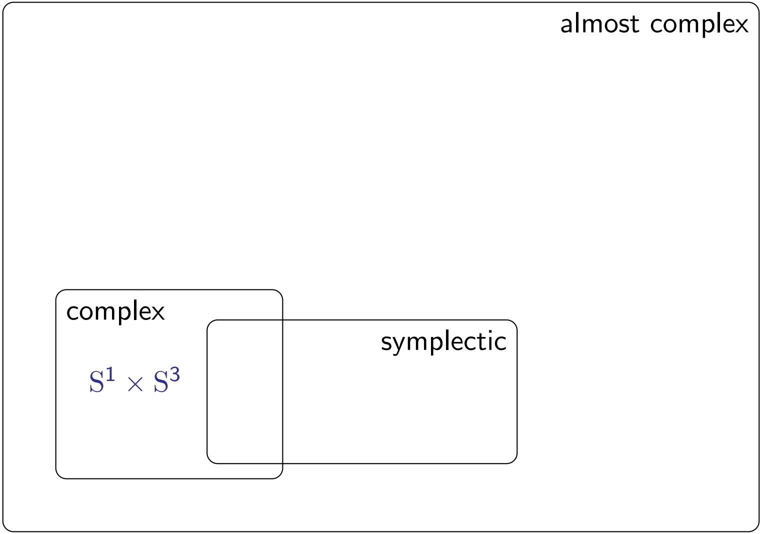

almost complex
complex
symplectic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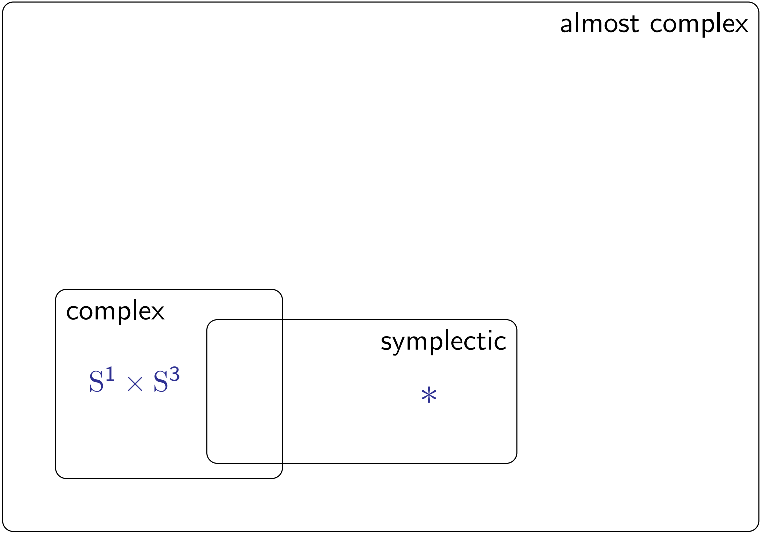
$S^1 \times S^3$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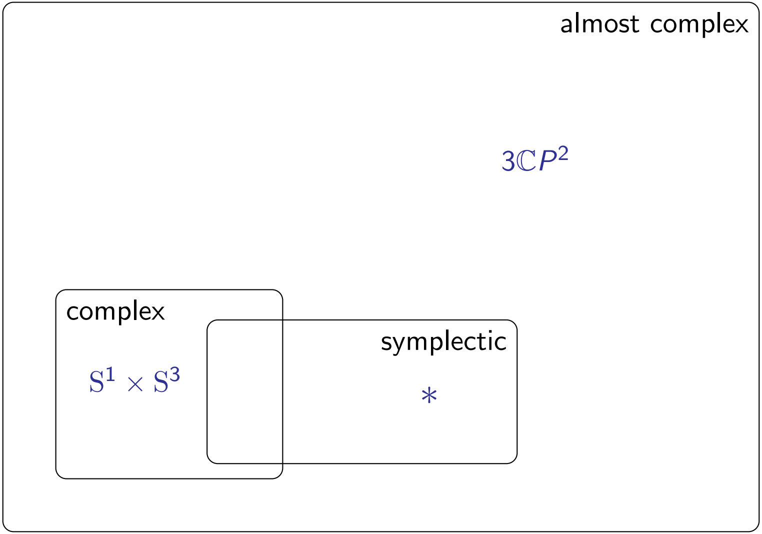
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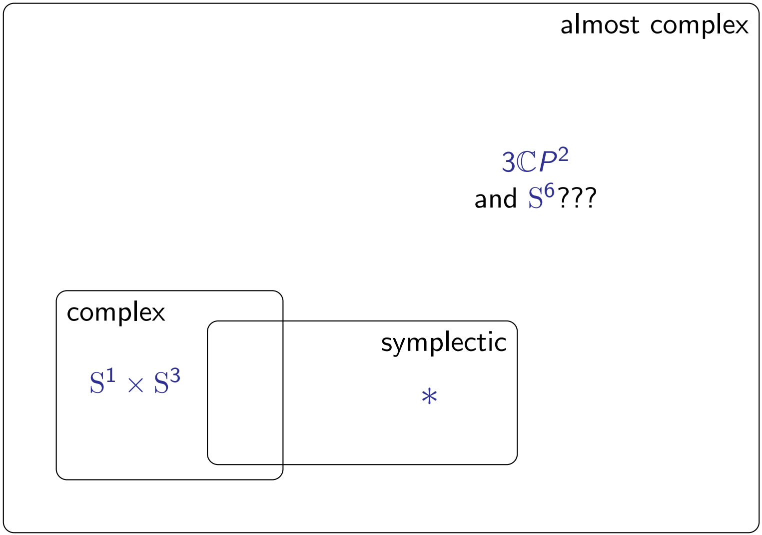

almost complex

$3\mathbb{C}P^2$

and S^6???

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symplectic

$*$

TM

TM

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$J^2 = -\,\mathrm{Id}$

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skew-symmetric

Generalized complex geometry (Hitchin-Gualtieri'03)

$$TM + T^*M$$

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Here, $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$,
involutive...
but do we even have
a bracket on $\Gamma(L)$?

The Dorfman bracket

$$[X+\alpha, Y+\beta] = [X, Y] + \mathcal{L}_X \beta - \imath_Y d\alpha$$

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Def.: generalized complex structure
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An equivalent formulation

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A generalized complex structure $\mathcal{J}$ is equivalent to a

lagrangian and involutive $L \subset T_{\mathbb{C}}M + T^*_{\mathbb{C}}M$ such that $L \cap \overline{L} = \{0\}$

An invariant for $\mathcal{J}$

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Two examples:

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$\underset{\text{type}}{\longrightarrow}$

symp. ● ● ● ● ● cplx.

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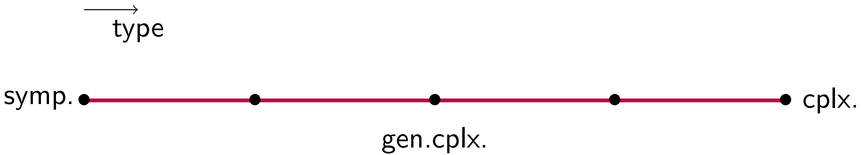


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- *The type determines (up to equivalence) the structure at each point.*
- *At each point there are some symplectic directions and some transversal complex directions.*
- **But the type may vary within a manifold!** *Preserving the parity and upper continuously. No unique local model.*

Why generalized complex geometry?

1. Complex and symplectic become the same structure

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- Interaction of complex and symplectic in mirror symmetry

- Extended deformation space of Barannikov and Kontsevich
(complex structures are deformed into symplectic ones)

- Other, like coisotropic A-branes...

2. Provides a new language, more suitable in some cases

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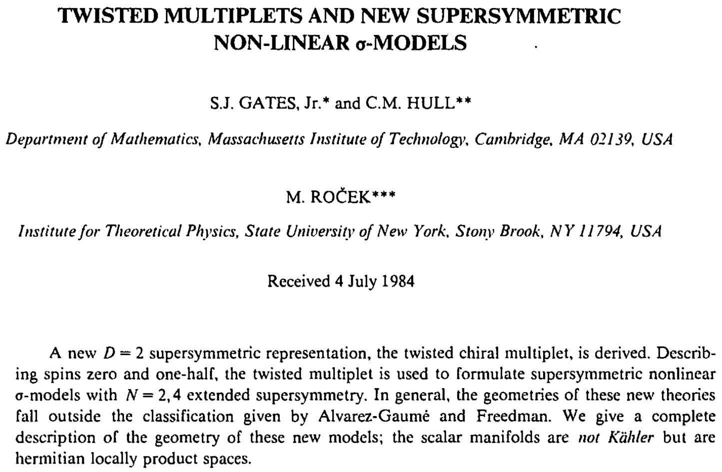
Bihermitian geometry'84



TWISTED MULTIPLETS AND NEW SUPERSYMMETRIC NON-LINEAR σ-MODELS

S.J. GATES, Jr.* and C.M. HULL**

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M. ROČEK***

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Received 4 July 1984

A new $D = 2$ supersymmetric representation, the twisted chiral multiplet, is derived. Describing spins zero and one-half, the twisted multiplet is used to formulate supersymmetric nonlinear σ-models with $N = 2,4$ extended supersymmetry. In general, the geometries of these new theories fall outside the classification given by Alvarez-Gaumé and Freedman. We give a complete description of the geometry of these new models; the scalar manifolds are *not Kähler* but are hermitian locally product spaces.

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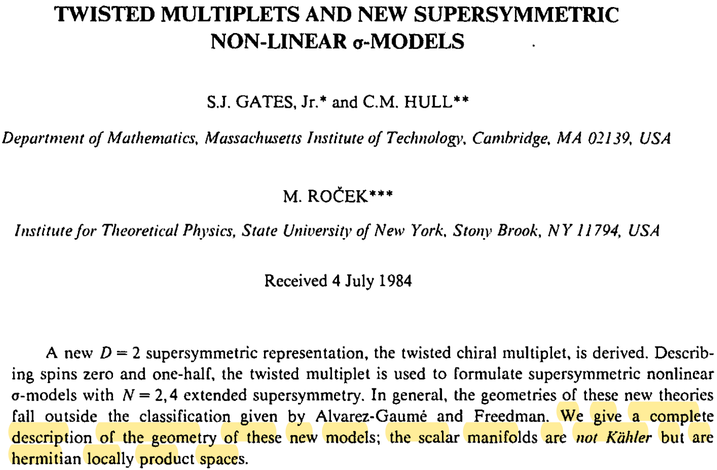

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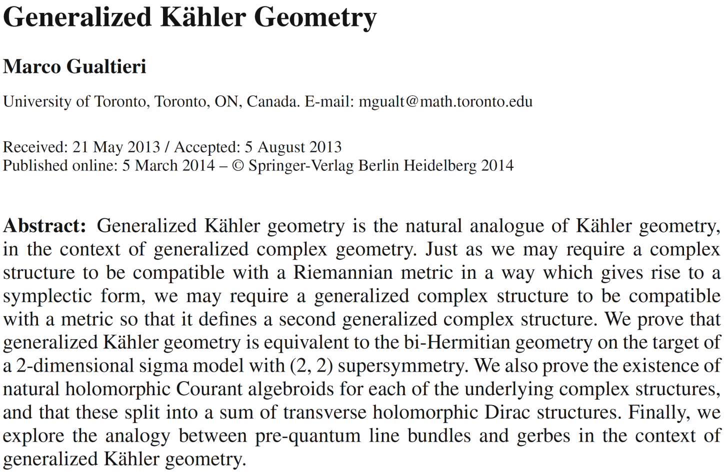

Generalized Kähler Geometry

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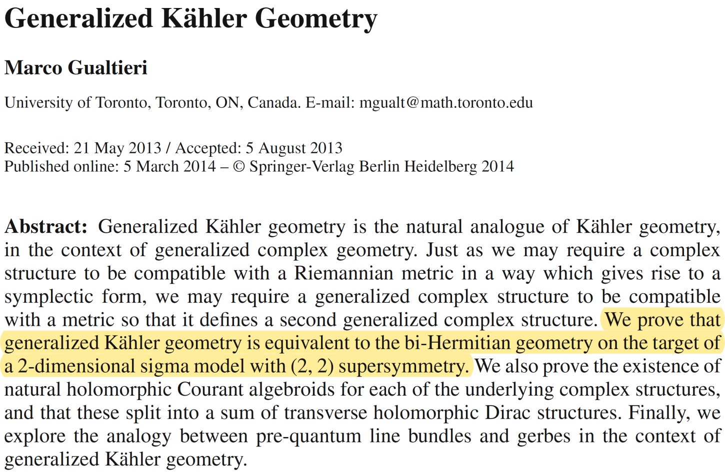
2. Provides a new language, more suitable in some cases

Generalized Kähler geometry'04



Generalized Kähler Geometry

Marco Gualtieri

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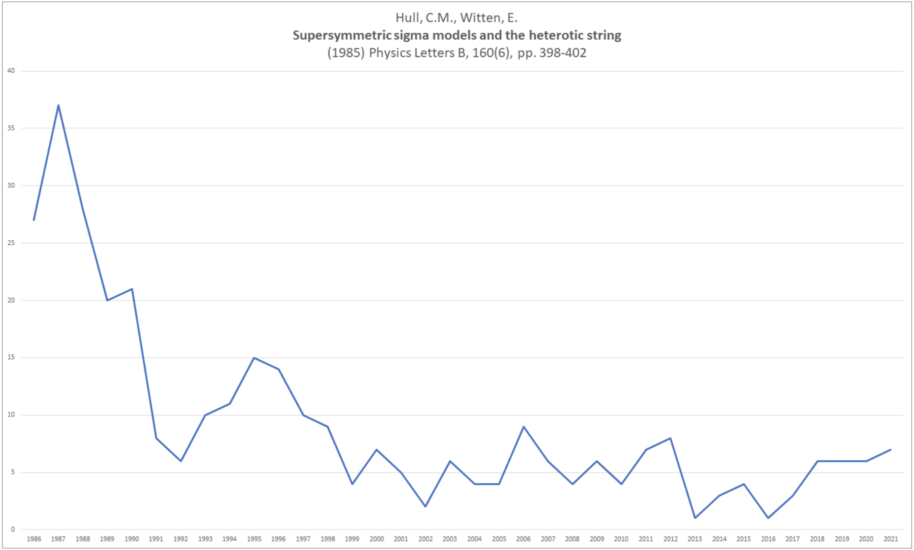
Hull, C.M., Witten, E.
Supersymmetric sigma models and the heterotic string
(1985) Physics Letters B, 160(6), pp. 398-402
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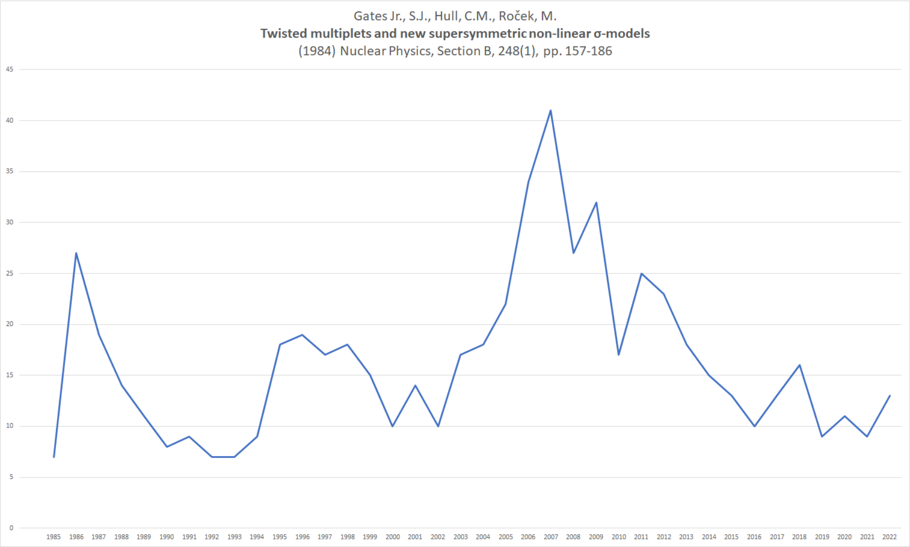
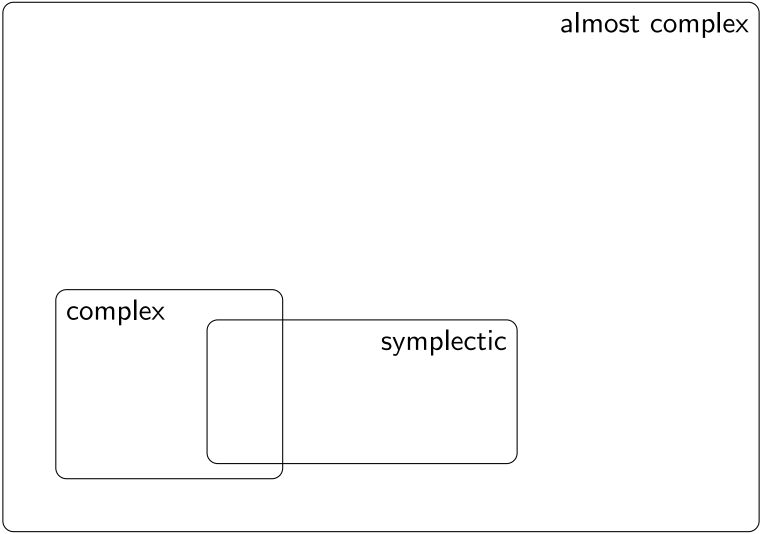


Gates Jr., S.J., Hull, C.M., Roček, M.

Twisted multiplets and new supersymmetric non-linear σ-models

(1984) Nuclear Physics, Section B, 248(1), pp. 157-186

3. Genuinely new structures

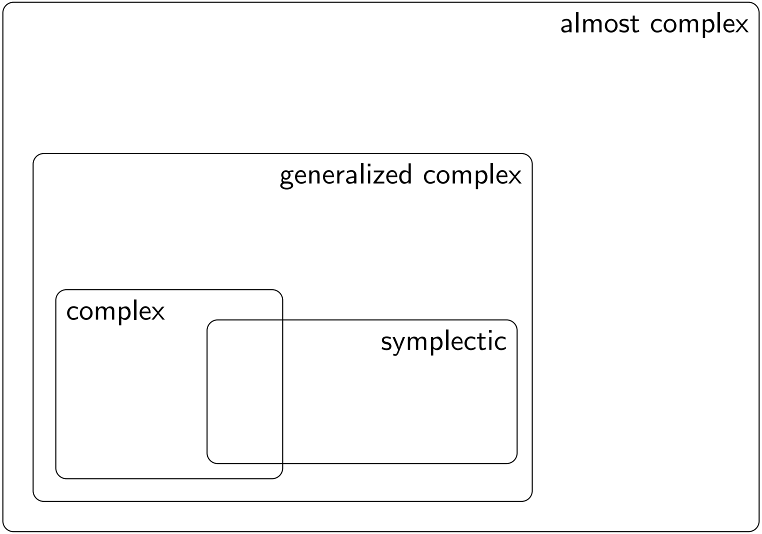


almost complex

complex

symplectic

3. Genuinely new structures



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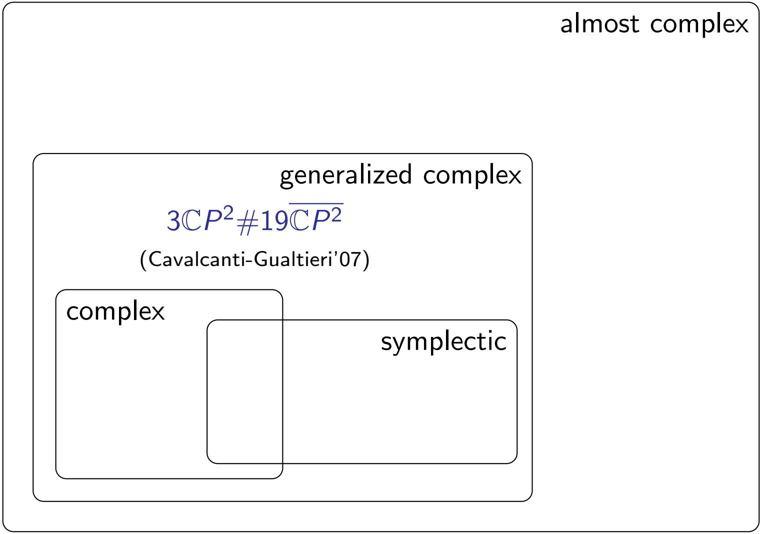

almost complex

generalized complex

$3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P^2}$

(Cavalcanti-Gualtieri'07)

complex

symplectic

A step back to Dirac structures (Courant'90, Weinstein)

symplectic (M, ω)

$\omega : TM \xrightarrow{\sim} T^*M$ or $\pi = \omega^{-1} : T^*M \xrightarrow{\sim} TM$

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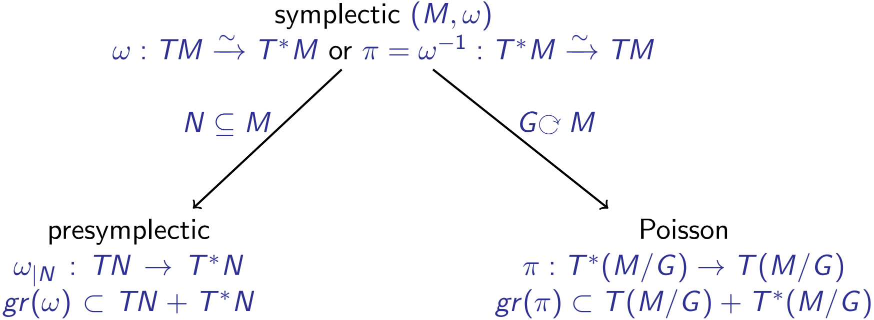

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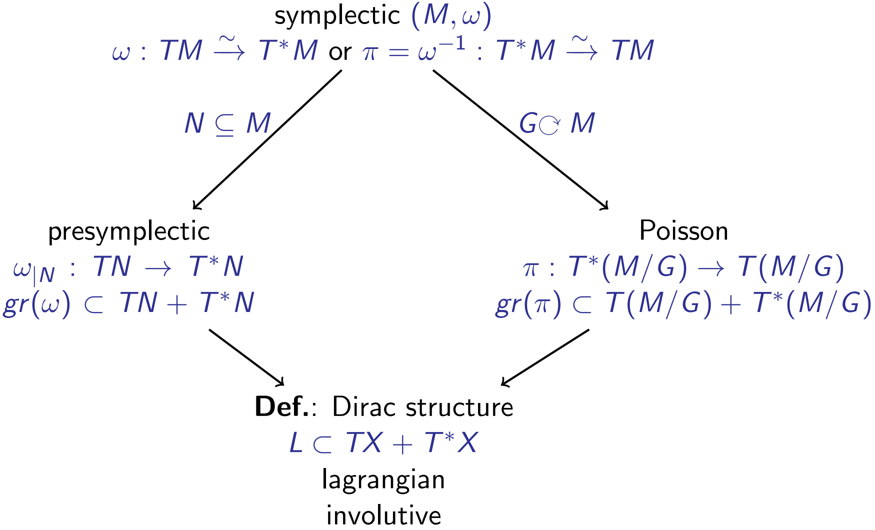

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Def.: Dirac structure

$L \subset TX + T^*X$

lagrangian

involutive

Dirac structures geometrically speaking

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$$\begin{aligned} \omega &: TM \to T^*M \\ d\omega &= 0 \end{aligned}$$

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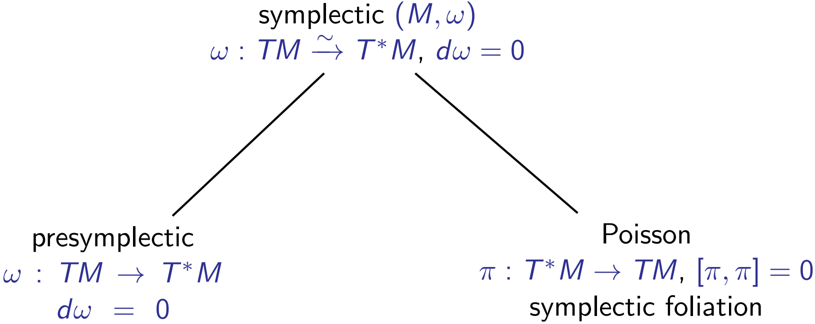

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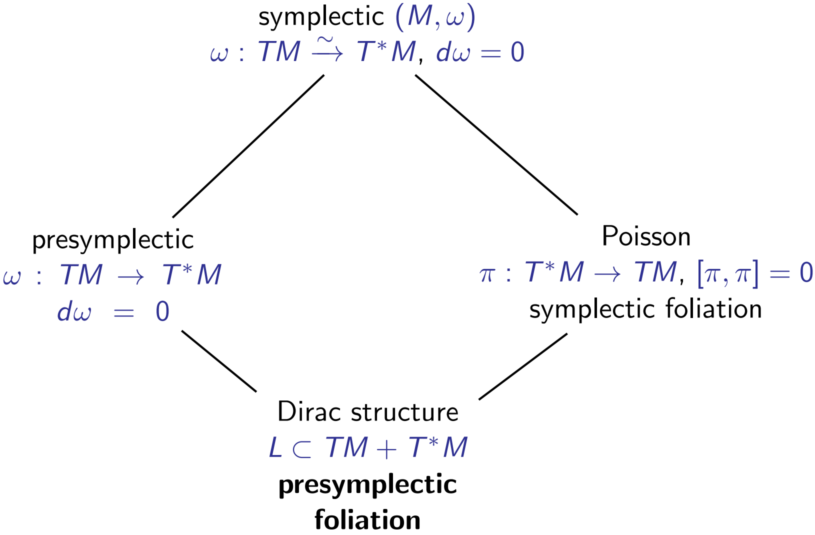

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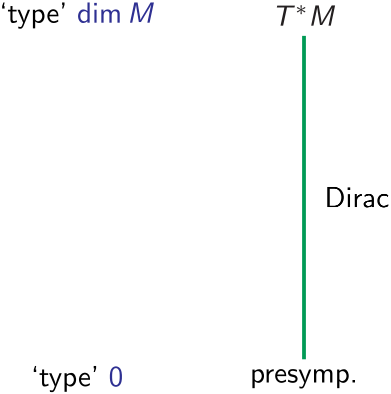

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Dirac

‘type’ 0 presymp.

Local description for generalized complex structures

Theorem (Bailey)

Locally a generalized complex structure is a symplectic foliation with a transverse holomorphic Poisson structure.

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The symplectic foliation, a Poisson structure!, was always there:

$$\mathcal{J} = \begin{pmatrix} A & \pi \\ B & C \end{pmatrix}$$

that is, $\pi : T^*M \to TM$.

Submanifolds of symplectic $\rightsquigarrow$ presymplectic $\rightsquigarrow$ Dirac

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Symplectic and complex $\rightsquigarrow$ generalized complex

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What invariant or invariants describe them?

Agüero'20, Bursztyn, R.

(Agüero, R.: *Complex Dirac structures: invariants and local structure*, to appear in Comm. Math. Phys.)

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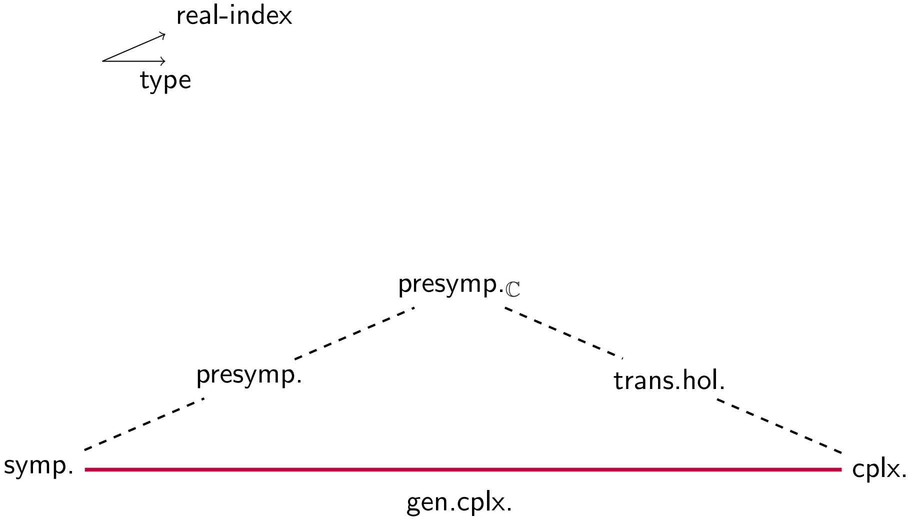
For constant order, a complex Dirac has associated a real Dirac.

$\underset{\text{type}}{\longrightarrow}$

symp.

cplx.

gen.cplx.



real-index

type

presymp.$_{\mathbb{C}}$

presymp.

trans.hol.

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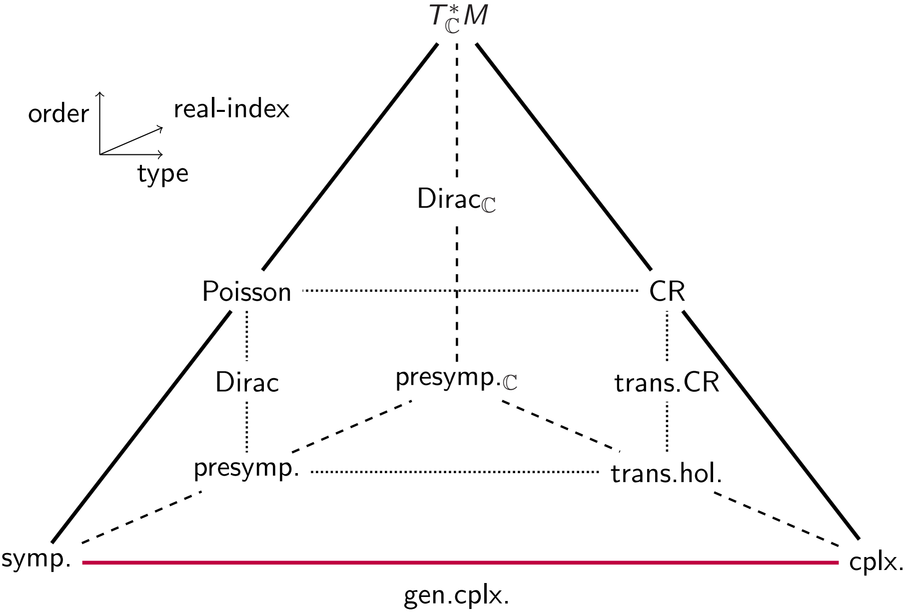
$T^*_{\mathbb{C}}M$
order
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Dirac$_{\mathbb{C}}$
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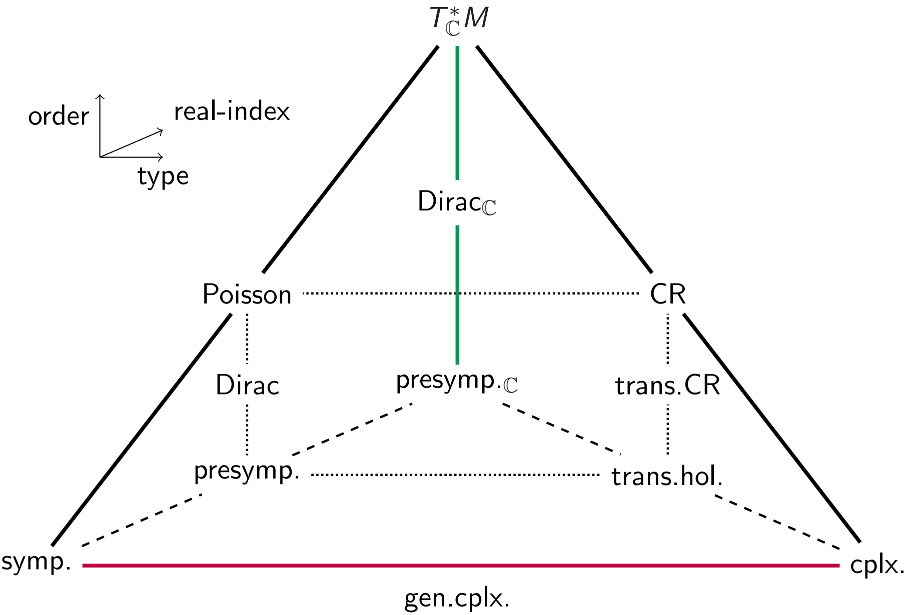
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Live

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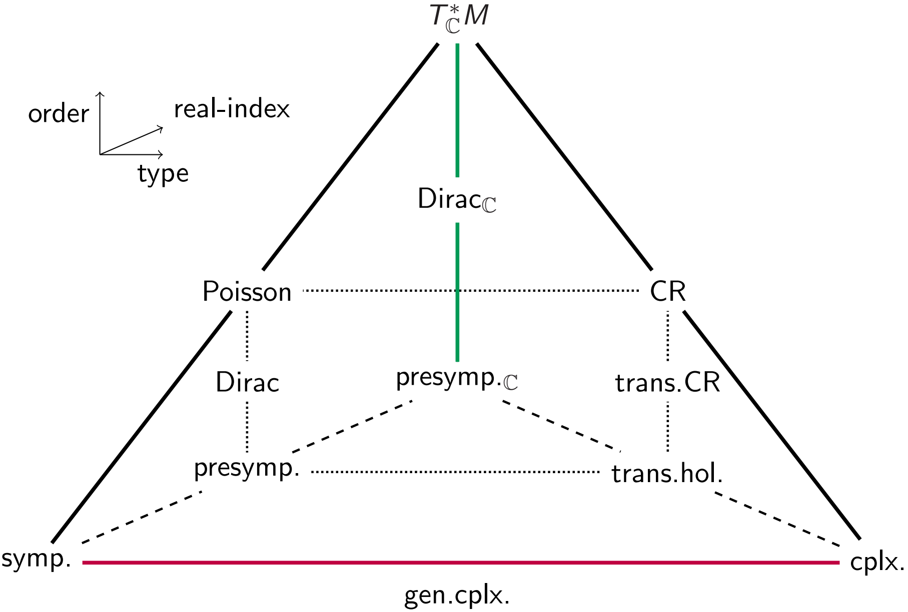
- Challenging and beautiful.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Thank you very much!

Danke shön!

Moltes gràcies!