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## Complex Dirac Structures.

A voyage from linear algebra to generalized geometry.

## Bachelor's Thesis in Mathematics

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#### Abstract

In this thesis we study complex Dirac structures mainly from a linear viewpoint. Motivated by the similarities that arise between linear symplectic and linear complex structures, we present generalized linear algebra and linear generalized complex structures as a means to encompass them both. In turn, these structures are a particular case of linear complex Dirac structures, which we study through three invariants: the real index, the order and the type. Ultimately, we give an overview on how our linear study integrates to generalized geometry and we show with a couple of examples that the invariants can vary over a manifold.


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## Introduction

In differential geometry we deal with smooth manifolds through their tangent bundle $T$, whose sections are endowed with the Lie bracket. Generalized geometry changes this mindset: in contrast, the main role is played by the generalized tangent bundle $T \oplus T^{*}$, which is equipped with the canonical pairing $\langle$,$\rangle and with$ an extension of the Lie bracket to its sections, namely the Dorfman bracket [,]. If we take a step further and complexify this setup, we enter the realm of generalized complex geometry. Specifically, in generalized complex geometry we work with the vector bundle $\left(T \oplus T^{*}\right)_{\mathbb{C}}$ and extend $\mathbb{C}$-linearly both $\langle$,$\rangle and [,].$

Generalized complex geometry was initially explored in [Hit03] by Hitchin and more generally established by Gualtieri shortly afterwards in [Gua03]. Due to its bonds with theoretical physics (see for instance [Zab06] or [KL07]), it had a rapid breakthrough and has been an active field of research ever since.

Generalized complex structures are introduced with the idea of encompassing two structures from differential geometry: symplectic and complex structures. Interestingly, though, if we restrict a generalized complex structure to a submanifold we may not get a generalized complex structure. This suggests studying a bigger class of structures called complex Dirac structures. A complex Dirac structure $L$ on a smooth manifold is defined as a subbundle of $\left(T \oplus T^{*}\right)_{\mathbb{C}}$ that is maximally isotropic for the canonical pairing and such that its sections are involutive for the Dorfman bracket, this is $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$. In particular, generalized complex structures are complex Dirac structures that satisfy $L \cap \bar{L}=0$.

Even though all these structures are purely geometrical, many of their features can be discovered in terms of (generalized!) linear algebra by simply restricting to each fiber. The main objective of this thesis is to give a linear approach of the aforementioned phenomena (Chapters 1 and 2). Lastly, we will conclude with an overview on how some of our pointwise study integrates to geometric structures and what the purely geometric phenomena are (Chapter 3).

In the first chapter, we study linear symplectic and linear complex structures. Our driving force is to show their common properties that inspire our move to generalized linear algebra. Furthermore, linear complex structures are a suitable background to introduce the realification and complexification of a vector space. These constructions give us a viewpoint of linear complex structures in terms of their $+i$-eigenspace.

In the second chapter, we introduce the complexified generalized vector space $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ with its canonical pairing $\langle$,$\rangle and linear generalized complex structures.$ Through some independent work, in Subsection 2.2 .3 we see that a linear generalized complex structure is equivalently given by a maximally isotropic subspace $L$ of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ that satisfies $L \cap \bar{L}=0$. Afterwards, we give an alternative description of maximally isotropic subspaces. With this description we independently see that if we restrict a maximally isotropic subspace of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ to a subspace $U$ of $V$ we obtain a maximally isotropic subspace of $\left(U \oplus U^{*}\right)_{\mathbb{C}}$ (Proposition 2.24). Since this is not the case for linear generalized complex structures, we drop the condition $L \cap \bar{L}=0$ and refer to maximally isotropic subspaces as linear complex Dirac structures. Up to this point, our main guideline is [Rub] and the majority of our results are adapted from [Gua03]. To conclude our linear approach, in Section 2.3 we follow [Agu20] and [AR22] to study the real index, the order and the type, three integers that suffice to classify linear complex Dirac structures.

Finally, the third chapter translates some of our previous work into geometry. We start with a discussion on the integrability condition of complex structures that is missed in the linear approach. Afterwards, we introduce generalized geometry and the properties of the Dorfman bracket to finally define complex Dirac structures. To reflect on the role played by manifolds, we conclude the dissertation showing that the type (Example 3.12) and the order (Example 3.13) of a complex Dirac structure can vary over a manifold. The first example was adapted from [Gua03] with our tools while the second was developed independently.

Before we begin, let us remark that throughout the thesis we will only work with finite-dimensional vector spaces over a field that is either $\mathbb{R}$ or $\mathbb{C}$. Consequently, we will use $\mathbb{K}$ to refer to any of this two fields and normally we will not specify that a vector space is finite-dimensional. Similarly, in the last chapter about geometry we assume that anything is smooth without further mention. Ultimately, we denote the trivial vector space or bundle just as a 0 without brackets.

## Chapter 1

## Linear algebra

Our starting point is to study linear symplectic and linear complex structures.

### 1.1. Preliminaries

To begin with, we recall some basic notions that will be useful later on.

Definition 1.1 (Annihilator of a subspace).
Let $V$ be a vector space over $\mathbb{K}$. Given a subspace $U$ of $V$, the annihilator of $U$ is the subspace

$$
\operatorname{Ann} U:=\left\{\alpha \in V^{*}: \alpha(U)=0\right\} \text { of } V^{*} .
$$

## Proposition 1.2.

Let $V$ be a vector space over $\mathbb{K}$ and $U$ be a subspace of $V$, we have

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{K}} V=\operatorname{dim}_{\mathbb{K}} U+\operatorname{dim}_{\mathbb{K}} \operatorname{Ann} U . \tag{1.1}
\end{equation*}
$$

Proof. In case $U$ is a proper subspace of $V \neq 0$, take a basis $\left(u_{k}\right)_{k=1}^{m}$ of $U$ and extend it to a basis $\left(u_{k}\right)_{k=1}^{n}$ of $V$, for certain integers $n>m \geq 1$. It is easy to show that $\left(u^{m+k}\right)_{k=1}^{n-m}$ is a basis of $\operatorname{Ann} U$, which gives equality (1.1).

Definition 1.3 (Orthogonal complement of a subspace).
Let $V$ be a vector space over $\mathbb{K}$ and $B: V \times V \rightarrow \mathbb{K}$ be a bilinear map. Given a subspace $U$ of $V$, the orthogonal complement of $U$ by $B$ is the subspace

$$
U^{B}:=\{v \in V: B(U, v)=0\} \text { of } V .
$$

Remark 1.4. One could also define the orthogonal complement through the condition $B(v, U)=0$. However, both conditions are equivalent in case of a symmetric or a skew-symmetric bilinear map.

Definition 1.5 (Non-degenerate bilinear map).
Let $V$ be a vector space over $\mathbb{K}$, a bilinear map $B: V \times V \rightarrow \mathbb{K}$ is called nondegenerate if any non-zero $v \in V$ gives that $B(\cdot, v) \in V^{*}$ is also non-zero.

Note that there is a one-to-one correspondence between linear maps $V \rightarrow V^{*}$ and bilinear maps $V \times V \rightarrow \mathbb{K}$. Indeed, a bilinear map $B: V \times V \rightarrow \mathbb{K}$ induces a linear map

$$
\begin{align*}
B^{b}: V & \rightarrow V^{*}  \tag{1.2}\\
v & \mapsto B(\cdot, v)
\end{align*}
$$

whilst a linear map $f: V \rightarrow V^{*}$ induces a bilinear map

$$
\begin{align*}
f^{\natural}: V \times V & \rightarrow \mathbb{K}  \tag{1.3}\\
(u, v) & \mapsto(f(v))(u) .
\end{align*}
$$

Since we have $B=\left(B^{\mathrm{b}}\right)^{\mathfrak{h}}$ and $B=\left(f^{\natural}\right)^{\mathrm{b}}$, the correspondence is one-to-one. What is more, $B$ is non-degenerate if and only if $B^{b}$ is injective, and as we exclusively work with finite-dimensional vector spaces and $\operatorname{dim}_{\mathbb{K}} V^{*}=\operatorname{dim}_{\mathbb{K}} V$ holds, $B$ is non-degenerate if and only if $B^{b}$ is an isomorphism.

## Proposition 1.6.

Let $V$ be a vector space over $\mathbb{K}, U$ be a subspace of $V$ and $B: V \times V \rightarrow \mathbb{K}$ be a non-degenerate bilinear map, we have

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{K}} V=\operatorname{dim}_{\mathbb{K}} U+\operatorname{dim}_{\mathbb{K}} U^{B} \tag{1.4}
\end{equation*}
$$

Proof. On the one hand, from the induced isomorphism (1.2) one can easily verify the equality $B^{\mathrm{b}}\left(U^{B}\right)=$ Ann $U$ by double inclusion. On the other hand, the restriction epimorphism $\left.\right|_{U}: V^{*} \rightarrow U^{*}$ trivially gives $\left.\operatorname{Ker}\right|_{U}=\operatorname{Ann} U$. As a consequence, the composition $\left.\right|_{U} \circ B^{b}: V \rightarrow U^{*}$ is also an epimorphism and satisfies

$$
\operatorname{Ker}\left(\left.\right|_{U} \circ B^{b}\right)=\left(B^{b}\right)^{-1}\left(\left.\operatorname{Ker}\right|_{U}\right)=\left(B^{b}\right)^{-1}(\operatorname{Ann} U)=\left(B^{b}\right)^{-1}\left(B^{b}\left(U^{B}\right)\right)=U^{B}
$$

which from the equality $\operatorname{dim}_{\mathbb{K}} U^{*}=\operatorname{dim}_{\mathbb{K}} U$ and the First Isomorphism Theorem gives equality (1.4).

### 1.2. Linear symplectic structures

Definition 1.7 (Linear symplectic structure).
Let $V$ be a vector space over $\mathbb{K}$, a linear symplectic structure $\omega$ on $V$ is a bilinear $\operatorname{map} \omega: V \times V \rightarrow \mathbb{K}$ that is non-degenerate and skew-symmetric.

Remark 1.8. It is well-known that since char $\mathbb{K} \neq 2$ skew-symmetry is equivalent to alternation.

It follows that a one-dimensional vector space $V$ does not admit a linear symplectic structure because any alternate bilinear map $\omega: V \times V \rightarrow \mathbb{K}$ must be trivial and hence degenerate. What is more, we will show that a vector space admits a linear symplectic structure if and only if it is even-dimensional. In order to give a characterization of linear symplectic structures we require a previous result.

## Lemma 1.9.

Let $V$ be a vector space over $\mathbb{K}$, $U$ be a subspace of $V$ and $\omega$ be a linear symplectic structure on $V$, we have $\left(U^{\omega}\right)^{\omega}=U$. In addition, the equality $U \oplus U^{\omega}=V$ holds if and only if $\left.\omega\right|_{U}$ is a linear symplectic structure on $U$.

Proof. In the first place and as $U^{\omega}$ is a subspace of $V$, from equality (1.4) we get

$$
\operatorname{dim}_{\mathbb{K}} U+\operatorname{dim}_{\mathbb{K}} U^{\omega}=\operatorname{dim}_{\mathbb{K}} V=\operatorname{dim}_{\mathbb{K}} U^{\omega}+\operatorname{dim}_{\mathbb{K}}\left(U^{\omega}\right)^{\omega},
$$

which gives $\operatorname{dim}_{\mathbb{K}} U=\operatorname{dim}_{\mathbb{K}}\left(U^{\omega}\right)^{\omega}$. Furthermore, since by skew-symmetry we have $\omega\left(U^{\omega}, u\right)=0$ for any $u \in U$, we get $U \subseteq\left(U^{\omega}\right)^{\omega}$ and thus the equality $U=\left(U^{\omega}\right)^{\omega}$.

In the second place, observe that $U \oplus U^{\omega}=V$ is equivalent to $U \cap U^{\omega}=0$ due to equality (1.4), which means that $\omega(U, u) \neq 0$ holds for any non-zero $u \in U$. In turn, the latter corresponds to $\left.\omega\right|_{U}$ being non-degenerate and, since the restriction is still bilinear and skew-symmetric, to $\left.\omega\right|_{U}$ being a symplectic structure on $U$.

## Theorem 1.10.

Let $V$ be a vector space over $\mathbb{K}$, there exists a linear symplectic structure $\omega$ on $V$ if and only if $\operatorname{dim}_{\mathbb{K}} V=2 n$ for a certain integer $n \geq 0$. In addition, if $n \geq 1$ there is a basis $\mathcal{B}$ of $V$ such that the induced isomorphism $\omega^{b}$ from (1.2) gives

$$
M_{2 n \times 2 n}\left(\omega^{b}, \mathcal{B}, \mathcal{B}^{*}\right)=\left[\begin{array}{cc}
0 & \operatorname{Id}_{n}  \tag{1.5}\\
-\operatorname{Id}_{n} & 0
\end{array}\right] .
$$

Proof. Let $V$ be a vector space over $\mathbb{K}$ together with a linear symplectic structure $\omega$. In case $V \neq 0$, take a non-zero vector $v_{1} \in V$. By non-degeneracy, $\omega\left(\cdot, v_{1}\right)$ is non-zero as well, so there exists $u_{1} \in V$ such that $\omega\left(u_{1}, v_{1}\right)=1$. In particular, $u_{1}$ must be non-zero and by alternation a linearly independent vector to $v_{1}$. Defining now $V_{1}$ as the linear span of $u_{1}$ and $v_{1}$, the restriction $\left.\omega\right|_{V_{1}}$ results to be a linear symplectic structure on $V_{1}$. Indeed, for any non-zero vector $\lambda u_{1}+\mu v_{1}, \lambda, \mu \in \mathbb{K}$, either $\lambda$ or $\mu$ is non-zero, so assuming $\lambda \neq 0$ without loss of generality we obtain

$$
\left.\omega\right|_{V_{1}}\left(v_{1}, \lambda u_{1}+\mu v_{1}\right)=\lambda \neq 0
$$

and hence $\left.\omega\right|_{V_{1}}$ is non-degenerate. Therefore, by Lemma 1.9 the equality

$$
V=V_{1} \oplus V_{1}^{\omega}=\left(V_{1}^{\omega}\right)^{\omega} \oplus V_{1}^{\omega}
$$

holds and $\omega_{1}:=\left.\omega\right|_{V_{1}^{\omega}}$ is a linear symplectic structure on $V_{1}^{\omega}$.
Likewise, observe that in case $V_{1}^{\omega} \neq 0$ we get a decomposition $V_{1}^{\omega}=V_{2} \oplus V_{2}^{\omega_{1}}$, where $V_{2}$ is the linear span of a couple of linearly independent non-zero vectors $u_{2}, v_{2} \in V_{1}^{\omega}$ that satisfy $\omega_{1}\left(u_{2}, v_{2}\right)=1$, and a linear symplectic structure on $V_{2}^{\omega_{1}}$ that we call $\omega_{2}:=\left.\omega_{1}\right|_{V_{2}^{\omega_{1}}}$. Again, in case $V_{2}^{\omega_{1}} \neq 0$ we proceed analogously. With this construction, we reduce the dimension by two units at each step. Thus, since we restrict to finite-dimensional vector spaces, clearly for a certain integer $n \geq 1$ we eventually get to a decomposition $V_{n-1}^{\omega_{n-2}}=V_{n} \oplus V_{n}^{\omega_{n-1}}$, where we set $\omega_{0}:=\omega$ and $V_{0}^{\omega_{-1}}:=V$, that satisfies $\operatorname{dim}_{\mathbb{K}} V_{n}^{\omega_{n-1}}<2$. Additionally, as $\left.\omega\right|_{V_{n}^{\omega_{n-1}}}$ is a linear symplectic structure on $V_{n}^{\omega_{n-1}}$ this vector space cannot be one-dimensional and instead we have $V_{n}^{\omega_{n-1}}=0$.

In conclusion, in case $V \neq 0$ there exists a decomposition $V=V_{1} \oplus \cdots \oplus V_{n}$ such that $\operatorname{dim}_{\mathbb{K}} V_{j}=2, j=1, \ldots, n$, and thus $\operatorname{dim}_{\mathbb{K}} V=2 n$ for a certain integer $n \geq 1$. What is more, this construction results in a basis

$$
\mathcal{B}=\left(u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{n}\right)
$$

of $V$ for which it is easy to verify that (1.5) holds.
Conversely, let $V$ be a vector space over $\mathbb{K}$ such that $\operatorname{dim}_{\mathbb{K}} V=2 n$ for a certain integer $n \geq 0$. On the one hand, in case $n=0$ the trivial bilinear map is a linear symplectic structure on $V$. On the other hand, in case $n \geq 1$ take a basis $\mathcal{B}$ of $V$ and define an isomorphism $f: V \rightarrow V^{*}$ by (1.5). Thereby, the induced bilinear map $\omega:=f^{\natural}$ from (1.3) is a linear symplectic structure on $V$ and by construction (1.5) holds.

Remark 1.11. Observe that given any basis $\mathcal{B}=\left(e_{i}\right)_{i=1}^{2 n}$ in fact we have

$$
M_{2 n \times 2 n}\left(\omega^{b}, \mathcal{B}, \mathcal{B}^{*}\right)=\left[\omega\left(e_{i}, e_{j}\right)\right]_{i, j=1}^{2 n},
$$

where the right-hand side is the usual matrix for bilinear maps.

### 1.3. Linear complex structures

Definition 1.12 (Linear complex structure).
Let $V$ be a vector space over $\mathbb{K}$, a linear complex structure on $V$ is a linear map $J: V \rightarrow V$ that satisfies $J^{2}=-$ Id.

### 1.3.1. Realification and complexification

Our first approach is to establish a one-to-one correspondence between vector spaces over $\mathbb{C}$ and vector spaces over $\mathbb{R}$ together with a linear complex structure. With this aim, we define the realification of a complex vector space.

Definition 1.13 (Realification of a complex vector space).
Let $V$ be a vector space over $\mathbb{C}$. The realification of $V$, which we denote as $V_{\mathbb{R}}$, is the vector space over $\mathbb{R}$ consisting of the abelian group of $V$ and its product by a scalar restricted to $\mathbb{R}$.

Remark 1.14. Note that as abelian groups or sets we have $V_{\mathbb{R}}=V$.

In turn, the multiplication by $i$ from the product by a scalar • of $V$ becomes a linear complex structure on $V_{\mathbb{R}}$. Indeed,

$$
\begin{aligned}
J: \quad V_{\mathbb{R}} & \rightarrow V_{\mathbb{R}} \\
v & \mapsto i \cdot v
\end{aligned}
$$

gives $J^{2}=-\mathrm{Id}$. Conversely, let $V$ be a vector space over $\mathbb{R}$ together with a linear complex structure $J$. We can extend the product by a scalar • of $V$ to $\mathbb{C}$ as

$$
\begin{equation*}
(a+b i) \cdot v:=a \cdot v+b \cdot J(v) \tag{1.6}
\end{equation*}
$$

for any $a, b \in \mathbb{R}$ and $v \in V$. Accordingly, the abelian group of $V$ and the extended product form a vector space over $\mathbb{C}$. Lastly, it is easy to verify that the correspondence is one-to-one.

Analogously to Definition 1.13, one may be tempted to call this last construction the complexification of a real vector space. However, this is not a proper definition since we will see next that not any vector space over $\mathbb{R}$ admits a linear complex structure.

Remark 1.15. In contrast, it is relevant to note that any vector space $V$ over $\mathbb{C}$ does admit a linear complex structure since the multiplication by $i$ is a linear complex structure on $V$ itself.
$\nabla$

## Theorem 1.16.

Let $V$ be a vector space over $\mathbb{R}$, there exists a linear complex structure $J$ on $V$ if and only if $\operatorname{dim}_{\mathbb{R}} V=2 n$ for a certain integer $n \geq 0$. In addition, if $n \geq 1$ there is a basis $\mathcal{B}$ of $V$ such that we have

$$
M_{2 n \times 2 n}(J, \mathcal{B}, \mathcal{B})=\left[\begin{array}{cc}
0 & \mathrm{Id}_{n}  \tag{1.7}\\
-\mathrm{Id}_{n} & 0
\end{array}\right]
$$

Proof. Let $V$ be a vector space over $\mathbb{R}$ together with a linear complex structure $J$. In case $V \neq 0$, as $\operatorname{det} J \in \mathbb{R}$ the equalities

$$
(\operatorname{det} J)^{2}=\operatorname{det} J^{2}=\operatorname{det}(-\mathrm{Id})=(-1)^{\operatorname{dim}_{\mathbb{R}} V}
$$

imply $\operatorname{dim}_{\mathbb{R}} V=2 n$ for a certain integer $n \geq 1$. In this instance, take any non-zero $v_{1} \in V$ and consider $J v_{1}$. In case $n \geq 2$, take any non-zero $v_{2} \in V$ which cannot be written as a linear combination of $v_{1}$ and $J v_{1}$ and consider $J v_{2}$. Inductively, we can go on until getting a set

$$
B=\left\{v_{1}, J v_{1}, \ldots, v_{n}, J v_{n}\right\}
$$

of non-zero vectors which satisfies that $v_{k}$ cannot be written as a linear combination of $v_{1}, J v_{1}, \ldots, v_{k-1}$ and $J v_{k-1}$, for $k=2, \ldots, n$. Notice that if $B$ were a linearly independent set, by ordering it as

$$
\mathcal{B}=\left(J v_{1}, \ldots, J v_{n}, v_{1}, \ldots, v_{n}\right)
$$

we would get a basis of $V$ for which it is easy to verify that (1.7) holds. To prove the latter, we will show that

$$
B_{k}=\left\{v_{1}, J v_{1}, \ldots, v_{k}, J v_{k}\right\}
$$

is a linearly independent set for $k=1, \ldots, n$ by induction.

With regard to the case $k=1$, by taking a linear combination $\lambda_{1} v_{1}+\mu_{1} J v_{1}=0$, $\lambda_{1}, \mu_{1} \in \mathbb{R}$, and applying $J$ to each side we get $\lambda_{1} J v_{1}-\mu_{1} v_{1}=0$. If we multiply these equations by $\lambda_{1}$ and $-\mu_{1}$ respectively and add them afterwards we obtain $\left(\lambda_{1}^{2}+\mu_{1}^{2}\right) v_{1}=0$, which is only possible if $\lambda_{1}=\mu_{1}=0$.

With respect to the case $2 \leq k \leq n$, by taking a linear combination

$$
\lambda_{1} v_{1}+\mu_{1} J v_{1}+\cdots+\lambda_{k} v_{k}+\mu_{k} J v_{k}=0
$$

with $\lambda_{1}, \mu_{1}, \ldots, \lambda_{k}, \mu_{k} \in \mathbb{R}$, and analogously applying $J$ to each side we get

$$
\lambda_{1} J v_{1}-\mu_{1} v_{1}+\cdots+\lambda_{k} J v_{k}-\mu_{k} v_{k}=0
$$

If we multiply these equations by $\lambda_{k}$ and $-\mu_{k}$ respectively and add them, in this case we obtain

$$
\left(\lambda_{k} \lambda_{1}+\mu_{k} \mu_{1}\right) v_{1}+\left(\lambda_{k} \mu_{1}-\mu_{k} \lambda_{1}\right) J v_{1}+\cdots+\left(\lambda_{k}^{2}+\mu_{k}^{2}\right) v_{k}=0 .
$$

Since by construction $v_{k}$ is not a linear combination of $v_{1}, J v_{1}, \ldots, v_{k-1}$ and $J v_{k-1}$, we have $\lambda_{k}^{2}+\mu_{k}^{2}=0$, which is only possible if $\lambda_{k}=\mu_{k}=0$. Finally substituting these into the first linear combination, as by induction hypothesis $B_{k-1}$ is a linearly independent set, we also obtain

$$
\lambda_{1}=\mu_{1}=\cdots=\lambda_{k-1}=\mu_{k-1}=0
$$

Conversely, let $V$ be a vector space over $\mathbb{R}$ such that $\operatorname{dim}_{\mathbb{R}} V=2 n$ for a certain integer $n \geq 1$. On the one hand, in case $n=0$ the trivial linear map is a linear complex structure on $V$. On the other hand, in case $n \geq 1$ by taking any basis $\mathcal{B}$ of $V$ and defining a linear map $J$ by (1.7) we get that $J$ is a linear complex structure on $V$ such that (1.7) holds.

## Corollary 1.17.

Let $V$ be a vector space over $\mathbb{C}$, we have $\operatorname{dim}_{\mathbb{R}} V_{\mathbb{R}}=2 \operatorname{dim}_{\mathbb{C}} V$.
Proof. In case $V \neq 0$, take a basis $\left(e_{k}\right)_{k=1}^{n}$ of $V$ for a certain integer $n \geq 1$. Recalling that the multiplication by $i$ is a linear complex structure on $V_{\mathbb{R}}$, which is a vector space over $\mathbb{R}$, it is easy to verify that we are able to take $v_{k}:=e_{k}$ in the construction of the previous proof, for $k=1, \ldots, n$. Therefore, we obtain that

$$
\mathcal{B}=\left(i e_{1}, \ldots, i e_{n}, e_{1}, \ldots, e_{n}\right)
$$

is a basis of $V_{\mathbb{R}}$ and the result follows.

As a consequence of Theorem 1.16, if we want to associate a vector space over $\mathbb{C}$ to any given vector space $V$ over $\mathbb{R}$ through a linear complex structure we need to somehow work with an even-dimensional vector space. For this, what we do is doubling the dimension by using $V \oplus V$ instead of $V$.

Definition 1.18 (Complexification of a real vector space).
Let $V$ be a vector space over $\mathbb{R}$, consider $V \oplus V$ and the linear complex structure

$$
\begin{aligned}
J: V \oplus V & \rightarrow V \oplus V \\
(u, w) & \mapsto(-w, u) .
\end{aligned}
$$

The complexification of $V$, which we denote as $V_{\mathbb{C}}$, is the vector space over $\mathbb{C}$ consisting of the abelian group of $V \oplus V$ and its product by a scalar extended to $\mathbb{C}$ as (1.6) for any $a, b \in \mathbb{R}$ and $v \in V \oplus V$.

The next result is of straightforward proof and contrasts Corollary 1.17.

## Proposition 1.19.

Let $V$ be a vector space over $\mathbb{R}$, any basis of $V$ is also a basis of $V_{\mathbb{C}}$ and particularly we have $\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}=\operatorname{dim}_{\mathbb{R}} V$.

Remark 1.20. As a mean to facilitate computations, we introduce a formal element $i$, denote $u+i w:=(u, w)$ and identify $V$ inside either $V \oplus V$ or $V_{\mathbb{C}}$ as the first component. In this way, everything works as one would expect, namely the notation $i w$ corresponds to the multiplication $i \cdot(0, w)$.

Thereby, the following definition arises naturally.

Definition 1.21 (Conjugate and real part of a subspace).
Let $V$ be a vector space over $\mathbb{R}$. Given a subspace $L$ of $V_{\mathbb{C}}$, the conjugate of $L$ is the subspace

$$
\bar{L}:=\left\{u-i w \in V_{\mathbb{C}}: u+i w \in L\right\}
$$

of $V_{\mathbb{C}}$ and the real part of $L$ is the subspace $\operatorname{Re} L:=L_{\mathbb{R}} \cap V$ of $V$.

Again, we omit the proof of the next result as it is straightforward.

## Proposition 1.22.

Let $V$ be a vector space over $\mathbb{R}$ and $L$ be a subspace of $V_{\mathbb{C}}$, the conjugate vectors of any basis of $L$ form a basis of $\bar{L}$ and particularly we have $\operatorname{dim}_{\mathbb{C}} \bar{L}=\operatorname{dim}_{\mathbb{C}} L$.

Finally, the following result will be useful in the last section of Chapter 2.

## Proposition 1.23.

Let $V$ be a vector space over $\mathbb{R}$ and $L$ be a subspace of $V_{\mathbb{C}}$, the equalities $\bar{L}=L$ and $(\operatorname{Re} L)_{\mathbb{C}}=L$ are equivalent.

Proof. Let $L$ be a subspace of $V_{\mathbb{C}}$ such that $\bar{L}=L$. On the one hand, it is easy to verify that the inclusion $(\operatorname{Re} L)_{\mathbb{C}} \subseteq L$ holds regardless of the hypothesis. On the other hand, take $u+i w \in L$. Since $u-i w \in \bar{L} \subseteq L$, we have

$$
u=\frac{1}{2}(u+i w+u-i w), w=-\frac{i}{2}(u+i w-(u-i w)) \in L
$$

and as a consequence $u, w \in \operatorname{Re} L$, which gives $u+i w \in(\operatorname{Re} L)_{\mathbb{C}}$.
Conversely, let $L$ be a subspace of $V_{\mathbb{C}}$ such that $(\operatorname{Re} L)_{\mathbb{C}}=L$. On the one hand, take $u-i w \in \bar{L}$. Since $u+i w \in L \subseteq(\operatorname{Re} L)_{\mathbb{C}}$, we have $u, w \in \operatorname{Re} L \subseteq L$ and thus $u-i w \in L$. On the other hand, Proposition 1.22 gives $\bar{L}=L$.

### 1.3.2. Characterization of linear complex structures over $\mathbb{R}$

Let $J$ be a linear complex structure on a vector space $V$ over $\mathbb{R}$. Since $J^{2}=-\mathrm{Id}$ holds, the minimal polynomial of $J$ is $x^{2}+1$ and thus its roots are $\pm i$. This suggests studying $J$ from the view of $V_{\mathbb{C}}$, so we extend the linear map $J$ to $V_{\mathbb{C}}$ as

$$
\begin{array}{rccc}
J: & V_{\mathbb{C}} & \rightarrow & V_{\mathbb{C}} \\
u+i w & \mapsto & J u+i J w .
\end{array}
$$

Accordingly, this extension also called $J$ is a linear complex structure on $V_{\mathbb{C}}$. Recall that $J$ in principle has no relation with the multiplication by $i$ used to define $V_{\mathbb{C}}$, which is a linear complex structure on $V_{\mathbb{C}}$ itself but comes from a linear complex structure on $V \oplus V$. Since $J^{2}=-\mathrm{Id}$ holds, however, $J$ has a $+i$-eigenspace and a $-i$-eigenspace, which we denote as $V^{1,0}$ and $V^{0,1}$ respectively and that result in a decomposition $V_{\mathbb{C}}=V^{1,0} \oplus V^{0,1}$. If we define

$$
L:=\{v-i J v: v \in V\},
$$

it is easy to show by double inclusion that we have $L=V^{1,0}$ and also

$$
\bar{L}=\{v+i J v: v \in V\}=V^{0,1} .
$$

This means that our decomposition is $V_{\mathbb{C}}=L \oplus \bar{L}$ and thus any $u+i w \in V_{\mathbb{C}}$ has unique $l_{1}, l_{2} \in V^{1,0}$ such that $u+i w=l_{1}+\overline{l_{2}}$ and

$$
\begin{equation*}
J(u+i w)=J\left(l_{1}+\overline{l_{2}}\right)=i l_{1}-i \bar{l}_{2} . \tag{1.8}
\end{equation*}
$$

Finally, from Corollary 1.17, Proposition 1.19 and Proposition 1.22 we get

$$
\operatorname{dim}_{\mathbb{R}} L_{\mathbb{R}}=2 \operatorname{dim}_{\mathbb{C}} L=\operatorname{dim}_{\mathbb{C}} L+\operatorname{dim}_{\mathbb{C}} \bar{L}=\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}=\operatorname{dim}_{\mathbb{R}} V
$$

Particularly, we have seen the following result.

## Proposition 1.24.

Let $V$ be a vector space over $\mathbb{R}, J$ be a linear complex structure on $V$ and $L$ be the $+i$-eigenspace of $J$, we have that $L$ is a subspace of $V_{\mathbb{C}}$ that satisfies $L \cap \bar{L}=0$ and $\operatorname{dim}_{\mathbb{R}} L_{\mathbb{R}}=\operatorname{dim}_{\mathbb{R}} V$.

To conclude, we show that this information suffices to determine $J$ uniquely.

## Proposition 1.25.

Let $V$ be a vector space over $\mathbb{R}$. Given a subspace $L$ of $V_{\mathbb{C}}$ that satisfies $L \cap \bar{L}=0$ and $\operatorname{dim}_{\mathbb{R}} L_{\mathbb{R}}=\operatorname{dim}_{\mathbb{R}} V$, there exists a unique linear complex structure $J$ on $V$ such that its $+i$-eigenspace is $L$.

Proof. Firstly, observe that $\{l+\bar{l}: l \in L\}$ is a subspace of $V$ of dimension equal to $\operatorname{dim}_{\mathbb{R}} L_{\mathbb{R}}=\operatorname{dim}_{\mathbb{R}} V$. Therefore, we obtain the equality

$$
V=\{l+\bar{l}: l \in L\} .
$$

Furthermore, as $L$ is a subspace of $V_{\mathbb{C}}$, given any $l \in L$ we have $i l \in L$. Since we also have $-i \bar{l}=\bar{i} \in \bar{L}$, we get $i l-i \bar{l} \in V$ and hence we can define a linear map

$$
\begin{array}{cccc}
J: \quad V & \rightarrow & V \\
l+\bar{l} & \mapsto & i l-i \bar{l} .
\end{array}
$$

Note that $J$ is well-defined thanks to the hypothesis $L \cap \bar{L}=0$. In this way, $J$ is clearly a linear complex structure on $V$ and satisfies $V^{1,0}=L$.

Finally, uniqueness comes precisely from the condition $V^{1,0}=L$. Indeed, given a linear complex structure $J$ on $V$, its extension to $V_{\mathbb{C}}$ is fully determined by $V^{1,0}$ as equality (1.8) holds. In other words, all linear complex structures on $V$ which give $V^{1,0}=L$ have the same extension to $V_{\mathbb{C}}$ and thus coincide.

## Chapter 2

## Generalized linear algebra

This chapter introduces generalized linear algebra with the idea of encompassing linear symplectic and linear complex structures. Its main goal is to approach complex Dirac structures from a linear viewpoint before advancing to geometry in the last chapter.

### 2.1. The generalized vector space

Definition 2.1 (Generalized vector space).
Let $V$ be a vector space over $\mathbb{K}$, the generalized vector space of $V$ is the vector space $V \oplus V^{*}$ equipped with the bilinear map

$$
\begin{aligned}
\langle,\rangle:\left(V \oplus V^{*}\right) \times\left(V \oplus V^{*}\right) & \rightarrow \mathbb{K} \\
(X+\alpha, Y+\beta) & \mapsto \frac{1}{2}(\beta(X)+\alpha(Y)),
\end{aligned}
$$

called the canonical pairing.

Remark 2.2. It is standard to use $X+\alpha:=(X, \alpha)$ and $Y+\beta:=(Y, \beta)$ to denote arbitrary elements of $V \oplus V^{*}$.

The canonical pairing is clearly symmetric. In case $V \neq 0$, by taking any basis $\left(v_{i}\right)_{i=1}^{n}$ of $V$ for a certain integer $n \geq 1$ and its dual basis $\left(v^{i}\right)_{i=1}^{n}$ of $V^{*}$, we obtain a basis

$$
\mathcal{B}=\left(v_{1}+v^{1}, \ldots, v_{n}+v^{n}, v_{1}-v^{1}, \ldots, v_{n}-v^{n}\right):=\left(e_{i}\right)_{i=1}^{n}
$$

of $V \oplus V^{*}$ such that we have

$$
\left[\left\langle e_{i}, e_{j}\right\rangle\right]_{i, j=1}^{2 n}=\left[\begin{array}{cc}
\operatorname{Id}_{n} & 0  \tag{2.1}\\
0 & -\mathrm{Id}_{n}
\end{array}\right] .
$$

Particularly, the canonical pairing is non-degenerate and we get the following.

Remark 2.3. As it is standard, we denote $U^{\perp}:=U^{\langle,\rangle}$(recall Definition 1.3). $\quad \nabla$

## Proposition 2.4.

Let $V$ be a vector space over $\mathbb{K}$ and $L$ be a subspace of $V \oplus V^{*}$, we have $\left(L^{\perp}\right)^{\perp}=L$.
Proof. By replacing skew-symmetry for symmetry, Lemma 1.9 applies.

### 2.2. Linear generalized complex structures

### 2.2.1. Generalization of linear symplectic and linear complex structures over $\mathbb{R}$

In the previous chapter, some similar properties of linear symplectic and linear complex structures emerged in the form of Theorem 1.10 and Theorem 1.16. However, whereas the first holds both for vector spaces over $\mathbb{R}$ and $\mathbb{C}$, the second does only for vector spaces over $\mathbb{R}$ (recall Remark 1.15). Therefore, we are interested in generalizing both classes of structures just in the real case.

Definition 2.5 (Linear generalized complex structure).
Let $V$ be a vector space over $\mathbb{R}$, a linear generalized complex structure on $V$ is a linear map

$$
\mathcal{J}: V \oplus V^{*} \rightarrow V \oplus V^{*}
$$

that satisfies $\mathcal{J}^{2}=-$ Id and that is skew-symmetric for the canonical pairing, this is $\langle\mathcal{J} u, v\rangle+\langle u, \mathcal{J} v\rangle=0$ for any $u, v \in V \oplus V^{*}$.

## Proposition 2.6.

Let $V$ be a vector space over $\mathbb{R}$. Given a linear map $\mathcal{J}: V \oplus V^{*} \rightarrow V \oplus V^{*}$, the next statements are equivalent.
(a) $\mathcal{J}$ is skew-symmetric for the canonical pairing.
(b) We have $\langle\mathcal{J} v, v\rangle=0$ for any $v \in V \oplus V^{*}$.

Besides, if $\mathcal{J}^{2}=-\mathrm{Id}$, the following statement is equivalent to the previous ones.
(c) We have $\langle\mathcal{J} u, \mathcal{J} v\rangle=\langle u, v\rangle$ for any $u, v \in V \oplus V^{*}$.

Proof. We will see $(a) \Leftrightarrow(b)$ and $(a) \Leftrightarrow(c)$.
$(a) \Rightarrow(b)$. Given $v \in V \oplus V^{*}$, since char $\mathbb{R} \neq 2$ we have

$$
2\langle\mathcal{J} v, v\rangle=\langle\mathcal{J} v, v\rangle+\langle v, \mathcal{J} v\rangle=0 \Rightarrow\langle\mathcal{J} v, v\rangle=0 .
$$

$(a) \Leftarrow(b)$. Given $u, v \in V \oplus V^{*}$, we have

$$
\begin{aligned}
\langle\mathcal{J} u, v\rangle+\langle u, \mathcal{J} v\rangle & =0+\langle\mathcal{J} u, v\rangle+\langle\mathcal{J} v, u\rangle+0 \\
& =\langle\mathcal{J} u, u\rangle+\langle\mathcal{J} u, v\rangle+\langle\mathcal{J} v, u\rangle+\langle\mathcal{J} v, v\rangle=\langle\mathcal{J}(u+v), u+v\rangle=0 .
\end{aligned}
$$

$(a) \Rightarrow(c)$. Given $u, v \in V \oplus V^{*}$, we have

$$
\langle\mathcal{J} u, \mathcal{J} v\rangle=-\left\langle u, \mathcal{J}^{2} v\right\rangle=\langle u, v\rangle .
$$

$(a) \Leftarrow(c)$. Given $u, v \in V \oplus V^{*}$, we have

$$
\langle\mathcal{J} u, v\rangle+\langle u, \mathcal{J} v\rangle=\langle\mathcal{J} u, v\rangle-\left\langle\mathcal{J}^{2} u, \mathcal{J} v\right\rangle=\langle\mathcal{J} u, v\rangle-\langle\mathcal{J} u, v\rangle=0 .
$$

The following example shows how in the real case linear symplectic and linear complex structures can be regarded as linear generalized complex structures.

## Example 2.7.

Given an even-dimensional vector space $V$ over $\mathbb{R}$, we know from Theorem 1.10 and Theorem 1.16 that it admits a linear symplectic structure $\omega$ and a linear complex structure $J$. In this case, consider the linear maps

$$
\mathcal{J}_{\omega}, \mathcal{J}_{J}: V \oplus V^{*} \rightarrow V \oplus V^{*}
$$

defined by

$$
\mathcal{J}_{\omega}(X+\alpha):=-\left(\omega^{b}\right)^{-1}(\alpha)+\omega^{b}(X) \text { and } \mathcal{J}_{J}(X+\alpha):=-J(X)+J^{*}(\alpha),
$$

where $\omega^{b}: V \rightarrow V^{*}$ is the isomorphism given in (1.2). It is appropriate to denote

$$
\left[\begin{array}{cc}
0 & -\left(\omega^{b}\right)^{-1} \\
\omega^{b} & 0
\end{array}\right]:=\mathcal{J}_{\omega} \text { and }\left[\begin{array}{cc}
-J & 0 \\
0 & J^{*}
\end{array}\right]:=\mathcal{J}_{J} \text {. }
$$

After verifying from the definition that $J^{*}$ is a linear complex structure on $V^{*}$, a straightforward computation shows $\mathcal{J}_{\omega}^{2}=\mathcal{J}_{J}^{2}=-$ Id. Furthermore, we have

$$
\begin{aligned}
2\left\langle\mathcal{J}_{\omega}(X+\alpha), X+\alpha\right\rangle & =\alpha\left(-\left(\omega^{b}\right)^{-1}(\alpha)\right)+\left(\omega^{b}(X)\right)(X) \\
& =-\omega\left(\left(\omega^{b}\right)^{-1}(\alpha),\left(\omega^{b}\right)^{-1}(\alpha)\right)+\omega(X, X)=0+0=0,
\end{aligned}
$$

where we used that $\omega$ is alternate (Remark 1.8), and

$$
\begin{aligned}
2\left\langle\mathcal{J}_{J}(X+\alpha), X+\alpha\right\rangle & =\alpha(-J(X))+\left(J^{*}(\alpha)\right)(X) \\
& =-\alpha(J(X))+\alpha(J(X))=0,
\end{aligned}
$$

where we used the definition of $J^{*}$. Thus, by Proposition 2.6 we have that $\mathcal{J}_{\omega}$ and $\mathcal{J}_{J}$ are linear generalized complex structures.

### 2.2.2. Characterization of linear generalized complex structures

Once seen these examples, note that any linear generalized complex structure $\mathcal{J}$ on a vector space $V$ over $\mathbb{R}$ is particularly a linear complex structure on $V \oplus V^{*}$. As $V \oplus V^{*}$ is also a vector space over $\mathbb{R}$, Proposition 1.24 gives that $L:=\left(V \oplus V^{*}\right)^{1,0}$ is a subspace of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ that satisfies $\operatorname{dim} L_{\mathbb{R}}=\operatorname{dim}_{\mathbb{R}}\left(V \oplus V^{*}\right)$ and $L \cap \bar{L}=0$.

We will see that the extra condition of linear generalized complex structures, namely skew-symmetry for the canonical pairing, gives us now more information about $L$. This motivates the following definition.

Definition 2.8 (Isotropic subspace).
Let $V$ be a vector space over $\mathbb{K}$, a subspace $U$ of $V \oplus V^{*}$ is isotropic if $U \subseteq U^{\perp}$. $\diamond$

Remark 2.9. Given a vector space $V$ over $\mathbb{R}$ one can canonically identify $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ and $V_{\mathbb{C}} \oplus\left(V_{\mathbb{C}}\right)^{*}$. When we talk about the canonical pairing, in the background we work with $V_{\mathbb{C}} \oplus\left(V_{\mathbb{C}}\right)^{*}$ but to keep it simple we will always refer to $\left(V \oplus V^{*}\right)_{\mathbb{C}} . \quad \nabla$

A straightforward computation shows the next first result.

## Proposition 2.10.

Let $V$ be a vector space over $\mathbb{R}$ and $L$ be an isotropic subspace of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$, we have that $\bar{L}$ is also isotropic.

Our next step is to prove analogous results to Proposition 1.24 and Proposition 1.25 for linear generalized complex structures.

## Proposition 2.11.

Let $V$ be a vector space over $\mathbb{R}, \mathcal{J}$ be a linear generalized complex structure on $V$ and $L$ be the $+i$-eigenspace of $\mathcal{J}$, we have that $L$ is an isotropic subspace of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ that satisfies $L \cap \bar{L}=0$ and $\operatorname{dim}_{\mathbb{C}} L=\operatorname{dim}_{\mathbb{R}} V$.

Proof. Corollary 1.17 gives $\operatorname{dim}_{\mathbb{R}} L_{\mathbb{R}}=2 \operatorname{dim}_{\mathbb{C}} L$. Since $\operatorname{dim}_{\mathbb{R}} V=\operatorname{dim}_{\mathbb{R}} V^{*}$, we also have $\operatorname{dim}_{\mathbb{R}}\left(V \oplus V^{*}\right)=2 \operatorname{dim}_{\mathbb{R}} V$ and due to Proposition 1.24 all that remains to be proved is the isotropy of $L$.

If we take $v \in L$, for any $u \in L$ we get

$$
i\langle u, v\rangle=\langle i u, v\rangle=\langle\mathcal{J} u, v\rangle=-\langle u, \mathcal{J} v\rangle=-\langle u, i v\rangle=-i\langle u, v\rangle
$$

Yet again from char $\mathbb{K} \neq 2$ it follows that $\langle u, v\rangle=0$, which gives $v \in L^{\perp}$ and hence $L \subseteq L^{\perp}$ as intended.

## Example 2.12.

Let us consider $\mathcal{J}_{J}$ and $\mathcal{J}_{\omega}$ from Example 2.7. We define

$$
L_{\omega}:=\left\{X-i \omega(X): X \in V_{\mathbb{C}}\right\} \text { and } L_{J}:=V^{0,1} \oplus\left(V^{*}\right)^{1,0}
$$

where $V^{1,0}$ and $\left(V^{*}\right)^{0,1}$ refer to the $-i$ and $+i$-eigenspace of $J$ and $J^{*}$, respectively. From Corollary 1.17, Proposition 1.19 and Proposition 1.24 we have

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{C}} L_{\omega} & =\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}=\operatorname{dim}_{\mathbb{R}} V \text { and } \\
\operatorname{dim}_{\mathbb{C}} L_{J} & =\operatorname{dim}_{\mathbb{C}} V^{1,0}+\operatorname{dim}_{\mathbb{C}}\left(V^{*}\right)^{0,1}=2 \operatorname{dim}_{\mathbb{C}} V^{1,0}=\operatorname{dim}_{\mathbb{R}}\left(V^{1,0}\right)_{\mathbb{R}}=\operatorname{dim}_{\mathbb{R}} V
\end{aligned}
$$

By Proposition 2.11, this means that $\operatorname{dim}_{\mathbb{C}} L_{J}$ and $\operatorname{dim}_{\mathbb{C}} L_{\omega}$ equal the dimension of the $+i$-eigenspace of $\mathcal{J}_{J}$ and $\mathcal{J}_{\omega}$, respectively. Since a straightforward computation shows that $L_{J}$ and $L_{\omega}$ are subspaces of the corresponding $+i$-eigenspaces, we get in fact an equality. Finally, one can verify that indeed both $L_{J}$ and $L_{\omega}$ are isotropic subspaces of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$.

## Proposition 2.13.

Let $V$ be a vector space over $\mathbb{R}$. Given an isotropic subspace $L$ of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ that satisfies $L \cap \bar{L}=0$ and $\operatorname{dim}_{\mathbb{C}} L=\operatorname{dim}_{\mathbb{R}} V$, there exists a unique linear generalized complex structure $\mathcal{J}$ on $V$ such that its $+i$-eigenspace is $L$.

Proof. Analogously to the preceding proof, the equalities $\operatorname{dim}_{\mathbb{R}} L_{\mathbb{R}}=2 \operatorname{dim}_{\mathbb{C}} L$ and $\operatorname{dim}_{\mathbb{R}}\left(V \oplus V^{*}\right)=2 \operatorname{dim}_{\mathbb{R}} V$ along with Proposition 1.25 give a unique linear complex structure $\mathcal{J}$ on $V \oplus V^{*}$ for which $\left(V \oplus V^{*}\right)^{1,0}=L$ holds. Therefore, all that is left to show is the skew-symmetry of $\mathcal{J}$ for the canonical pairing.

With this aim, recall that we have

$$
V \oplus V^{*}=\{l+\bar{l}: l \in L\}
$$

and as a consequence we can write any $v \in V \oplus V^{*}$ as $v=l+\bar{l}$ for a certain $l \in L$. Thereby, we get

$$
\langle\mathcal{J}(l+\bar{l}), l+\bar{l}\rangle=\langle i l-i \bar{l}, l+\bar{l}\rangle=i\langle l, l\rangle+i\langle l, \bar{l}\rangle-i\langle\bar{l}, l\rangle-i\langle\bar{l}, \bar{l}\rangle=i\langle l, l\rangle-i\langle\bar{l}, \bar{l}\rangle .
$$

As $L$ is isotropic, Proposition 2.10 gives $\langle l, l\rangle=\langle\bar{l}, \bar{l}\rangle=0$ and thus $\langle\mathcal{J} v, v\rangle=0$.

### 2.2.3. Maximally isotropic subspaces

We have seen in Proposition 2.11 and Proposition 2.13 that a linear generalized complex structure on a vector space $V$ over $\mathbb{R}$ is equivalently given by an isotropic subspace $L$ of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ such that $L \cap \bar{L}=0$ and $\operatorname{dim}_{\mathbb{C}} L=\operatorname{dim}_{\mathbb{R}} V$. We will now work towards a first characterization of isotropic subspaces of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ whose dimension equals $\operatorname{dim}_{\mathbb{R}} V$. Even though the outcome is well-known, let us remark that the core of this study, this is the complex case of Lemma 2.14 and Proposition 2.16, was proved independently.

In the first place, given a vector space $V$ over $\mathbb{K}$ note that since the canonical pairing is bilinear and non-degenerate from equality (1.4) we get

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{K}} L+\operatorname{dim}_{\mathbb{K}} L^{\perp}=\operatorname{dim}_{\mathbb{K}}\left(V \oplus V^{*}\right)=2 \operatorname{dim}_{\mathbb{K}} V \tag{2.2}
\end{equation*}
$$

for any subspace $L$ of $V \oplus V^{*}$. In the second place, we need the following result.

## Lemma 2.14.

Let $V$ be a vector space over $\mathbb{K}$ and $L$ be a subspace of $V \oplus V^{*}$ such that $L \subsetneq L^{\perp}$, there exists a non-zero $v \in L^{\perp}$ such that $v \notin L$ and $\langle v, v\rangle=0$.

Proof. To begin with, say $C$ is the non-zero space that satisfies $L^{\perp}=L \oplus C$. We see first that $\operatorname{dim}_{\mathbb{K}} C=1$ is not possible, otherwise the equality

$$
\operatorname{dim}_{\mathbb{K}} L+\operatorname{dim}_{\mathbb{K}} L^{\perp}=2 \operatorname{dim}_{\mathbb{K}} L+\operatorname{dim}_{\mathbb{K}} C=2 \operatorname{dim}_{\mathbb{K}} L+1
$$

would contradict equality (2.2). Therefore, we have $\operatorname{dim}_{\mathbb{K}} C \geq 2$. Moreover, from equality (2.2) we also get

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{K}} L^{\perp}>\operatorname{dim}_{\mathbb{K}} L=2 \operatorname{dim}_{\mathbb{K}} V-\operatorname{dim}_{\mathbb{K}} L^{\perp} \Longrightarrow \operatorname{dim}_{\mathbb{K}} L^{\perp}>\operatorname{dim}_{\mathbb{K}} V \tag{2.3}
\end{equation*}
$$

We distinguish now between the real and the complex case.
On the one hand, suppose that $V \neq 0$ is a vector space over $\mathbb{R}$, take a basis $\left(v_{i}\right)_{i=1}^{n}$ and its dual basis $\left(v^{i}\right)_{i=1}^{n}$ for a certain integer $n \geq 1$, and define $P$ as the
linear span of $v_{1}+v^{1}, \ldots, v_{n}+v^{n}$ and $N$ as the linear span of $v_{1}-v^{1}, \ldots, v_{n}-v^{n}$. Clearly, we have

$$
\operatorname{dim}_{\mathbb{R}} P=\operatorname{dim}_{\mathbb{R}} N=\operatorname{dim}_{\mathbb{R}} V=n
$$

As a result, from equality (2.3) we get

$$
\begin{aligned}
2 \operatorname{dim}_{\mathbb{R}} V & =\operatorname{dim}_{\mathbb{R}}\left(V \oplus V^{*}\right) \geq \operatorname{dim}_{\mathbb{R}}\left(L^{\perp}+P\right)=\operatorname{dim}_{\mathbb{R}} L^{\perp}+\operatorname{dim}_{\mathbb{R}} P-\operatorname{dim}_{\mathbb{R}} L^{\perp} \cap P \\
& >2 \operatorname{dim}_{\mathbb{R}} V-\operatorname{dim}_{\mathbb{R}} L^{\perp} \cap P
\end{aligned}
$$

which gives $\operatorname{dim}_{\mathbb{R}} L^{\perp} \cap P>0$ and $L^{\perp} \cap P \neq 0$. Analogously, we get $L^{\perp} \cap N \neq 0$. Consequently, there exist non-zero vectors $u_{1} \in L^{\perp} \cap P$ and $w_{1} \in L^{\perp} \cap N$. What is more, since $L^{\perp}=L \oplus C$, there are decompositions $u_{1}=u_{2}+u$ and $w_{1}=w_{2}+w$ with $u_{2}, w_{2} \in L \subsetneq L^{\perp}$ and $u, w \in C$. Thus, we get

$$
\begin{aligned}
\langle u, u\rangle & =\left\langle u_{1}-u_{2}, u_{1}-u_{2}\right\rangle=\left\langle u_{1}, u_{1}\right\rangle-\left\langle u_{1}, u_{2}\right\rangle-\left\langle u_{2}, u_{1}\right\rangle+\left\langle u_{2}, u_{2}\right\rangle \\
& =\left\langle u_{1}, u_{1}\right\rangle+0+0+0=\left\langle u_{1}, u_{1}\right\rangle>0
\end{aligned}
$$

and in the same way also

$$
\begin{aligned}
\langle w, w\rangle & =\left\langle w_{1}-w_{2}, w_{1}-w_{2}\right\rangle=\left\langle w_{1}, w_{1}\right\rangle-\left\langle w_{1}, w_{2}\right\rangle-\left\langle w_{2}, w_{1}\right\rangle+\left\langle w_{2}, w_{2}\right\rangle \\
& =\left\langle w_{1}, w_{1}\right\rangle+0+0+0=\left\langle w_{1}, w_{1}\right\rangle<0
\end{aligned}
$$

where in each last step we used that a non-zero $v \in P$ satisfies $\langle v, v\rangle>0$ whereas a non-zero $v \in N$ satisfies $\langle v, v\rangle<0$ (recall (2.1)). Thereby, $u$ and $w$ are non-zero and linearly independent. Besides, the polynomial

$$
P(x):=\langle u, u\rangle x^{2}+2\langle w, u\rangle x+\langle w, w\rangle \in \mathbb{R}[x]
$$

has at least a root $\lambda \in \mathbb{R}$. By construction, $v:=w+\lambda u \in C$ is non-zero and satisfies $\langle v, v\rangle=0$.

On the other hand, suppose that $V$ is a vector space over $\mathbb{C}$. At the beginning of the proof we saw $\operatorname{dim}_{\mathbb{C}} C \geq 2$, so we can take two linearly independent non-zero vectors $u, w \in C$. Then, the polynomial

$$
P(x):=\langle u, u\rangle x^{2}+2\langle w, u\rangle x+\langle w, w\rangle \in \mathbb{C}[x]
$$

has at least a root $\lambda \in \mathbb{C}$. Again, by construction $v:=w+\lambda u \in C$ is non-zero and satisfies $\langle v, v\rangle=0$.

Remark 2.15. As it is standard in algebra, an isotropic subspace is called maximally isotropic if it is not strictly contained in an isotropic subspace.

## Proposition 2.16.

Let $V$ be a vector space over $\mathbb{K}$. Given a subspace $L$ of $V \oplus V^{*}$, the next statements are equivalent.
(a) $L$ is isotropic and $\operatorname{dim}_{\mathbb{K}} L=\operatorname{dim}_{\mathbb{K}} V$.
(b) $L=L^{\perp}$.
(c) $L$ is maximally isotropic.

Proof. We will see $(a) \Leftrightarrow(b)$ and $(b) \Leftrightarrow(c)$.
$(a) \Rightarrow(b)$. It follows directly from the hypothesis $L \subseteq L^{\perp}$ and from the equality $\operatorname{dim}_{\mathbb{K}} L=\operatorname{dim}_{\mathbb{K}} L^{\perp}$ given by the hypothesis $\operatorname{dim}_{\mathbb{K}} L=\operatorname{dim}_{\mathbb{K}} V$ and equality (2.2).
$(a) \Leftarrow(b)$. Similarly, isotropy and $\operatorname{dim}_{\mathbb{K}} L=\operatorname{dim}_{\mathbb{K}} L^{\perp}$ follow from the hypothesis and $\operatorname{dim}_{\mathbb{K}} L=\operatorname{dim}_{\mathbb{K}} V$ does from the latter and equality (2.2).
$(b) \Rightarrow(c)$. Once again, $L$ is clearly isotropic. Given now an isotropic subspace $M$ of $V \oplus V^{*}$ such that $L \subseteq M$, which implies $M^{\perp} \subseteq L^{\perp}$, the chain of inclusions

$$
M^{\perp} \subseteq L^{\perp}=L \subseteq M \subseteq M^{\perp}
$$

holds and gives $L=M$ as intended.
$(b) \Leftarrow(c)$. As $L$ is isotropic, we have $L \subseteq L^{\perp}$. By contradiction, suppose $L \subsetneq L^{\perp}$. From Lemma 2.14 there exists a non-zero $v \in L^{\perp}$ such that $v \notin L$ and $\langle v, v\rangle=0$. If we call $S$ the linear span of $v$, we obtain an isotropic subspace $L \oplus S$ that strictly contains $L$ and hence a contradiction to the hypothesis.

## Corollary 2.17.

Let $V$ be a vector space over $\mathbb{R}$. A subspace $L$ of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ is maximally isotropic if and only if it is isotropic and $\operatorname{dim}_{\mathbb{C}} L=\operatorname{dim}_{\mathbb{R}} V$.

Proof. Take the vector space $V_{\mathbb{C}}$, which is a vector space over $\mathbb{C}$. The result is direct from the complex version of Proposition 2.16 together with Proposition 1.19.

Therefore, from Proposition 2.11 and Proposition 2.13 we get that any linear generalized complex structure on a vector space $V$ over $\mathbb{R}$ is equivalently given by a maximally isotropic subspace $L$ of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ that satisfies $L \cap \bar{L}=0$. To conclude this subsection, we show an analogous result to Theorem 1.10 and Theorem 1.16 for linear generalized complex structures.

## Theorem 2.18.

Let $V$ be a vector space over $\mathbb{R}$, there exists a linear generalized complex structure $\mathcal{J}$ on $V$ if and only if $\operatorname{dim}_{\mathbb{R}} V=2 n$ for a certain integer $n \geq 0$.

Proof. Let $V$ be a vector space over $\mathbb{R}$ together with a linear generalized complex structure $\mathcal{J}$. In case $V \neq 0$, take any non-zero $v_{1} \in V$. From Proposition 2.6 we have $\left\langle\mathcal{J} v_{1}, v_{1}\right\rangle=0$ and as $\left\langle v_{1}, v_{1}\right\rangle=0$ holds it also gives $\left\langle\mathcal{J} v_{1}, \mathcal{J} v_{1}\right\rangle=0$. Moreover, since $\mathcal{J}$ is in particular a linear generalized complex structure on $V \oplus V^{*}$ the same argument given in the proof of Theorem 1.16 shows that $v_{1}$ and $\mathcal{J} v_{1}$ are linearly independent. Therefore, their linear span, say $U_{1}$, is a two-dimensional isotropic subspace of $V \oplus V^{*}$. If $U_{1}$ is maximally isotropic, we stop here.

Otherwise, by Proposition 2.16 we have $U_{1} \subsetneq U_{1}^{\perp}$ and from the real version of Lemma 2.14 there is a non-zero vector $v_{2} \in U_{1}^{\perp}$ such that $v_{2} \notin U_{1}$ and $\left\langle v_{2}, v_{2}\right\rangle=0$. Then, as before we have $\left\langle\mathcal{J} v_{2}, v_{2}\right\rangle=0$ and $\left\langle\mathcal{J} v_{2}, \mathcal{J} v_{2}\right\rangle=0$. Moreover, our choice gives $\left\langle U_{1}, v_{2}\right\rangle=0$ and since by construction $\mathcal{J} U_{1}=U_{1}$, it gives

$$
\left\langle\mathcal{J} v_{2}, U_{1}\right\rangle=-\left\langle v_{2}, \mathcal{J} U_{1}\right\rangle=-\left\langle v_{2}, U_{1}\right\rangle=0
$$

as well. In addition, once again by the same argument of Theorem 1.16 we have that $v_{1}, \mathcal{J} v_{1}, v_{2}$ and $\mathcal{J} v_{2}$ are linearly independent, and hence their linear span, say $U_{2}$, is a four-dimensional isotropic subspace of $V \oplus V^{*}$. Since $\operatorname{dim}_{\mathbb{R}} U_{2}=\operatorname{dim}_{\mathbb{R}} U_{1}+2$ and we only work with finite-dimensional vector spaces, if we go on inductively sooner or later we will end up with an isotropic subspace $U_{n}$, for a certain integer $n \geq 1$, which is maximally isotropic. By construction, we have $\operatorname{dim}_{\mathbb{R}} U_{n}=2 n$ and the real version of Proposition 2.16 gives

$$
\operatorname{dim}_{\mathbb{R}} V=\operatorname{dim}_{\mathbb{R}} U_{n}=2 n .
$$

Finally, the converse has already been shown through Example 2.7.

### 2.2.4. A description of maximally isotropic subspaces

After Proposition 2.16, our next step is to give a more descriptive characterization of maximally isotropic subspaces.

Let $V$ be a vector space over $\mathbb{K}$ and $L$ be a subspace of $V \oplus V^{*}$. To begin with, observe that given $X+\beta \in L$ and $\alpha \in V^{*}$, we have

$$
\begin{equation*}
X+\alpha=X+\beta+(\alpha-\beta) \tag{2.4}
\end{equation*}
$$

and thus $X+\alpha \in L$ if and only if $\alpha-\beta \in L$, this is $\alpha-\beta \in L \cap V^{*}$. In our case of interest we have the following preliminary result, where we refer to the linear projection maps $V \oplus V^{*} \rightarrow V$ and $V \oplus V^{*} \rightarrow V^{*}$ as $\pi_{V}$ and $\pi_{V^{*}}$, respectively.

## Lemma 2.19.

Let $V$ be a vector space over $\mathbb{K}$ and $L$ be a subspace of $V \oplus V^{*}$, we have

$$
L \cap V^{*}=\operatorname{Ann}\left(\pi_{V} L^{\perp}\right) \text { and } L \cap V=\operatorname{Ann}\left(\pi_{V^{*}} L^{\perp}\right)
$$

Proof. Let us just see the first case as the second is analogous. On the one hand, take $\beta \in L \cap V^{*}$. For any $X \in \pi_{V} L^{\perp}$, there is one $\alpha \in V^{*}$ such that $X+\alpha \in L^{\perp}$. Thus, we have

$$
\beta(X)=2\langle X+\alpha, \beta\rangle=0,
$$

which gives $\beta \in \operatorname{Ann}\left(\pi_{V} L^{\perp}\right)$. On the other hand, take $\beta \in \operatorname{Ann}\left(\pi_{V} L^{\perp}\right)$. For any $X+\alpha \in L^{\perp}$, we have $X \in \pi_{V} L^{\perp}$ and thus

$$
2\langle X+\alpha, \beta\rangle=\beta(X)=0
$$

which together with Proposition 2.4 gives $\beta \in\left(L^{\perp}\right)^{\perp}=L$.

Therefore, in case $L$ is maximally isotropic this result gives that the linear map

$$
\begin{align*}
\varepsilon: & E \tag{2.5}
\end{align*} \rightarrow E^{*},
$$

where $E:=\pi_{V} L$ and $\alpha \in V^{*}$ is one such that $X+\alpha \in L$, is well-defined. Indeed, by Proposition 2.16 we have $L=L^{\perp}$. Thus, if $X \in E$ and $\alpha, \beta \in V^{*}$ are ones such that $X+\alpha, X+\beta \in L$ from Proposition 2.16 and Lemma 2.19 we get

$$
\alpha-\beta \in L \cap V^{*}=\operatorname{Ann}\left(\pi_{V} L^{\perp}\right)=\operatorname{Ann} E,
$$

which gives $\left.\alpha\right|_{E}=\left.\beta\right|_{E}$. On top of that, for any $X \in E$ there is one $\alpha \in V^{*}$ such that $X+\alpha \in L$ and since $L$ is isotropic we get

$$
\left.\alpha\right|_{E}(X)=\alpha(X)=\langle X+\alpha, X+\alpha\rangle=0,
$$

which gives that the induced bilinear map $\varepsilon^{\natural}$ from (1.3) is skew-symmetric (recall Remark 1.8). Finally, by defining the subspace

$$
\begin{equation*}
L(E, \varepsilon):=\left\{X+\alpha \in E \oplus V^{*}:\left.\alpha\right|_{E}=\varepsilon(X)\right\} \tag{2.6}
\end{equation*}
$$

of $V \oplus V^{*}$ we obtain the following result.

## Proposition 2.20.

Let $V$ be a vector space over $\mathbb{K}$ and $L$ be a maximally isotropic subspace $L$ of $V \oplus V^{*}$, we have $L=L(E, \varepsilon)$, where $E:=\pi_{V} L$ and $\varepsilon$ is the linear map (2.5).

Proof. On the one hand, our construction gives $L \subseteq L(E, \varepsilon)$. On the other hand, take $X+\alpha \in L(E, \varepsilon)$. As $X \in E$, there is one $\beta \in V^{*}$ such that $X+\beta \in L$ and thus we get $\varepsilon(X)=\left.\beta\right|_{E}$. Thereby, we have $\left.\alpha\right|_{E}=\left.\beta\right|_{E}$ and from Proposition 2.16 and Lemma 2.19 we get

$$
\alpha-\beta \in \operatorname{Ann} E=\operatorname{Ann} \pi_{V} L^{\perp}=L \cap V^{*} \subseteq L,
$$

which together with equality (2.4) gives $X+\alpha \in L$.
Observe now that one can define the subspace (2.6) of $V \oplus V^{*}$ starting with any subspace $E$ of $V$ and any linear map $\varepsilon: E \rightarrow E^{*}$. Correspondingly, the converse result follows.

## Proposition 2.21.

Let $V$ be a vector space over $\mathbb{K}$. Given a subspace $E$ of $V$ and a linear map $\varepsilon: E \rightarrow E^{*}$ such that the induced bilinear map $\varepsilon^{\natural}$ is skew-symmetric, the subspace $L:=L(E, \varepsilon)$ of $V \oplus V^{*}$ is maximally isotropic, $E=\pi_{V} L$ and $\varepsilon$ is the linear map (2.5).

Proof. Firstly, take $Y+\beta \in L$. Since $\varepsilon^{\natural}$ is alternate, for any $X+\alpha \in L$ we have

$$
\begin{aligned}
2\langle X+\alpha, Y+\beta\rangle & =\beta(X)+\alpha(Y)=\left.\beta\right|_{E}(X)+\left.\alpha\right|_{E}(Y) \\
& =(\varepsilon(X))(X)+(\varepsilon(Y))(X)=\varepsilon^{\natural}(X, X)+\varepsilon^{\natural}(Y, Y)=0+0=0
\end{aligned}
$$

and thus $L$ is isotropic. Secondly, it is easy to verify that we have Ann $E \subseteq L(E, \varepsilon)$ and that the succession

$$
\begin{align*}
0 \longrightarrow \text { Ann } E \longrightarrow \begin{array}{l}
L(E, \varepsilon) \\
\\
X+\alpha \\
\longmapsto
\end{array} \quad X & \longrightarrow \tag{2.7}
\end{align*}
$$

is exact, namely $\operatorname{Ker} \pi_{V}=\operatorname{Ann} E$ and $\pi_{V}(L(E, \varepsilon))=E$. In particular, the equality (1.1) together with the First Isomorphism Theorem gives

$$
\operatorname{dim}_{\mathbb{K}} L(E, \varepsilon)=\operatorname{dim}_{\mathbb{K}} E+\operatorname{dim}_{\mathbb{K}} \operatorname{Ann} E=\operatorname{dim}_{\mathbb{K}} V
$$

and thus from Proposition 2.16 we obtain that $L$ is maximally isotropic. Finally, by construction we get $E=\pi_{V} L$ and that $\varepsilon$ is given by (2.5).

### 2.2.5. Restriction to a subspace

Given a proper subspace $U$ of $V$ and a maximally isotropic subspace $L$ of $V \oplus V^{*}$, we will now use the last results to study $L$ from the viewpoint of $U \oplus U^{*}$. Our reason for this is to see what a linear generalized complex structure on $V$ induces on $U$. Let us remark that this section was inspired by Section 6 of [Bur13] and developed independently.

Remark 2.22. For a proper subspace $U$ of $V$, there is a restriction map $\left.\right|_{U}: V^{*} \rightarrow U^{*}$ and thus $U^{*}$ is a quotient rather than a subset of $V^{*}$.

This observation implies that we cannot just consider $\left(U \oplus U^{*}\right) \cap L$ because this intersection is not well-defined. However, one can easily show by double inclusion that we have

$$
U^{*}=\left\{\left.\alpha\right|_{U}: \alpha \in V^{*}\right\} \text { and hence } U \oplus U^{*}=\left\{X+\left.\alpha\right|_{U}: X+\alpha \in U \oplus V^{*}\right\} .
$$

Thereby, we define the following.

Definition 2.23 (Restriction of a maximally isotropic subspace).
Let $V$ be a vector space over $\mathbb{K}$ and $L$ be a maximally isotropic subspace of $V \oplus V^{*}$. Given a subspace $U$ of $V$, the restriction of $L$ to $U \oplus U^{*}$ is the subspace

$$
L_{U}:=\left\{X+\left.\alpha\right|_{U}: X+\alpha \in\left(U \oplus V^{*}\right) \cap L\right\} \text { of } U \oplus U^{*} .
$$

In particular, we recover $L$ by taking $U=V$, this is $L_{V}=L$. We show next that $L_{U}$ is in fact a maximally isotropic subspace of $U \oplus U^{*}$.

## Proposition 2.24.

Let $V$ be a vector space over $\mathbb{K}$, $U$ be a subspace of $V$ and $L$ be a maximally isotropic subspace of $V \oplus V^{*}$, the subspace $L_{U}$ of $U \oplus U^{*}$ is maximally isotropic.

Proof. Recall that from Proposition 2.20 we have $L=L(E, \varepsilon)$, where $E:=\pi_{V} L$ and $\varepsilon$ is the linear map (2.5). To begin with, note that the linear map

$$
\begin{array}{rl}
\varepsilon_{U}: E \cap U & E \xrightarrow{\varepsilon} E^{*} \\
\alpha & \longrightarrow(E \cap U)^{*} \\
& \left.\longmapsto \alpha\right|_{E \cap U}
\end{array}
$$

induces a skew-symmetric bilinear map $\varepsilon_{U}^{\natural}$ from (1.3). Since $E \cap U$ is a subspace of $U$, Proposition 2.21 gives that $L\left(E \cap U, \varepsilon_{U}\right)$ is a maximally isotropic subspace of $U \oplus U^{*}$. Therefore, it suffices to show that we have $L_{U}=L\left(E \cap U, \varepsilon_{U}\right)$.

By double inclusion, on the one hand take $X+\left.\alpha\right|_{U} \in L_{U}$, namely

$$
X+\alpha \in\left(U \oplus V^{*}\right) \cap L
$$

In the first place, we have $X \in U$. In the second place, we have $X+\alpha \in L$, which gives $X \in E$ and $\left.\alpha\right|_{E}=\varepsilon(X)$. Consequently, we get $X \in E \cap U$ and

$$
\left.\left(\left.\alpha\right|_{U}\right)\right|_{E \cap U}=\left.\alpha\right|_{E \cap U}=\left.\left(\left.\alpha\right|_{E}\right)\right|_{E \cap U}=\left.(\varepsilon(X))\right|_{E \cap U}=\varepsilon_{U}(X),
$$

which give $X+\left.\alpha\right|_{U} \in L\left(E \cap U, \varepsilon_{U}\right)$. On the other hand, take

$$
X+\beta \in L\left(E \cap U, \varepsilon_{U}\right)
$$

We have $X \in E \cap U$ and that $\beta \in U^{*}$ satisfies $\left.\beta\right|_{E \cap U}=\varepsilon_{U}(X)$. In this instance, the linear map

$$
\begin{aligned}
\gamma: E+U & \longrightarrow \mathbb{K} \\
Y+Z & \longmapsto(\varepsilon(X))(Y)+\beta(Z)
\end{aligned}
$$

is well-defined. Indeed, let $Y_{1}, Y_{2} \in E$ and $Z_{1}, Z_{2} \in U$ be such that $Y_{1}+Z_{1}=Y_{2}+Z_{2}$. We have

$$
Y_{1}-Y_{2}=Z_{2}-Z_{1} \in E \cap U
$$

and hence we get

$$
\begin{aligned}
\beta\left(Z_{2}-Z_{1}\right) & =\left.\beta\right|_{E \cap U}\left(Z_{2}-Z_{1}\right)=\left(\varepsilon_{U}(X)\right)\left(Z_{2}-Z_{1}\right) \\
& =\left.(\varepsilon(X))\right|_{E \cap U}\left(Z_{2}-Z_{1}\right)=(\varepsilon(X))\left(Z_{2}-Z_{1}\right)=(\varepsilon(X))\left(Y_{1}-Y_{2}\right),
\end{aligned}
$$

which gives

$$
(\varepsilon(X))\left(Y_{1}\right)+\beta\left(Z_{1}\right)=(\varepsilon(X))\left(Y_{2}\right)+\beta\left(Z_{2}\right) .
$$

Take now one $\alpha \in V^{*}$ such that $\left.\alpha\right|_{E+U}=\gamma$ and consider $X+\alpha$. In the first place, since $X \in E \cap U$, we have $X+\alpha \in U \oplus V^{*}$. In the second place, we have $X \in E$ and that $\alpha \in V^{*}$ satisfies

$$
\left.\alpha\right|_{E}=\left.\left(\left.\alpha\right|_{E+U}\right)\right|_{E}=\left.\gamma\right|_{E}=\varepsilon(X),
$$

which gives $X+\alpha \in L$ and thus we have

$$
X+\alpha \in\left(U \oplus V^{*}\right) \cap L
$$

and $X+\left.\alpha\right|_{U} \in L_{U}$. Finally, we have

$$
\left.\alpha\right|_{U}=\left.\left(\left.\alpha\right|_{E+U}\right)\right|_{U}=\left.\gamma\right|_{U}=\beta,
$$

which gives $X+\beta \in L_{U}$.

At this point, consider any vector space $V$ over $\mathbb{R}$ with $\operatorname{dim}_{\mathbb{R}} V=2 n$ for a certain integer $n \geq 1$. From Theorem 2.18, we know that $V$ admits a linear generalized complex structure $\mathcal{J}$ and Proposition 2.11 together with Proposition 2.16 assure that its $+i$-eigenspace $L$ is a maximally isotropic subspace of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ such that $L \cap \bar{L}=0$. Consider now any subspace $U$ of $V$ with $\operatorname{dim}_{\mathbb{R}} U=2 n-1$. Firstly, $U_{\mathbb{C}}$ is a subspace of $V_{\mathbb{C}}$ and the complex version of Proposition 2.24 gives that $L_{U_{\mathbb{C}}}$ is a maximally isotropic subspace of $\left(U \oplus U^{*}\right)_{\mathbb{C}}$ (recall Remark 2.9). Secondly, we have $L_{U_{\mathrm{C}}} \cap \overline{L_{U_{\mathrm{C}}}} \neq 0$. Indeed, $L_{U_{\mathrm{C}}} \cap \overline{L_{U_{\mathrm{C}}}}=0$ would give by Proposition 2.13 that there is a linear generalized complex structure $\mathcal{J}_{U}$ of $U$, which contradicts Theorem 2.18.

Therefore, this suggests that we should set aside the condition $L \cap \bar{L}=0$ and study the bigger class of maximally isotropic subspaces.

### 2.3. Linear complex Dirac structures

The following definition is just a renaming of maximally isotropic subspaces.
Definition 2.25 (Linear complex Dirac structure).
Let $V$ be a vector space over $\mathbb{R}$. A linear complex Dirac structure on $V$ is a maximally isotropic subspace $L$ of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$.

In order to give an overview of linear complex Dirac structures, we will study three integers that characterize them. Throughout this last section of the chapter, our main guideline is Section 3.1 of [Agu20] and we also take some information from Section 4 of [AR22].

Definition 2.26 (The invariants).
Let $V$ be a vector space over $\mathbb{R}, L$ be a linear complex Dirac structure on $V$ and $E:=\pi_{V_{\mathrm{C}}} L$, the invariants of $L$ are the following non-negative integers.

- The real index of $L$ is $r:=\operatorname{dim}_{\mathbb{C}}(L \cap \bar{L})$.
- The order of $L$ is $s:=\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}-\operatorname{dim}_{\mathbb{C}}(E+\bar{E})$.
- The type of $L$ is $t:=\operatorname{dim}_{\mathbb{C}}(E+\bar{E})-\operatorname{dim}_{\mathbb{C}} E$.

The real index, introduced in [Gua03], is the most basic invariant. We have the following fundamental result, analogous to Theorem 2.18.

## Theorem 2.27.

Let $V$ be a vector space over $\mathbb{R}$, there exists a linear complex Dirac structure $L$ on $V$ with real index $r$ if and only if $\operatorname{dim}_{\mathbb{R}} V=2 n+r$ for a certain integer $n \geq 0$.

The proof of the direct statement is developed in subsection 3.1.2 and concluded in Corollary 3.8 of [Agu20]. Moreover, Example 3.15 or Example 3.16 also from [Agu20] suffice to prove the converse.

In order to study some properties of the invariants, we have to introduce some associated subspaces. Given a linear complex Dirac structure $L$ on a vector space $V$ over $\mathbb{R}$, recall Definition 1.21 and consider the subspaces

$$
\begin{align*}
E & :=\pi_{V_{\mathbb{C}}} L \text { of } V_{\mathbb{C}}, \\
K & :=\operatorname{Re}(L \cap \bar{L}) \text { of } V \oplus V^{*}, \\
\Delta_{0} & :=\pi_{V} K \text { of } V,  \tag{2.8}\\
\Delta & :=\operatorname{Re}(E \cap \bar{E}) \text { of } V, \\
D & :=\operatorname{Re}(E+\bar{E}) \text { of } V .
\end{align*}
$$

In the first place, from Proposition 1.19 and Proposition 1.23 we have

$$
\begin{align*}
& \operatorname{dim}_{\mathbb{R}} K=\operatorname{dim}_{\mathbb{C}}(L \cap \bar{L}), \quad \operatorname{dim}_{\mathbb{R}} \operatorname{Re}(L+\bar{L})=\operatorname{dim}_{\mathbb{C}}(L+\bar{L})  \tag{2.9}\\
& \operatorname{dim}_{\mathbb{R}} \Delta=\operatorname{dim}_{\mathbb{C}}(E \cap \bar{E}), \quad \text { and } \operatorname{dim}_{\mathbb{R}} D=\operatorname{dim}_{\mathbb{C}}(E+\bar{E})
\end{align*}
$$

In the second place, we have the next results.

## Lemma 2.28.

Let $V$ be a vector space over $\mathbb{R}, L$ be a linear complex Dirac structure on $V$ and $K$ and $D$ be the associated subspaces of $L$ from (2.8), we have $D=\pi_{V} K^{\perp}$.

Proof. Firstly, we will show that the equality

$$
\begin{equation*}
K^{\perp}=\operatorname{Re}(L+\bar{L}) \tag{2.10}
\end{equation*}
$$

holds. On the one hand, take $u \in \operatorname{Re}(L+\bar{L})$. We have $u \in L+\bar{L}$ (recall Remark 1.14) and hence there are $u_{1} \in L$ and $u_{2} \in \bar{L}$ such that $u=u_{1}+u_{2}$. Furthermore, since $L$ is isotropic, we have that $\bar{L}$ is isotropic as well (Proposition 2.10). Thereby, for any $w \in K$ we have $w \in L \cap \bar{L}$ and we get

$$
\langle w, u\rangle=\left\langle w, u_{1}+u_{2}\right\rangle=\left\langle w, u_{1}\right\rangle+\left\langle w, u_{2}\right\rangle=0+0=0,
$$

which gives $u \in K^{\perp}$ and thus $\operatorname{Re}(L+\bar{L}) \subseteq K^{\perp}$. On the other hand, from equalities (1.4) and (2.9), Corollary 2.17 and Proposition 1.22 we get

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{R}} K^{\perp} & =\operatorname{dim}_{\mathbb{R}}\left(V \oplus V^{*}\right)-\operatorname{dim}_{\mathbb{R}} K=2 \operatorname{dim}_{\mathbb{R}} V-\operatorname{dim}_{\mathbb{C}}(L \cap \bar{L}) \\
& =2 \operatorname{dim}_{\mathbb{R}} V-\left(2 \operatorname{dim}_{\mathbb{C}} L-\operatorname{dim}_{\mathbb{C}}(L+\bar{L})\right) \\
& =2 \operatorname{dim}_{\mathbb{R}} V-\left(2 \operatorname{dim}_{\mathbb{R}} V-\operatorname{dim}_{\mathbb{R}} \operatorname{Re}(L+\bar{L})\right)=\operatorname{dim}_{\mathbb{R}} \operatorname{Re}(L+\bar{L}),
\end{aligned}
$$

which gives equality (2.10). Secondly, the chain of equalities

$$
\operatorname{Re}(E+\bar{E})=\operatorname{Re}\left(\pi_{V_{\mathbb{C}}} L+\pi_{V_{\mathbb{C}}} \bar{L}\right)=\operatorname{Re}\left(\pi_{V_{\mathbb{C}}}(L+\bar{L})\right)=\pi_{V} \operatorname{Re}(L+\bar{L})
$$

holds, which together with equality (2.10) gives $D=\pi_{V} K^{\perp}$.

## Corollary 2.29 .

Let $V$ be a vector space over $\mathbb{R}, L$ be a linear complex Dirac structure on $V$ and $K$ be the associated subspace of $L$ from (2.8), we have that $K$ is isotropic.

Proof. It is direct from inclusion $\operatorname{Re}(L \cap \bar{L}) \subseteq \operatorname{Re}(L+\bar{L})$ and equality (2.10).

## Proposition 2.30.

Let $V$ be a vector space over $\mathbb{R}, L$ be a linear complex Dirac structure on $V$ and $\Delta_{0}$, $\Delta$ and $D$ be the associated subspaces of $L$ from (2.8), we have $\Delta_{0} \subseteq \Delta \subseteq D$.

Proof. It is easy to verify that the chain of inclusions

$$
\begin{aligned}
\pi_{V} K & =\pi_{V} \operatorname{Re}(L \cap \bar{L})=\operatorname{Re}\left(\pi_{V_{\mathbb{C}}}(L \cap \bar{L})\right) \subseteq \operatorname{Re}\left(\left(\pi_{V_{\mathbb{C}}} L\right) \cap\left(\pi_{V_{\mathbb{C}}} \bar{L}\right)\right) \\
& =\operatorname{Re}(E \cap \bar{E}) \subseteq \operatorname{Re}(E+\bar{E})
\end{aligned}
$$

holds, which gives $\Delta_{0} \subseteq \Delta \subseteq D$.

Once seen this results for the associated subspaces, with regard to the invariants we obtain the following result.

## Proposition 2.31.

Let $V$ be a vector space over $\mathbb{R}$, $L$ be a linear complex Dirac structure on $V$ with real index $r$, order s and type $t$ and $\Delta_{0}$ and $\Delta$ be the associated subspaces of $L$ from (2.8), the following equalities hold.
(a) $\operatorname{dim}_{\mathbb{R}} \Delta_{0}=r-s$.
(b) $\operatorname{dim}_{\mathbb{R}} \Delta=\operatorname{dim}_{\mathbb{R}} V-s-2 t$.

## Proof.

(a). It is easy to verify that the succession

$$
0 \longrightarrow K \cap V^{*} \longrightarrow K \xrightarrow{\pi_{V}} \Delta_{0} \longrightarrow 0
$$

is exact, which together with the First Isomorphism Theorem gives

$$
\operatorname{dim}_{\mathbb{R}} K-\operatorname{dim}_{\mathbb{R}} K \cap V^{*}=\operatorname{dim}_{\mathbb{R}} \Delta_{0} .
$$

Besides, Lemma 2.19 and Lemma 2.28 give

$$
K \cap V^{*}=\operatorname{Ann}\left(\pi_{V} K^{\perp}\right)=\operatorname{Ann} D
$$

and hence from equalities (1.1) and (2.9) together with Proposition 1.19 we get

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{R}} \Delta_{0} & =\operatorname{dim}_{\mathbb{R}} K-\operatorname{dim}_{\mathbb{R}}\left(K \cap V^{*}\right)=\operatorname{dim}_{\mathbb{C}}(L \cap \bar{L})-\operatorname{dim}_{\mathbb{R}} \operatorname{Ann} D \\
& =r-\left(\operatorname{dim}_{\mathbb{R}} V-\operatorname{dim}_{\mathbb{R}} D\right)=r-\left(\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}-\operatorname{dim}_{\mathbb{C}}(E+\bar{E})\right)=r-s .
\end{aligned}
$$

(b). From Proposition 1.19 and Proposition 1.22, we get

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{R}} \Delta & =\operatorname{dim}_{\mathbb{C}}(E \cap \bar{E})=2 \operatorname{dim}_{\mathbb{C}} E-\operatorname{dim}_{\mathbb{C}}(E+\bar{E})=\operatorname{dim}_{\mathbb{C}} E-t \\
& =\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}-\left(\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}-\operatorname{dim}_{\mathbb{C}} E\right)-t=\operatorname{dim}_{\mathbb{R}} V-(s+t)-t \\
& =\operatorname{dim}_{\mathbb{R}} V-s-2 t .
\end{aligned}
$$

## Corollary 2.32.

Let $V$ be a vector space over $\mathbb{R}$ and $L$ be a linear complex Dirac structure of $V$ with real index $r$, order $s$ and type $t$, the following inequalities hold.
(a) $0 \leq r \leq \operatorname{dim}_{\mathbb{R}} V$.
(b) $0 \leq s \leq r$.
(c) $0 \leq t \leq \frac{1}{2}\left(\operatorname{dim}_{\mathbb{R}} V-r\right)$.

Proof. Trivially, we have $r, s, t \geq 0$. Moreover, (a) follows directly from Theorem 2.27 and $(b)$ does from Proposition $2.31(a)$ and the fact $\operatorname{dim}_{\mathbb{R}} \Delta_{0} \geq 0$.
(c). From Proposition 2.30 and Proposition 2.31 we get

$$
\begin{gathered}
r-s=\operatorname{dim}_{\mathbb{R}} \Delta_{0} \leq \operatorname{dim}_{\mathbb{R}} \Delta=\operatorname{dim}_{\mathbb{R}} V-s-2 t=\operatorname{dim}_{\mathbb{R}} V-r+r-s-2 t \\
\Longrightarrow 0 \leq \operatorname{dim}_{\mathbb{R}} V-r-2 t \Longrightarrow t \leq \frac{1}{2}\left(\operatorname{dim}_{\mathbb{R}} V-r\right)
\end{gathered}
$$

## Example 2.33.

Note that due to Proposition 2.11 and Proposition 2.13 linear generalized complex structures correspond to linear complex Dirac structures with real index $r=0$ and hence by Corollary 2.32 also of order $s=0$. In regard of the type, we will look at the complex Dirac structures from Example 2.12. Let $V$ be a vector space over $\mathbb{R}$ such that $\operatorname{dim}_{\mathbb{R}} V=2 n$ for a certain integer $n \geq 0$.

On the one hand, let $\omega$ be a linear symplectic structure on $V$ and take the linear complex Dirac structure $L_{\omega}$. Clearly, we have $E=V_{\mathbb{C}}$ and thus the type of $L_{\omega}$ is

$$
t=\operatorname{dim}_{\mathbb{C}}(E+\bar{E})-\operatorname{dim}_{\mathbb{C}} E=\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}-\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}=0
$$

On the other hand, let $J$ be a linear complex structure on $V$ and take the linear complex Dirac structure $L_{J}$. We have $E=V^{0,1}$, where $V^{0,1}$ is the $-i$-eigenspace of $J$, and thus Corollary, 1.17 Proposition 1.22 and Proposition 1.24 give that the type of $L_{J}$ is

$$
\begin{aligned}
t & =\operatorname{dim}_{\mathbb{C}}\left(V^{0,1}+V^{1,0}\right)-\operatorname{dim}_{\mathbb{C}} V^{0,1}=2 \operatorname{dim}_{\mathbb{C}} V^{1,0}-\operatorname{dim}_{\mathbb{C}} V^{1,0} \\
& =\operatorname{dim}_{\mathbb{C}} V^{1,0}=\frac{1}{2} \operatorname{dim}_{\mathbb{R}}\left(V^{1,0}\right)_{\mathbb{R}}=\frac{1}{2} \operatorname{dim}_{\mathbb{R}} V
\end{aligned}
$$

and hence we get from Corollary 2.32 that this structures have extreme type.

To conclude this chapter, let us remark that the invariants we worked with are relevant because they suffice to fully classify linear complex Dirac structures (see Proposition 3.18 of [Agu20] or Proposition 4.15 of [AR22]). Finally, from Corollary 2.32 we get that any possible combination of the invariants is an integer triplet of a tetrahedron (see Figure 2.1, original from [AR22] and where one can find more information about the labelled structures we have not talked about).


Figure 2.1: Tetrahedron representing the linear structures encompassed by linear complex Dirac structures.

## Chapter 3

## Geometry and generalized geometry

As we highlighted in the introduction, the bunch of structures we approached from a linear viewpoint make their mark in geometry. Therefore, in this chapter we integrate some of our study to geometry. This chapter is not as detailed as the previous ones and assumes that the reader has some familiarity with manifolds.

### 3.1. Geometry

### 3.1.1. Preliminaries

To begin with, we gather without much detail some standard definitions and properties that will be required afterwards. For further insight, see [Lee13].

Let $M$ be a manifold, we denote its tangent and cotangent vector bundles by $T$ and $T^{*}$, respectively. Consider any $X \in \Gamma(T)$. In the first place, the contraction by $X$ and the exterior derivative are linear maps

$$
\iota_{X}, d: \Omega^{\bullet}(M) \rightarrow \Omega^{\bullet}(M)
$$

that give $\iota_{X} \alpha \in \Omega^{k-1}(M)$ and $d \alpha \in \Omega^{k+1}(M)$ for any integer $k \geq 0$ and $\alpha \in \Omega^{k}(M)$. Particularly, for any $\alpha \in \Omega^{1}(M)$ and $f \in C^{\infty}(M)$ we have $\iota_{X} \alpha=\alpha(X)$ and

$$
\iota_{X} d f=(d f)(X)=X(f) .
$$

Moreover, for any integers $k, l \geq 0, \alpha \in \Omega^{k}(M)$ and $\beta \in \Omega^{l}(M)$ we have

$$
\begin{align*}
\iota_{X}(\alpha \wedge \beta) & =\left(\iota_{X} \alpha\right) \wedge \beta+(-1)^{k} \alpha \wedge\left(\iota_{X} \beta\right) \text { and }  \tag{3.1}\\
d(\alpha \wedge \beta) & =(d \alpha) \wedge \beta+(-1)^{k} \alpha \wedge(d \beta) . \tag{3.2}
\end{align*}
$$

Besides, it is relevant to note that the exterior derivative satisfies $d \circ d=0$.
Secondly, the Lie derivative of forms $L_{X}: \Omega^{\bullet}(M) \rightarrow \Omega^{\bullet}(M)$ satisfies "Cartan's magic formula", this is

$$
\begin{equation*}
L_{X}=d \circ \iota_{X}+\iota_{X} \circ d \tag{3.3}
\end{equation*}
$$

Thirdly, by understanding vector fields as derivations the bilinear map

$$
\begin{array}{rlc}
{[,]: \quad \Gamma(T) \times \Gamma(T)} & \rightarrow & \Gamma(T) \\
(X, Y) & \mapsto X \circ Y-Y \circ X
\end{array}
$$

gives that the couple $(\Gamma(T),[]$,$) is a Lie algebra, and hence is referred to as the Lie$ bracket. Furthermore, for any $X, Y \in \Gamma(T)$ the Lie bracket satisfies

$$
\begin{equation*}
\iota_{[X, Y]}=L_{X} \circ \iota_{Y}-\iota_{Y} \circ L_{X} \tag{3.4}
\end{equation*}
$$

and for any $f \in C^{\infty}(M)$ we have

$$
\begin{equation*}
[X, f Y]=(X f) Y+f[X, Y] \tag{3.5}
\end{equation*}
$$

Lastly, given an $n$-manifold $M$ for a certain integer $n \geq 0$, one extends $\mathbb{C}$-linearly the Lie bracket to $T_{\mathbb{C}}$ and the contraction, the exterior and the Lie derivative to

$$
\Omega_{\mathbb{C}}^{\bullet}(M):=\bigoplus_{k=0}^{n} \Gamma\left(\wedge^{k}\left(T_{\mathbb{C}}\right)^{*}\right)
$$

### 3.1.2. Complex and symplectic structures

Let $M$ be a manifold. In the place of a vector space, our starting point is now the tangent bundle $T$ of $M$. Thereby, the next definition naturally arises.

Definition 3.1 (Almost complex structure).
Let $M$ be a manifold, an almost complex structure $J$ on $M$ is a vector bundle map $J: T \rightarrow T$ that satisfies $J^{2}=-$ Id. A manifold together with an almost complex structure is called an almost complex manifold.

Nevertheless, from differential geometry we have the next definition.

Definition 3.2 (Complex manifold).
A complex $2 n$-manifold, for an integer $n \geq 0$, is a manifold defined by an atlas of charts to the unit disc of $\mathbb{C}^{n}$ and such that the transition maps are holomorphic. $\diamond$

It is straightforward to show that any complex manifold $M$ induces an almost complex structure on $M$. In a nutshell, it suffices to consider the local coordinate basis $\mathcal{B}$ at each fiber, define a map $J: T M \rightarrow T M$ pointwise by (1.7) and verify that indeed $J$ is a vector bundle map.

Therefore, any complex manifold induces an almost complex structure, but we would like both notions to be equivalent. The converse, however, does not hold in general and requires an extra integrability condition given by the NewlanderNirenberg Theorem, originally proved in [NN57]. Briefly, given an almost complex manifold $(M, J)$ of $+i$-eigenbundle $L$, the theorem claims that if $\Gamma(L)$ is involutive for the Lie bracket, this is

$$
\begin{equation*}
[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L) \tag{3.6}
\end{equation*}
$$

there is a unique holomorphic atlas that endows $M$ with the structure of a complex manifold and such that the induced almost complex structure is precisely $J$ (see [Wel80] for more details). Therefore, the involutivity of $\Gamma(L)$ for the Lie bracket enables us to recover a complex manifold just from the information coded in its tangent vector bundle. Consequently, we define the following.

Definition 3.3 (Complex structure).
Let $M$ be a manifold, a complex structure $J$ on $M$ is an almost complex structure on $M$ such that its $+i$-eigenbundle $L$ satisfies (3.6).

With regard to symplectic geometry, one has a similar situation. In this case, the corresponding integrability condition gives the next definition.

Definition 3.4 (Symplectic structure).
Let $M$ be a manifold, an almost symplectic structure $\omega$ on $M$ is a non-degenerate 2 -form $\omega \in \Omega^{2}(M)$. A symplectic structure $\omega$ on $M$ is a closed almost symplectic structure on $M$, this is $d \omega=0$.

### 3.2. Generalized geometry

In this last section, to get the full picture we combine generalized linear algebra with geometry. Let $M$ be a manifold, our starting point is the generalized tangent
bundle $T \oplus T^{*}$, which we denote as $\mathbb{T}$. This is the global version of the generalized vector space $V \oplus V^{*}$ we took in Chapter 2. In this context, the canonical pairing $\langle$,$\rangle is naturally defined on the fibers of \mathbb{T}$. As in the last chapter, we will introduce generalized geometry as a means to describe together both complex and symplectic structures. With this aim and following the discussion of the last section, one would like to encompass their integrability conditions.

### 3.2.1. The Dorfman Bracket

The joint description of symplectic and complex structures will be done with the Dorfman bracket.

Definition 3.5 (The Dorfman bracket).
Let $M$ be a manifold, the extension of the Lie bracket to $\Gamma(\mathbb{T})$ as

$$
[,]: \begin{align*}
\Gamma(\mathbb{T}) \times \Gamma(\mathbb{T}) & \rightarrow \Gamma(\mathbb{T})  \tag{3.7}\\
(X+\alpha, Y+\beta) & \mapsto[X, Y]+L_{X} \beta-\iota_{Y} d \alpha
\end{align*}
$$

is called the Dorfman bracket.

Clearly, the Dorfman bracket is bilinear. To see its properties, consider first the projection map $\pi_{T}: \Gamma(\mathbb{T}) \rightarrow \Gamma(T)$, the isomorphism $\langle,\rangle^{b}$ from (1.2) and define

$$
D:=\left(\langle,\rangle^{b}\right)^{-1} \circ\left(\pi_{T}\right)^{*} \circ d: C^{\infty}(M) \rightarrow \Gamma(\mathbb{T}) .
$$

## Lemma 3.6.

Let $M$ be a manifold, we have $D=2 d$.

Proof. Take $f \in C^{\infty}(M)$. In the first place, for any $X+\alpha \in \Gamma(\mathbb{T})$ we have

$$
\left(\left(\pi_{T}\right)^{*}(d f)\right)(X+\alpha)=d f\left(\pi_{T}(X+\alpha)\right)=d f(X)=2\langle X+\alpha, d f\rangle,
$$

which gives $\left(\pi_{T}\right)^{*} \circ d=2\langle,\rangle^{b} \circ d$ and thus $D=2 d$.

## Proposition 3.7.

Let $M$ be a manifold, for any $f \in C^{\infty}(M)$ and $u, v, w \in \Gamma(\mathbb{T})$ the following holds.
(a) $[u, v]+[v, u]=D\langle u, v\rangle$.
(b) $[u,[v, w]]=[[u, v], w]+[v,[u, w]]$.
(c) $\left(\pi_{T}(u)\right)(\langle v, w\rangle)=\langle[u, v], w\rangle+\langle v,[u, w]\rangle$.
(d) $\pi_{T}([u, v])=\left[\pi_{T}(u), \pi_{T}(v)\right]$.
(e) $[u, f v]=\left(\pi_{T}(u)\right)(f) v+f[u, v]$.

Proof. (d) is trivial and (a), (b), (c) and (e) are direct computations. Let us see (a) and $(e)$ and omit the rest. Take $u=X+\alpha, v=Y+\beta \in \Gamma(\mathbb{T})$.
(a) From equality (3.3), we have

$$
\begin{aligned}
& {[u, v]=[X, Y]+L_{X} \beta-\iota_{X} d \alpha=[X, Y]+d\left(\iota_{X} \beta\right)+\iota_{X} d \beta-\iota_{Y} d \alpha \text { and }} \\
& {[v, u]=[Y, X]+L_{Y} \alpha-\iota_{Y} d \beta=[Y, X]+d\left(\iota_{Y} \alpha\right)+\iota_{Y} d \alpha-\iota_{X} d \beta .}
\end{aligned}
$$

Since the Lie bracket is skew-symmetric, adding this equations we get

$$
\begin{align*}
{[u, v]+[v, u] } & =[X, Y]+[Y, X]+d\left(\iota_{Y} \beta\right)+d\left(\iota_{X} \alpha\right)  \tag{3.8}\\
& =[X, Y]-[X, Y]+d\left(\iota_{Y} \beta+\iota_{X} \alpha\right)=2 d\langle u, v\rangle,
\end{align*}
$$

which together with Lemma 3.6 gives the intended equality.
(e) Consider $f \in C^{\infty}(M)$. From equalities (3.1), (3.2) and (3.3), we have

$$
\begin{aligned}
L_{X}(f \beta)-\iota_{f Y} d \alpha & =\iota_{X} d(f \beta)+d\left(f \iota_{X} \beta\right)-d \alpha(f Y) \\
& =\iota_{X}(d f \wedge \beta+f d \beta)+\left(\iota_{X} \beta\right) d f+f d\left(\iota_{X} \beta\right)-f d \alpha(Y) \\
& =\left(\iota_{X} d f\right) \wedge \beta+f \iota_{X} d \beta+f d\left(\iota_{X} \beta\right)-f \iota_{Y} d \alpha \\
& =X(f) \beta+f\left(d\left(\iota_{X} \beta\right)+\iota_{X} d \beta-i_{Y} d \alpha\right) .
\end{aligned}
$$

Thus, from equality (3.5) we get

$$
\begin{aligned}
{[u, f v] } & =[X, f Y]+L_{X}(f \beta)-\iota_{f Y} d \alpha \\
& =X(f) Y+f[X, Y]+X(f) \beta+f\left(d\left(\iota_{X} \beta\right)+\iota_{X} d \beta-i_{Y} d \alpha\right) \\
& =X(f) v+f\left([X, Y]+d\left(\iota_{X} \beta\right)+\iota_{X} d \beta-i_{Y} d \alpha\right) \\
& =\left(\pi_{T}(u)\right)(f) v+f[u, v] .
\end{aligned}
$$

Remark 3.8. In contrast to the Lie bracket, Proposition 3.7 (a) gives that the Dorfman bracket is not skew-symmetric.

Remark 3.9. Proposition 3.7 gives that the quadruple ( $\mathbb{T},\langle\rangle,, \pi_{T},[$,$] ) is a Courant$ algebroid on $M$, a structure originally introduced in [LWX97] following Courant's theory of Dirac structures [Cou90]. Besides, we know from [Uch02] that equations $(d)$ and $(e)$ are in fact a consequence of equations $(a),(b)$ and $(c)$.

### 3.2.2. Complex Dirac structures

As the Lie bracket, the canonical pairing and the Dorfman bracket can naturally be extended $\mathbb{C}$-linearly to $\mathbb{T}_{\mathbb{C}}$ and preserve their properties. Therefore, on account of Proposition 2.11 and Proposition 2.13 we define the following.

Definition 3.10 (Complex Dirac structure and generalized complex structure). Let $M$ be a manifold, a complex Dirac structure on $M$ is a subbundle $L$ of $\mathbb{T}_{\mathbb{C}}$ that is maximally isotropic for the canonical pairing and such that $\Gamma(L)$ is involutive for the Dorfman bracket, this is (3.6). A generalized complex structure $L$ on $M$ is a complex Dirac structure on $M$ that additionally satisfies $L \cap \bar{L}=0$.

Generalized complex structures were originally introduced in [Hit03] and further developed in [Gua03]. As one would expect, the integrability condition of complex and symplectic structures translates accordingly. Let us verify the latter (check Example 3.34 of [Gua03] for the complex case).

## Example 3.11.

Given a $2 n$-manifold $M$, for a certain integer $n \geq 0$, on account of Example 2.7 and Example 2.12 any almost symplectic structure $\omega$ gives a maximally isotropic subbundle

$$
L_{\omega}:=\left\{X-i \omega^{b}(X): X \in T_{\mathbb{C}}\right\}
$$

of $\mathbb{T}_{\mathbb{C}}$. Given any $X, Y \in T_{\mathbb{C}}$, in the first place we have

$$
\left[X-i \omega^{b}(X), Y-i \omega^{b}(Y)\right]=[X, Y]-i L_{X} \omega^{b}(Y)+i \iota_{Y} d\left(\omega^{b}(X)\right)
$$

In the second place, from (3.3) and (3.4) we have

$$
\begin{aligned}
\omega^{b}([X, Y]) & =\iota_{[X, Y]} \omega=L_{X} \iota_{Y} \omega-\iota_{Y} L_{X} \omega=L_{X} \iota_{Y} \omega-\iota_{Y} d\left(\iota_{X} \omega\right)-\iota_{Y} \iota_{X} d \omega \\
& =L_{X} \omega^{b}(Y)-\iota_{Y} d\left(\omega^{b}(X)\right)-\iota_{Y} \iota_{X} d \omega .
\end{aligned}
$$

Thus, on the one hand $d \omega=0$ gives

$$
\begin{equation*}
\left[X-i \omega^{b}(X), Y-i \omega^{b}(Y)\right]=[X, Y]-i \omega^{b}([X, Y]) \in \Gamma(L) \tag{3.9}
\end{equation*}
$$

for any $X, Y \in T_{\mathbb{C}}$. On the other hand, if (3.9) holds for any $X, Y \in T_{\mathbb{C}}$ we obtain $d \omega=0$ and hence both conditions are equivalent.

After having translated the basics of our work with linear algebra to geometry, we will study a peculiar phenomenon that now emerges. In section 2.3, we introduced three integers associated to any linear complex Dirac structure, namely
the real index, the order and the type, which where the key for the classification of linear complex Dirac structures. For a complex Dirac structure on a manifold, however, these integers do not need to be the same at all points. We show next two examples where an invariant varies over a manifold. Let us note that the first example was adapted from Section 4.1 of [Gua03], where it is treated in terms of spinors (which we did not introduce) and translated to real coordinates whereas the second was inspired by the first one and independently deduced.

Example 3.12 (Type change with constant real index and order).
Consider $M=\mathbb{R}^{4}$ with global coordinates $(x, y, z, t)$. Let $\partial_{x}, \partial_{y}, \partial_{z}$ and $\partial_{t}$ be the coordinate vector fields and $d x, d y, d z$ and $d t$ be their dual 1-forms, respectively. Take the sections

$$
\begin{aligned}
& v_{1}:=\partial_{x}+i \partial_{y} \\
& v_{2}:=\partial_{z}+i \partial_{t} \\
& v_{3}:=(x+i y)\left(\partial_{x}-i \partial_{y}\right)-2(d z+i d t) \text { and } \\
& v_{4}:=(x+i y)\left(\partial_{z}-i \partial_{t}\right)+2(d x+i d y)
\end{aligned}
$$

of $\mathbb{T}_{\mathbb{C}}$ and define the vector subbundle $L$ of $\mathbb{T}_{\mathbb{C}}$ as the linear span of $v_{1}(p), v_{2}(p)$, $v_{3}(p)$ and $v_{4}(p)$, for each $p=(x, y, z, t) \in \mathbb{R}^{4}$.

Firstly, we will see that $L$ is a maximally isotropic subbundle of $\mathbb{T}_{\mathbb{C}}$. Indeed, fix any $p=(x, y, z, t) \in \mathbb{R}^{4}$. On the one hand, the vectors $l_{j}:=v_{j}(p)$, for $j=1, \ldots, 4$, are linearly independent and thus $\operatorname{dim}_{\mathbb{C}} L_{p}=4$. On the other hand, we have

$$
\begin{aligned}
\left\langle l_{1}, l_{1}\right\rangle & =\left\langle l_{2}, l_{2}\right\rangle=\left\langle l_{3}, l_{3}\right\rangle=\left\langle l_{4}, l_{4}\right\rangle=0 \\
\left\langle l_{1}, l_{2}\right\rangle & =\left\langle l_{1}, l_{3}\right\rangle=\left\langle l_{2}, l_{4}\right\rangle=0 \\
\left\langle l_{1}, l_{4}\right\rangle & =(d x+i d y)\left(\partial_{x}+i \partial_{y}\right)=d x \partial_{x}-d y \partial_{y}=1-1=0 \\
\left\langle l_{2}, l_{3}\right\rangle & =-(d z+i d t)\left(\partial_{z}+i \partial_{t}\right)=-(1-1)=0 \text { and } \\
\left\langle l_{3}, l_{4}\right\rangle & =-(d z+i d t)\left((x+i y) \cdot\left(\partial_{z}-i \partial_{t}\right)\right)+(d x+i d y)\left((x+i y) \cdot\left(\partial_{x}-i \partial_{y}\right)\right) \\
& =-(x+i y)+(x+i y)=0 .
\end{aligned}
$$

Since the canonical pairing is bilinear and symmetric, the latter gives that $L_{p}$ is isotropic and thus by Proposition 2.16 we obtain that $L_{p}$ is maximally isotropic.

Secondly, we will show that $\Gamma(L)$ is involutive for the Dorfman bracket. Equalities (3.8) and (3.10) give

$$
\left[v_{1}, v_{1}\right]=\left[v_{2}, v_{2}\right]=\left[v_{3}, v_{3}\right]=\left[v_{4}, v_{4}\right]=0 \in \Gamma(L) .
$$

Moreover, since by construction $\Gamma(L)$ is a $C^{\infty}(M)$-module generated by $v_{1}, v_{2}, v_{3}$ and $v_{4}$, the computations

$$
\begin{aligned}
{\left[v_{1}, v_{2}\right]=} & {\left[\partial_{x}+i \partial_{y}, \partial_{z}+i \partial_{t}\right]=0 \in \Gamma(L) } \\
{\left[v_{1}, v_{3}\right]=} & {\left[\partial_{x}+i \partial_{y},(x+i y)\left(\partial_{x}-i \partial_{y}\right)\right]+L_{\partial_{x}+i \partial_{y}}(-2(d z+i d t)) } \\
= & \left(\left(\partial_{x}+i \partial_{y}\right)(x+i y)\right) \cdot\left(\partial_{x}-i \partial_{y}\right)+0=(1-1)\left(\partial_{x}-i \partial_{y}\right)=0 \in \Gamma(L), \\
{\left[v_{2}, v_{4}\right]=} & {\left[\partial_{z}+i \partial_{t},(x+i y)\left(\partial_{z}-i \partial_{t}\right)\right]+L_{\partial_{z}+i \partial_{t}}(2(d x+i d y))=0 \in \Gamma(L), } \\
{\left[v_{1}, v_{4}\right]=} & {\left[\partial_{x}+i \partial_{y},(x+i y)\left(\partial_{z}-i \partial_{t}\right)\right]+L_{\partial_{x}+i \partial_{y}}(2(d x+i d y)) } \\
= & 0+d\left(2(d x+i d y)\left(\partial_{x}+i \partial_{y}\right)\right)=d(2(1-1))=0 \in \Gamma(L), \\
{\left[v_{2}, v_{3}\right]=} & {\left[\partial_{z}+i \partial_{t},(x+i y)\left(\partial_{x}-i \partial_{y}\right)\right]+L_{\partial_{z}+i \partial_{t}}(-2(d z+i d t))=0 \in \Gamma(L) \text { and } } \\
{\left[v_{3}, v_{4}\right]=} & {\left[(x+i y)\left(\partial_{x}-i \partial_{y}\right),(x+i y)\left(\partial_{z}-i \partial_{t}\right)\right] } \\
& +L_{(x+i y)}\left(\partial_{x}-i \partial_{y}\right)(2(d x+i d y))-\iota_{(x+i y)}\left(\partial_{z}-i \partial_{t}\right) d(-2(d z+i d t)) \\
= & 2(x+i y)\left(\partial_{z}-i \partial_{t}\right)-(x+i y)\left[\partial_{z}-i \partial_{t},(x+i y)\left(\partial_{x}-i \partial_{y}\right)\right] \\
& +d\left(2(d x+i d y)\left((x+i y) \cdot\left(\partial_{x}-i \partial_{y}\right)\right)\right)+0 \\
= & 2(x+i y)\left(\partial_{z}-i \partial_{t}\right)+0+d(4(x+i y)) \\
= & 2(x+i y)\left(\partial_{z}-i \partial_{t}\right)+4(d x+i d y)=2 v_{4} \in \Gamma(L) .
\end{aligned}
$$

together with Proposition 3.7 (a) and $(e)$ suffice to prove $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$.
Thus, $L$ is a complex Dirac structure on $\mathbb{R}^{4}$. Particularly, it is easy to verify that $L \cap \bar{L}=0$ holds and thus $L$ is a generalized complex structure. Hence, at any point $p \in \mathbb{R}^{4}$ the real index is $r_{p}=0$ and by Corollary 2.32 the order is also $s_{p}=0$.

Finally, let us look at the type. Consider the subbundle $E:=\pi_{T_{\mathbb{C}}} L$ of $T_{\mathbb{C}}$. At any point $p=(x, y, z, t) \in \mathbb{R}^{4}$, we get that $E_{p}$ is the linear span of

$$
\partial_{x}+i \partial_{y}, \quad \partial_{z}+i \partial_{t}, \quad(x+i y)\left(\partial_{x}-i \partial_{y}\right) \text { and }(x+i y)\left(\partial_{z}-i \partial_{t}\right) .
$$

Therefore, it is clear that we get

$$
\operatorname{dim}_{\mathbb{C}} E_{p}= \begin{cases}2 & \text { if } x=y=0  \tag{3.11}\\ 4 & \text { otherwise }\end{cases}
$$

Since it is easy to verify that in any case we have $\operatorname{dim}_{\mathbb{C}}\left(E_{p}+\overline{E_{p}}\right)=4$, the type is $t_{p}=2$ in case $x=y=0$ and $t_{p}=0$ otherwise. Thus, if we come back to Figure 2.1 this can be interpreted as a structure that is generally equivalent to a symplectic structure but "blows up" to a complex structure at $x=y=0$.

Example 3.13 (Order change with constant real index and type).
Consider $M=\mathbb{R}^{4}$ with global coordinates $(x, y, z, t)$ but take now the sections

$$
\begin{aligned}
& v_{1}:=\partial_{x}-\partial_{y} \\
& v_{2}:=\partial_{z}-\partial_{t} \\
& v_{3}:=(x+y)\left(\partial_{x}+\partial_{y}\right)-2(d z+d t) \text { and } \\
& v_{4}:=(x+y)\left(\partial_{z}+\partial_{t}\right)+2(d x+d y)
\end{aligned}
$$

of $\mathbb{T}_{\mathbb{C}}$ and define the vector subbundle $L$ of $\mathbb{T}_{\mathbb{C}}$ as the linear span of $v_{1}(p), v_{2}(p)$, $v_{3}(p)$ and $v_{4}(p)$, for each $p=(x, y, z, t) \in \mathbb{R}^{4}$.

Analogously to Example 3.12, one can show that indeed $L$ is a complex Dirac structure on $\mathbb{R}^{4}$ (again, we obtain $\left[v_{3}, v_{4}\right]=2 v_{4}$ and all other brackets vanish).

Fix any $p=(x, y, z, t) \in \mathbb{R}^{4}$. In this case, trivially we have $L_{p}=\overline{L_{p}}$ and $E_{p}=\overline{E_{p}}$. This way, on the one hand by and Proposition 2.11 the real index is

$$
r_{p}=\operatorname{dim}_{\mathbb{C}}\left(L_{p} \cap \overline{L_{p}}\right)=4
$$

On the other hand, we have

$$
\operatorname{dim}_{\mathbb{C}}\left(E_{p}+\overline{E_{p}}\right)=\operatorname{dim}_{\mathbb{C}} E_{p}= \begin{cases}2 & \text { if } x=y=0  \tag{3.12}\\ 4 & \text { otherwise }\end{cases}
$$

and thus we get that the type is $t_{p}=0$ (which holds with Lemma 2.32) and that the order is $s_{p}=2$ in case $y+x=0$ and $s_{p}=0$ otherwise. Thus, if we come back to Figure 2.1 this can be interpreted as a structure that lies in the vertical edge of the tetrahedron and that is generally equivalent to a presymplectic structure but "blows up" to a different structure at $x+y=0$.

To conclude, given a manifold $M$ and a complex Dirac structure $L$ of $M$, one can define its real index, order and type as functions $M \rightarrow \mathbb{Z}_{\geq 0}$. For some information on how this functions behave, check for instance Lemma 4.12 of [AR22]. Therefore, at each point of $M$ the invariants are an integer triplet of a tetrahedron (recall Figure 2.1), but interestingly we have just shown that this triplet does not need to be the same at all points of $M$. Therefore, complex Dirac structures can describe changing geometric structures.

## References

[Agu20] Dan Aguero. "Complex Dirac structures with constant real index". Available at https://impa.br/wp-content/uploads/2020/09/tese_dout_ Dan-Aguero.pdf. PhD thesis. IMPA, 2020.
[AR22] Dan Aguero and Roberto Rubio. "Complex Dirac structures: invariants and local structure". In: Commun. Math. Phys. 396.2 (2022), pp. 623646. ISSN: 0010-3616. DOI: $10.1007 / \mathrm{s} 00220-022-04471-1$.
[Bur13] Henrique Bursztyn. "A brief introduction to Dirac manifolds". In: Geometric and topological methods for quantum field theory. Papers based on the presentations at the 6th summer school, Villa de Leyva, Colombia, July 6-23, 2009. Cambridge: Cambridge University Press, 2013, pp. 138. ISBN: 978-1-107-02683-4; 978-1-139-20864-2.
[Cou90] Theodore James Courant. "Dirac manifolds". In: Trans. Am. Math. Soc. 319.2 (1990), pp. 631-661. ISSN: 0002-9947. DOI: 10.2307/2001258.
[Gua03] Marco Gualtieri. "Generalized complex geometry". Available at https: //arxiv. org/pdf/math/0401221.pdf. PhD thesis. University of Oxford, June 2003.
[Hit03] Nigel Hitchin. "Generalized Calabi-Yau manifolds". In: Q. J. Math. 54.3 (2003), pp. 281-308. ISSN: 0033-5606. DOI: 10.1093/qjmath/54.3. 281.
[KL07] Anton Kapustin and Yi Li. "Topological sigma-models with $H$-flux and twisted generalized complex manifolds". In: Adv. Theor. Math. Phys. 11.2 (2007), pp. 261-290. ISSN: 1095-0761. DOI: 10.4310/ATMP . 2007. v11.n2.a3.
[Lee13] John M. Lee. Introduction to smooth manifolds. 2nd revised ed. Vol. 218. Grad. Texts Math. New York, NY: Springer, 2013. ISBN: 978-1-4419-9981-8; 978-1-4419-9982-5. DOI: 10.1007/978-1-4419-9982-5.
[LWX97] Zhangju Liu, Alan Weinstein, and Ping Xu. "Manin triples for Lie bialgebroids". In: J. Differ. Geom. 45.3 (1997). Issn: 0022-040X. Doi: 10. 4310/jdg/1214459842.
[NN57] A. Newlander and Louis Nirenberg. "Complex analytic coordinates in almost complex manifolds". In: Ann. Math. (2) 65 (1957), pp. 391404. ISSN: 0003-486X. DOI: 10.2307/1970051.
[Rub] Roberto Rubio. Generalized Geometry. URL: https://mat. uab.cat / ~rubio/generalizada/gg-uab.pdf.
[Uch02] Kyousuke Uchino. "Remarks on the definition of a Courant algebroid". English. In: Lett. Math. Phys. 60.2 (2002), pp. 171-175. ISSN: 03779017. DOI: 10.1023/A:1016179410273.
[Wel80] R. O. jun. Wells. Differential analysis on complex manifolds. 2nd ed. Vol. 65. Grad. Texts Math. Springer, Cham, 1980.
[Zab06] Maxim Zabzine. "Lectures on generalized complex geometry and supersymmetry." In: Arch. Math., Brno 42.5 (2006), pp. 119-146. ISSN: 0044-8753.

