INTEGRAL GEOMETRY OF COMPLEX SPACE FORMS

GIL SOLANES

These are the notes of a series of lectures given at the Erasmus Workshop *Hermitian Integral Geometry* that took place from July 17th to 19th, 2012 at the Goethe Universität Frankfurt.

Our aim is to describe how Alesker's theory of valuations has been used to determine the integral geometry of the complex space forms $\mathbb{C}^n$, $\mathbb{CP}^n$, and $\mathbb{CH}^n$. With this goal in mind, we focus on invariant valuations in isotropic spaces. Accordingly, many important results about valuations are presented in a weakened version. Nevertheless, their significance should remain visible even in this simplified form.

1. Affine isotropic spaces

1.1. **Classical theory.** Let $\mathcal{K}^n$ denote the space of compact non-empty convex bodies in $\mathbb{R}^n$. A functional $\varphi\colon \mathcal{K}^n \to \mathbb{R}$ is a *valuation* if

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B)$$

whenever $A, B, A \cup B \in \mathcal{K}^n$.

We denote by Val the space of continuous (with respect to the Haussdorf topology) translation invariant valuations, and by Val^G the subset of valuations invariant under a subgroup G of $GL(n)$.

Theorem 1.1 (Hadwiger)**.**

$$\mathrm{Val}^{SO(n)} = \langle \mu_0, \dots, \mu_n \rangle$$

where the valuations μ_i are the so-called intrinsic volumes *given by*

$$\mu_j(A) = \int_{\mathrm{Gr}_j} \mathrm{vol}_j(\pi_E(A)) dm_j(E) \tag{1}$$

where m_j is an $SO(n)$-invariant measure on the Grassmanian Gr_j.

We normalize m_j in such a way that $\mu_j(A) = \mathrm{vol}_j(A)$ when A has dimension j. For $j = 0, n$ it is understood that $\mu_0 = \chi$ and $\mu_n = \mathrm{vol}_n$.

When ∂A is smooth, the intrinsic volumes are given by integration of symmetric functions of the principal curvatures. More generally, there are canonical differential forms $\kappa_i \in \Omega^{n-1}(S\mathbb{R}^n)$ in the unit sphere bundle of $\mathbb{R}^n$ such that, for any $A \in \mathcal{K}^n$,

$$\mu_i(A) = \int_{N(A)} \kappa_i \tag{2}$$

where $N(A) \subset S\mathbb{R}^n$ is the set of outward pointing unit normal vectors to A, which is an oriented Lipschitz submanifold of $S\mathbb{R}^n$.

Hadwiger's theorem easily implies Blaschke's principal kinematic formula:

Theorem 1.2. *For $A, B \in \mathcal{K}^n$,*

$$\int_{\overline{SO(n)}} \chi(A \cap gB)\, dg = \sum_{i+j=n} \binom{n}{i}^{-1} \frac{\omega_i \omega_j}{\omega_n} \mu_i(A)\mu_j(B), \tag{3}$$

where $\overline{SO(n)} := SO(n) \ltimes \mathbb{R}^n$ and ω_k is the volume of the unit ball of dimension k.

The proof is based on the observation that the left hand side above is a valuation on both A, B. Application of Hadwiger's theorem yields (3) except for the constants. These can be found by examining the case of two spheres of different radii. With the same argument one shows that similar formulas exist when $\chi = \mu_0$ is replaced by any μ_k in (3):

$$\int_{\overline{SO(n)}} \mu_k(A \cap gB)\, dg = \sum_{i+j=n+k} c_{ik} \mu_i(A)\mu_j(B), \tag{4}$$

for certain constants c_{ik}.

1.2. **Irreducibility theorem.** A natural topology on Val is given by the norm

$$\|\phi\| = \sup\{|\phi(A)| : \ A \subset B(0,1)\}.$$

There is a natural action of $GL(n)$ on Val given by

$$g \cdot \phi(A) = \phi(g^{-1}A).$$

Theorem 1.3 (Alesker, [2])**.** *Let $\mathrm{Val}_i^\varepsilon$ denote the space of valuations ϕ such that $\phi(\lambda A) = \lambda^i(A)$ for $\lambda > 0$, and $\phi(-K) = \varepsilon\phi(K)$. Then*

$$\mathrm{Val} = \bigoplus_{\substack{i = 0, \ldots, n \\ \varepsilon = \pm}} \mathrm{Val}_i^\varepsilon$$

is a decomposition into $GL(n)$-irreducible components (minimal closed $GL(n)$-invariant subspaces).

Corollary 1.4 ([22])**.** *Let $G \leq O(n)$ act transitively on S^{n-1}.*

i) The space Val^G of continuous valuations invariant under $\bar{G} = G \ltimes \mathbb{R}^n$ consists of valuations μ of the form

$$\mu(A) = \int_{N(A)} \varphi + \lambda \,\mathrm{vol}(A)$$

with $\varphi \in \Omega^{n-1}(S\mathbb{R}^n)$ invariant under $\bar{G}$. In particular, Val^G is finite dimensional.

ii) Val^G is spanned by valuations of the form

$$\mu_A^G = \int_G \int_{\mathbb{R}^n} \chi(\cdot \cap (x - gA)) dx dg.$$

Proof. *i)* It is easy to see that each irreducible component $\mathrm{Val}_k^\varepsilon$ contains one element of the form

$$\mu(A) = \int_{N(A)} \varphi$$

with $\varphi \in \Omega^{n-1}(S\mathbb{R}^n)$ translation invariant. By the irreducibility theorem, elements of this form are dense in $\mathrm{Val}_k^\varepsilon$. i.e., each $\phi \in \mathrm{Val}_k^G$ can be approximated by a sequence ϕ_i of such valuations. By averaging $g\phi_i$ we get a sequence of valuations given by invariant differential forms which still approximate ϕ. Hence, the valuations in the statement are dense in Val^G. Since $\Omega^{n-1}(S\mathbb{R}^n)^{\bar{G}}$ is finite-dimensional, so is Val^G. Hence, these valuations span Val^G.

ii) follows similarly. □

Remark 1.5. *The φ above is not unique, but λ is unique:*

$$\lambda = \mathrm{vol}^*(\mu) := \lim_{R\to\infty} \frac{\mu(B_R)}{\mathrm{vol}(B_R)}$$

That Val^G is finite dimensional implies the existence of kinematic formulas in the style of (3): if $\varphi_1, \ldots, \varphi_N$ is a basis of Val^G, then

$$\int_{\bar{G}} \varphi_k(A \cap gB)\, dg = \sum_{i,j} c_{ij}^k \varphi_i(A)\varphi_j(B), \tag{5}$$

for certain constants c_{ij}^k. These constants can be encoded in a basis free way, by defining

$$\begin{aligned} k_G\colon \mathrm{Val}^G &\longrightarrow \mathrm{Val}^G \otimes \mathrm{Val}^G \\ \varphi_k &\longmapsto \sum_{i,j} c_{ij}^k \varphi_i \otimes \varphi_j. \end{aligned}$$

The groups G acting transitively on the sphere are known ([19, 26]):

$SO(n), U(n), SU(n), Sp(n), Sp(n)\cdot U(1), Sp(n)\cdot Sp(1), G_2, Spin(7), Spin(9)$.

The kinematic operator k_G is known in the cases $G = SO(n), U(n), SU(n), G_2, Spin(7)$ (cf. [17, 12, 13]).

1.3. **Algebraic Integral Geometry.**

Theorem 1.6. *There is a product on Val^G compatible with the linear structure, and characterized by the following property. For $A \in \mathcal{K}^n$ and $\varphi \in \mathrm{Val}^G$,*

$$\mu_A^G \cdot \varphi = \int_{\bar{G}} \varphi(\cdot \cap gA) dg. \tag{6}$$

This endows Val^G with the structure of a commutative algebra with a unit element: χ.

Proof. Suppose $\sum_i a_i \mu_{A_i}^G = 0$. Then

$$\begin{aligned} \sum_i a_i \mu_{A_i}^G \cdot \mu_B^G &= \sum_i a_i \int_{\bar{G}\times\bar{G}} \chi(\cdot \cap gA \cap hB) dg dh \\ &= \int_{\bar{G}} \sum_i a_i \mu_{A_i}^G(\cdot \cap hB) dh = 0. \end{aligned}$$

□

The product allows to define the so-called Poincaré duality $\mathrm{pd} : \mathrm{Val}^G \otimes \mathrm{Val}^G \to \mathbb{R}$ as follows

$$\mathrm{pd}(\psi, \varphi) = \mathrm{vol}^*(\psi \cdot \varphi).$$

Proposition 1.7. $\mathrm{pd} \in \mathrm{Val}^{G*} \otimes \mathrm{Val}^{G*} \equiv \mathrm{Hom}(\mathrm{Val}^G, \mathrm{Val}^{G*})$ *is inverse to* $k_G(\chi) \in \mathrm{Val}^G \otimes \mathrm{Val}^G \equiv \mathrm{Hom}(\mathrm{Val}^{G*}, \mathrm{Val}^G)$.

Proof. Note first that Val^{G*} is spanned by elements of the form $ev_A(\varphi) = \varphi(A)$ where $A \in \mathcal{K}^n$. Then to show that $\mathrm{pd} \circ k_G(\chi) = \mathrm{id}_{\mathrm{Val}^{G*}}$ we just need to check for $\varphi \in \mathrm{Val}^G$,

$$\langle \mathrm{pd} \circ k_G(\chi)(ev_A), \varphi\rangle = \mathrm{pd}(\mu_A^G, \varphi) = \mathrm{vol}^*(\varphi \cdot \mu_A^G) = \mathrm{vol}^*(k_G(\varphi)(A)) = \varphi(A).$$

□

In particular pd is non-degenerate. Note also that k_G is fully determined by pd and the product:

$$k_G(\chi) = \mathrm{pd}^{-1}, \qquad (\chi \otimes \mu) k_G(\chi) = (\mu \otimes \chi) k_G(\chi) = k_G(\mu).$$

Both facts are equivalent to the following statement, which has been called *fundamental theorem of algebraic integral geometry.*

Theorem 1.8 ([16])**.** *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Val}^G & \xrightarrow{\ k\ } & \mathrm{Val}^G \otimes \mathrm{Val}^G \\ \Big\downarrow{\scriptstyle \mathrm{pd}} & & \Big\downarrow{\scriptstyle \mathrm{pd}\otimes\mathrm{pd}} \\ \mathrm{Val}^{G*} & \xrightarrow{\ m^*\ } & \mathrm{Val}^{G*} \otimes \mathrm{Val}^{G*} \end{array}$$

Proof. Let $\varphi_1, \dots, \varphi_N$ be a basis of Val. Let $\kappa_G(\chi) = c_{ij}\varphi_i \otimes \varphi_j$, and $\mathrm{pd}(\varphi_i, \varphi_j) = c^{ij}$ where $(c^{ij}) = (c_{ij})^{-1}$. Then

$$(\mathrm{pd} \otimes \mathrm{pd}) \circ \kappa_G(\chi) = c_{ij}\, \mathrm{pd}(\varphi_i) \otimes \mathrm{pd}(\varphi_j),$$

whence

$$\langle (\mathrm{pd} \otimes \mathrm{pd}) \circ \kappa_G(\chi), \varphi_r \otimes \varphi_s\rangle = c_{ij} \cdot \langle \mathrm{pd}(\varphi_i) \otimes \mathrm{pd}(\varphi_j), \varphi_r \otimes \varphi_s\rangle = c_{ij} \cdot c^{ri} \cdot c^{sj} = c^{rs}$$

since (c_{ij}) is symmetric. But

$$\langle m^*(\mathrm{pd}(\chi)), \varphi_r \otimes \varphi_s\rangle = \langle \mathrm{pd}(\chi), \varphi_r \cdot \varphi_s\rangle = \mathrm{pd}(\varphi_r, \varphi_s) = c^{rs}.$$

□

1.4. **Fourier transform.** Klain has shown that the volume is the unique (up to a constant factor) even translation invariant valuation that vanishes on degenerate sets (simple). Hence, given $\phi \in \mathrm{Val}_k^+$ and $E \in \mathrm{Gr}_k$ there is a factor $\mathrm{Kl}_\phi(E)$ such that $\phi|_E = \mathrm{Kl}_\phi(E)\, \mathrm{vol}_k$. This defines a map

$$\mathrm{Kl} : \mathrm{Val}_k^+ \to C(\mathrm{Gr}_k).$$

This map is injective: if $\phi \in \mathrm{Val}_k^+$ has identically vanishing Klain function, then $\phi|_F$ is simple for every $F \in \mathrm{Gr}_{k+1}$. Hence $\phi|_F$ is a multiple of vol_{k+1}. But this contradicts homogeneity unless $\phi|_F = 0$ for every $F \in \mathrm{Gr}_{k+1}$. In other words, $\phi|_E$ is simple for every $E \in \mathrm{Gr}_{k+2}$. Repeating this argument we see that ϕ vanishes on spaces of any dimension.

Theorem 1.9 (Alesker,[3])**.** *There is a duality map* $\mathrm{Val}_k^+ \xrightarrow{\ \widehat{}\ } \mathrm{Val}_{n-k}^+$ *called* Fourier transform *characterized by*

$$\mathrm{Kl}_{\hat{\mu}}(E) = \mathrm{Kl}_\mu(E^\perp)$$

Let G be a group as before, and assume that $\mathrm{Val}^G \subset \mathrm{Val}^+$ (it turns out that this is always true). Let us show the existence of the restricted Fourier transform $\mathrm{Val}_k^G \stackrel{\widehat{}}{\longrightarrow} \mathrm{Val}_{n-k}^G$. First of all, as a consequence of the irreducibility theorem, Val^G is generated by elements of the form

$$\mu = \int_{\mathrm{Gr}_k} \mathrm{vol}(\pi_E(\cdot))dm(E)$$

where m is some G-invariant measure on Gr_k. Then

$$\widehat{\mu} = \int_{\mathrm{Gr}_k} \int_{E^\perp} \chi(\cdot \cap (x+E))dxdm(E)$$

1.5. **Two operators.** Let us consider the following operators $\tilde{L}, \tilde{\Lambda} : \mathrm{Val}^G \to \mathrm{Val}^G$, of degrees $+1, -1$ respectively:

$$\tilde{L}\phi := \mu_1 \cdot \phi, \tag{7}$$

$$\tilde{\Lambda}\phi := \left.\frac{d}{dt}\right|_{t=0} \phi(\cdot + tB), \tag{8}$$

where B is the unit ball of $\mathbb{R}^n$. We renormalize these operators by taking

$$L := \frac{2\omega_k}{\omega_{k+1}} \tilde{L}, \tag{9}$$

$$\Lambda := \frac{\omega_{n-k}}{\omega_{n-k+1}} \tilde{\Lambda} \tag{10}$$

on each homogeneous component Val_k^G.

Lemma 1.10 ([16])**.** *The Fourier transform intertwines the operators L, Λ:*

$$L \circ \widehat{} = \widehat{} \circ \Lambda \tag{11}$$

Proof. Let us write a generic $\phi \in \mathrm{Val}^G$ as a Crofton valuation:

$$\phi = \int_{\mathrm{Gr}_k} \mathrm{vol}_k(\pi_E(\cdot))dm_\phi(E)$$

for some G-invariant measure m_ϕ on Gr_k. The Fourier transform is

$$\hat{\phi} = \int_{\mathrm{Gr}_k} \int_{E^\perp} \chi(\cdot \cap (x+E))dxdm_\phi(E)$$

Recalling (1),

$$\begin{aligned}\hat{\phi} \cdot \mu_1 = \hat{\phi} \cdot \hat{\mu}_{n-1} &= \int_{\mathrm{Gr}_k \times \mathrm{Gr}_{n-1}} \int_{E^\perp + H^\perp} \chi(\cdot \cap (x+E) \cap (y+H))dxdydHdm_\phi(E) \\ &= \int_{\mathrm{Gr}_k \times \mathrm{Gr}_{n-1}} \int_{(E\cap H)^\perp} \chi(\cdot \cap (z + E\cap H))dz \sin(E,H)dHdm_\phi(E)\end{aligned}$$

Now

$$\widehat{\hat{\phi} \cdot \mu_1} = \int_{\mathrm{Gr}_k} \left(\int_{\mathrm{Gr}_{n-1}} \mathrm{vol}(\pi_{(E\cap H)^\perp}(\cdot)) \sin(E,H)dH \right) dm_\phi(E).$$

By $SO(k)$-invariance, the integral between brackets is a multiple of $\mu_{k-1}(\pi_E(\cdot))$, so

$$\begin{aligned}\widehat{\phi\cdot\mu_1} &= c\int_{\mathrm{Gr}_k}\mu_{k-1}(\pi_E(\cdot))dm_\phi(E)\\ &= c\,\frac{d}{dt}\Big|_{t=0}\int_{\mathrm{Gr}_k}\mu_k(\pi_E(\cdot+tB))dm_\phi(E)\\ &= c'\Lambda\phi.\end{aligned}$$

Since the constants are independent of ϕ it is enough to check them in the case $\phi=\mu_k$. $\square$

2. Complex affine space

Next we study $\mathrm{Val}^{U(n)}$, the space of translation, $U(n)$-invariant valuations. The results in this section come mainly from [23] and [17].

2.1. **Hermitian intrinsic volumes.** Let (z,ζ) denote a generic point of $T\mathbb{C}^n\simeq\mathbb{C}^n\times\mathbb{C}^n$. Consider the following differential 1-forms on $T\mathbb{C}^n$

$$\alpha=\langle dz,\zeta\rangle,\quad \beta=\langle dz,i\zeta\rangle,\quad \gamma=\langle d\zeta,i\zeta\rangle, \tag{12}$$

and the 2-forms

$$\theta_0=\frac{1}{2}d\gamma=\frac{1}{2}\langle id\zeta,d\zeta\rangle,\quad \theta_1=d\beta=\langle id\zeta,dz\rangle,\quad \theta_2=\frac{1}{2}\langle idz,dz\rangle,\quad \theta_s=-d\alpha.$$

(here the inner product of 1-forms is antisymmetrized). The restrictions of these forms to the sphere bundle $S\mathbb{C}^n\simeq\mathbb{C}^n\times S^{2n-1}$ generate the algebra of $U(n)-$invariant forms on that space. This was shown in [27] using invariant theory. Thus each element in $\mathrm{Val}^{U(n)}$ comes from a differential form of degree $2n-1$ obtained by exterior product of these forms. Since the contact form α and its exterior derivative $-\theta_s$ vanish identically on any normal cycle, it is enough to consider the products of $\beta,\gamma,\theta_0,\theta_1,\theta_2$.

Therefore, $\mathrm{Val}^{U(n)}$ is spanned by the valuations defined by the following differential forms

$$\beta_{k,q}:=c_{n,k,q}\,\beta\wedge\theta_0^{n-k+q}\wedge\theta_1^{k-2q-1}\wedge\theta_2^q, \tag{13}$$

$$\gamma_{k,q}:=\frac{c_{n,k,q}}{2}\,\gamma\wedge\theta_0^{n-k+q-1}\wedge\theta_1^{k-2q}\wedge\theta_2^q, \tag{14}$$

where the constants are chosen for later convenience as

$$c_{n,k,q}:=\frac{1}{q!(n-k+q)!(k-2q)!\omega_{2n-k}}.$$

For $k,q\geq 0$ integers with $\max\{0,k-n\}\leq q\leq\frac{k}{2}\leq n$, we set

$$\theta_{k,q}:=c_{n,k,q}\theta_0^{n+q-k}\wedge\theta_1^{k-2q}\wedge\theta_2^q$$

We define the *hermitian intrinsic volumes* $\mu_{k,q}\in\mathrm{Val}^{U(n)}$ by

$$\mu_{k,q}(A):=\int_{N_1(A)}\theta_{k,q},\qquad 0,k-n\leq q\leq\frac{k}{2} \tag{15}$$

where $N^1(A)\subset\mathbb{C}^n\oplus\mathbb{C}^n$ denotes the set of (z,ζ) such that $z\in A$, $\|\zeta\|\leq 1$ and $\zeta\cdot v\leq 0$ for every v in the tangent cone of A at x. Since $\partial N_1(K)=N(K)$, from Stokes' theorem one easily computes that

Proposition 2.1.

$$\mu_{k,q} = \int_{N(\cdot)} \beta_{k,q} = \int_{N(\cdot)} \gamma_{k,q}.$$

Note however that $\beta_{2q,q}$ and $\gamma_{n+q,q}$ are not defined.

We know that $\mu_{k,q}$ span $\mathrm{Val}^{U(n)}$. To see that they are independent, we look at their Klain functions.

Let $\mathrm{Gr}_{k,q}$ denote the $U(n)$-orbit of $\mathbb{C}^q \oplus \mathbb{R}^{k-2q}$ inside Gr_k. In other words, the elements of $\mathrm{Gr}_{k,q}$ are real linear k-planes containing a q-dimensional complex subspace, and an orthogonal subspace to it which is isotropic with respect to the Kähler form. The range for k, q is the same as in (15).

Theorem 2.2. *The valuations $\mu_{k,q} \in \mathrm{Val}_k^{U(n)}$ are uniquely determined by*

$$\mathrm{Kl}_{\mu_{k,q}}(E^{k,p}) = \delta_{q,p}, \qquad E^{k,p} \in \mathrm{Gr}_{k,p} \tag{16}$$

These valuations are invariant under the restriction map $r\colon \mathrm{Val}^{U(n)} \to \mathrm{Val}^{U(n-1)}$, except that $r(\mu_{2n,n}) = r(\mu_{2n,n-1}) = 0$ and $r(\mu_{k,k-n}) = 0$. They satisfy the relations

$$\widehat{\mu_{k,q}} = \mu_{2n-k,n-k+q}. \tag{17}$$

Proof. Let us evaluate the Klain function of $\mu_{k,q}$ on $E^{k,p} := \mathbb{C}^p \oplus \mathbb{R}^{k-2p} \in \mathrm{Gr}_k$. Let $e_1, \ldots, e_n; u_1, \ldots, u_n$ be the canonical basis of $T\mathbb{C}^n = \mathbb{C}^n \oplus \mathbb{C}^n$. Then $\mathrm{Kl}_{\mu_{k,q}}(E^{k,p})$ equals

$$\begin{aligned}
&\omega_{2n-k}\theta_{k,q}(e_1, ie_1, \ldots, e_p, ie_p, e_{p+1}, e_{p+2}, \ldots, e_{k-p}, iu_{p+1}, iu_{p+2}, \ldots, iu_{k-p},\\
&\qquad\qquad u_{k-p+1}, iu_{k-p+1}, \ldots, u_n, iu_n) \qquad (18)\\
&\quad = \omega_{2n-k}c_{n,k,q}\delta_q^p p!(n-k+p)!\theta_1^{k-2p}(e_{p+1}, e_{p+2}, \ldots, e_{k-p}, iu_{p+1}, iu_{p+2})\\
&\quad = \delta_q^p \qquad (19)
\end{aligned}$$

This proves (16). Since $(E^{k,q})^\perp = E^{2n-k,n-k+q}$, the relation (17) is immediate. □

We conclude that the hermitian intrinsic volumes comprise a basis of $\mathrm{Val}^{U(n)}$. In particular,

$$\dim \mathrm{Val}_k^{U(n)} = \dim \mathrm{Val}_{2n-k}^{U(n)} = \left\lfloor \frac{k}{2} \right\rfloor + 1, \qquad k \leq n.$$

It turns out that everything works a little better in terms of the following basis

Definition 2.3. *The following are called Tasaki valuations*

$$\tau_{k,q} = \sum_{i=q}^{\lfloor k/2 \rfloor} \binom{i}{q} \mu_{k,i}$$

2.2. **Principal kinematic formula.**

Lemma 2.4.

$$L\tau_{k,p} = (k-2p+1)\,\tau_{k+1,p}, \tag{20}$$

$$\Lambda\tau_{k,p} = (2n-2p-k+1)\,\tau_{k-1,p} + (k-2p+1)\,\tau_{k-1,p-1}. \tag{21}$$

Proof. The formulas above are equivalent to

$$\Lambda\mu_{k,q} = 2(n-k+q+1)\mu_{k-1,q} + (k-2q+1)\mu_{k-1,q-1}, \tag{22}$$

$$L\mu_{k,q} = 2(q+1)\mu_{k+1,q+1} + (k-2q+1)\mu_{k+1,q}. \tag{23}$$

It is not hard to show that if $\mu \in \mathrm{Val}$ is obtained by integration over $N_1(\cdot)$ of a differential form ψ then $\tilde{\Lambda}\mu$ is obtained by integration of the Lie derivative $\mathcal{L}_T\psi$ with respect to the Reeb vector field T. It is easy to compute that

$$\mathcal{L}_T\theta_0 = 0, \quad \mathcal{L}_T\theta_1 = 2\theta_0, \quad \mathcal{L}_T\theta_2 = \theta_1.$$

This yields

$$\mathcal{L}_T\theta_{k,q} = \frac{\omega_{2n-k+1}}{\omega_{2n-k}}\left(2(n-k+q+1)\theta_{k-1,q} + (k-2q+1)\theta_{k-1,q-1}\right).$$

From this, (22) follows at once. Relation (23) follows from (22) using equation (17) and (11). □

Theorem 2.5. *Let us consider the generators*

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}(2,\mathbb{R})$. *The map*

$$\begin{aligned} H &\mapsto (2k-2n)\,\mathrm{id} \\ X &\mapsto L \\ Y &\mapsto \Lambda \end{aligned}$$

defines a representation of $\mathfrak{sl}(2,\mathbb{R})$ *on* $\mathrm{Val}^{U(n)}$.

Proof. Easy to check using Lemma 2.4. □

Every $\mathfrak{sl}(2,\mathbb{R})$-representation is the direct sum of irreducible representations (minimal invariant proper subspaces). Each of these irreducible components is of the form

$$V = \bigoplus_{i=0}^{r} V_{2i-r}$$

for some $r \in \mathbb{N}$, where V_j is the eigenspace of H of eigenvector j. Moreover, these eigenspaces are 1-dimensional and $V_j = X^j(V_{-r})$. Note in particular that $V_{-r} \subset \ker Y$ and $V_r \subset \ker X$. Hence, each irreducible component corresponds to a 1-dimensional subspace of elements π with $Y(\pi) = 0$. These are called *primitive elements.*

In the case of $\mathrm{Val}^{U(n)}$, the homogeneous components $\mathrm{Val}_k^{U(n)}$ are precisely the eigenspaces of H with eigenvalues $2k-2n$. Looking at the dimensions of $\mathrm{Val}_k^{U(n)}$ we see that there exists a unique (up to a multiplicative constant) primitive valuation in $\mathrm{Val}^{U(n)}$ in each even degree not larger than n.

For $0 \leq 2r \leq n$ we put

$$\pi_{2r,r} := (-1)^r(2n-4r+1)!! \sum_{i=0}^{r} (-1)^i \frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!}\,\tau_{2r,i} \tag{24}$$

which, by Lemma 2.4, is a primitive valuation of degree $2r$. For $2r \leq k \leq 2n-2r$ we define

$$\pi_{k,r} := L^{k-2r}\pi_{2r,r} \tag{25}$$

$$= (-1)^r(2n-4r+1)!!\sum_{i=0}^{r}(-1)^i\frac{(k-2i)!}{(2r-2i)!}\frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!}\tau_{k,i} \tag{26}$$

by (20). These valuations comprise a basis of $\mathrm{Val}^{U(n)}$.

Proposition 2.6.

$$\pi_{k,r}\cdot\pi_{2n-k,s} = 0, \quad r \neq s.$$

Proof. Say $r > s$, then

$$\pi_{k,r}\cdot\pi_{2n-k,s} = L^{k-2r}\pi_{2r,r}\cdot L^{2n-k-2s}\pi_{2s,s} = C\cdot L^{2n-2r-2s}\pi_{2r,r}\cdot\pi_{2s,s} = 0 \tag{27}$$

since $L^{2n-4r+1}\pi_{2r,r} = 0$. □

Theorem 2.7. *Let a_{nkr} be given by $\pi_{kr}\cdot\pi_{2n-k,r} = a_{nkr}^{-1}\,\mathrm{vol}_{2n}$, and put $p := \min\{\lfloor\frac{k}{2}\rfloor, \lfloor n-\frac{k}{2}\rfloor\}$. Then*

$$k_{U(n)}(\chi) = \sum_{k=0}^{2n}\sum_{r=0}^{p} a_{nkr}\pi_{kr}\otimes\pi_{2n-k,r}.$$

We refer to [17] for the actual computation of the constants. It requires a detailed study of the algebra structure of $\mathrm{Val}^{U(n)}$, which is the subject of the following subsection.

2.3. Algebra structure of $\mathrm{Val}^{U(n)}$.

The algebra structure of $\mathrm{Val}^{SO(n)}$ is very simple

$$\mathrm{Val}^{SO(n)} \equiv \frac{\mathbb{R}[t]}{(t^{n+1})}.$$

The generator t is usually taken as $t = \frac{2}{\pi}\mu_1$.

Since $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ we obviously have $t \in \mathrm{Val}^{U(n)}$. In other words $\mathrm{Val}^{SO(2n)} \subset \mathrm{Val}^{U(n)}$. Recalling that t is a multiple of the mean width, it is natural to define its complex analog as

$$s := \int_{\mathrm{Gr}_1^{\mathbb{C}}}\mathrm{area}(\pi_E(\cdot))d_E m$$

where $\mathrm{Gr}_1^{\mathbb{C}} = \mathbb{C}\mathrm{P}^n$ is the space of complex directions, and m is the Haar measure, normalized so that $s(D^1_{\mathbb{C}}) = 1$ where $D^1_{\mathbb{C}} \subset E$ denotes the unit disk inside some linear direction $E \in \mathrm{Gr}_1^{\mathbb{C}}$.

Theorem 2.8 ([1])**.**

$$s^k = \frac{1}{4^k\omega_{2k}}\sum_{q=0}^{k}\binom{2q}{q}\binom{2k-2q}{k-q}\binom{k}{q}^{-1}\tau_{2k,q} \tag{28}$$

The proof is based on the comparison of the first variation of the two sides of the equation.

Theorem 2.9.
$$\tau_{k,q} = \frac{\pi^k}{\omega_k (k-2q)!(2q)!} t^{k-2q} u^q \tag{29}$$
where $u = \frac{2}{\pi}\mu_{2,1} = 4s - t^2$.

Proof. The case $k = 2q$ follows from (28), and the following consequence of (20):
$$t^j \tau_{k,p} = \frac{\omega_{k+j}}{\pi^j \omega_k} \frac{(k-2p+j)!}{(k-2p)!} \tau_{k+j,p}.$$
The remainig cases follow then also from this equation. □

A useful consequence of the relation above is
$$s \cdot \mu_{k,q} = \frac{(k-2q+1)(k-2q+2)}{2\pi(k+2)} \mu_{k+2,q} + \frac{2(q+1)(k-q+1)}{\pi(k+2)} \mu_{k+2,q+1}. \tag{30}$$

Theorem 2.10 ([23])**.**
$$\mathrm{Val}^{U(n)} = \frac{\mathbb{R}[s,t]}{(\mu_{n+1,0}, \mu_{n+2,0})} \tag{31}$$
where
$$\mu_{k,0} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{\pi^k}{\omega_k (k-2i)!(2i)!} t^{k-2i} u^i.$$

Proof. Let I be the kernel of the projection $\mathbb{R}[s,t] \to \mathrm{Val}^{U(n)}$. It follows from (29) that $\mu_{n+1,0}, \mu_{n+2,0} \in I$. Let us show that these polynomial are relatively prime. Indeed, suppose $\mu_{n+1,0} = d_{n+1} \cdot \omega, \mu_{n+2,0} = d_{n+2} \cdot \omega$ with d_{n+1}, d_{n+2} relatively prime and $0 < k := \deg \omega \leq n$. Then one checks by counting dimensions that the ideal (d_{n+1}, d_{n+2}) contains all the polynomials of degree $> 2n-2k$. In particular, every polynomial of degree $2n-k$ vanishes on $W := \mathbb{R}[s,t]/(d_{n+1}, d_{n+2})$.

Since $\deg \omega \leq n$ we have $\omega \neq 0$ in $\mathrm{Val}^{U(n)}$. Then, by Alesker-Poincaré duality there exists $g \in \mathbb{R}[s,t]$ of degree $2n-k$ such that $\omega \cdot g \neq 0$ in $\mathrm{Val}^{U(n)}$. This is a contradiction since the map
$$\mathbb{R}[s,t] \xrightarrow{\cdot \omega} \mathrm{Val}^{U(n)}$$
factors through W.

We conclude that $\mu_{n+1,0}, \mu_{n+2,0}$ are relatively prime. Then, it is easy to see that they generate I. □

3. Rank one symmetric spaces

The theory of valuations has been extended to manifolds in a series of papers by S. Alesker [5, 6, 7, 8, 11]. In general, there is no convenient convexity notion on general manifolds. Hence, valuations will be applied to smooth submanifolds with boundary or even with corners (locally modeled on $\{0\} \times [0,\infty)^k \subset \mathbb{R}^n$), altough the theory extends to much more general classes of sets.

Let M^n be a riemannian manifold, and let $\mathcal{P}(M)$ be the set of compact submanifolds with corners. Given $A \in \mathcal{P}(M)$ we define its *normal cycle* as the set $N(A)$ of its outward pointing unit normal vectors. Then, $N(A)$ is an $(n-1)-$dimensional Lipschitz submanifold of SM the unit tangent sphere bundle of M.

Definition 3.1. *A* valuation *on M is a functional $\mu\colon \mathcal{P}(M)\to\mathbb{R}$ of the form*

$$\mu(A)=\int_{N(A)}\omega+\int_A\eta$$

where $\omega\in\Omega^{n-1}(SM)$ and $\eta\in\Omega^n(M)$ are fixed. We denote by $\mathcal{V}(M)$ the space of valuations of M.

A curvature measure *on M associates to each $A\in\mathcal{P}(M)$ a Borel measure $\Phi(A,\cdot)$ by*

$$\Phi(A,U)=\int_{N(A)\cap\pi^{-1}U}\omega+\int_{A\cap U}\eta$$

where $\pi: SM\to M$ is the projection. We denote by $\mathcal{C}(M)$ the space of curvature measures of M.

Thus we have projections

$$\Omega^{m-1}(SM)\times\Omega^m(M)\xrightarrow{\text{integ}}\mathcal{C}(M)\xrightarrow{\text{glob}}\mathcal{V}(M).$$

Let now G be a group acting isometrically on M, and let us restrict to G-invariant differential forms. It is not hard to see that $\Omega^G(M),\Omega^G(SM)$ are finite dimensional if and only if M acts transitively on SM. We say in this case that M is isotropic. Isotropic spaces are classified: besides the affine ones there are only the rank one symmetric spaces. These are

$$M=S^n,\mathbb{R}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n,\mathbb{H}\mathrm{P}^n,\mathbb{O}\mathrm{P}^2,$$

and their non-compact (hyperbolic) duals. We restrict for simplicity to the compact case.

Theorem 3.2 ([21])**.** *There exists a linear map $K_G\colon\mathcal{C}^G(M)\to\mathcal{C}^G(M)\otimes\mathcal{C}^G(M)$ such that*

$$K_G(\Phi)(A,U,B,V)=\int_G\Phi(A\cap gB,U\cap gV)dg.$$

In particular there are kinematic formulas for valuations $k_G\colon\mathcal{V}^G(M)\to\mathcal{V}^G(M)\otimes\mathcal{V}^G(M)$.

In the case of real space forms $M=\mathbb{R}^n,S^n,\mathbb{R}P^n,H^n$, the globalization map glob: $\mathcal{C}^G(M)\to\mathcal{V}^G(M)$ is an isomorphism. Hence, K_G and k_G are equivalent. For instance, putting Δ_i for the curvature measures defined by κ_i,

$$\int_{\overline{SO(n)}}\Delta_0(A\cap gB,U\cap gV)\,dg=\sum_{i+j=n}\binom{n}{i}^{-1}\frac{\omega_i\omega_j}{\omega_n}\Delta_i(A,U)\Delta_j(B,V).\quad(32)$$

Given $A\in\mathcal{P}(M)$, let

$$\mu_A^G=\int_G\chi(\cdot\cap gA)dg\in\mathcal{V}^G.$$

These valuations span $\mathcal{V}^G$.

There is a product in $\mathcal{V}^G$ uniquely determined by

$$\mu_A^G \cdot \psi = \int_G \psi(\cdot \cap gA) dg \in \mathcal{V}^G, \qquad \psi \in \mathcal{V}^G.$$

There is also a $\mathcal{V}^G$-module structure on $\mathcal{C}^G$ fulfilling

$$\mu_A^G \cdot \Phi = \int_G \Phi(\cdot \cap gA, \cdot) dg$$

Assuming M compact, we define a Poincaré pairing by

$$\mathrm{pd}(\mu, \phi) = \frac{\mu \cdot \phi(M)}{\mathrm{vol}(M)}.$$

Again we have the fundamental theorem

Theorem 3.3. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{V}^G(M) & \xrightarrow{k} & \mathcal{V}^G(M) \otimes \mathcal{V}^G(M) \\ \Big\downarrow \mathrm{pd} & & \Big\downarrow \mathrm{pd} \otimes \mathrm{pd} \\ \mathcal{V}^G(M)^* & \xrightarrow{m^*} & \mathcal{V}^G(M)^* \otimes \mathcal{V}^G(M)^* \end{array}$$

Theorem 3.4 (Transfer principle, Howard [25])**.** *Let $M = G/H$ be a rank one symmetric space. There is a canonical isomorphism of vector spaces*

$$\tau : \mathcal{C}^G(M) \longrightarrow \mathrm{Curv}^H(T_x M)$$

where $\mathrm{Curv}^H = \mathcal{C}^{\bar{H}}$ *denotes the space of $\bar{H}$-invariant curvature measures. Moreover*

$$\begin{array}{ccc} \mathcal{C}^G(M) & \xrightarrow{K_G} & \mathcal{C}^G(M) \otimes \mathcal{C}^G(M) \\ \Big\downarrow \tau & & \Big\downarrow \tau\otimes\tau \\ \mathrm{Curv}^H & \xrightarrow{K_H} & \mathrm{Curv}^H \otimes \mathrm{Curv}^H \end{array}$$

There is a close relationship between K and the module structure:

$$K(\phi \cdot \Phi) = (\phi \otimes \chi) K(\Phi) = (\chi \otimes \phi) K(\Phi).$$

However, we can not recover K from the module structure. The best we have is

$$\bar{m} = (id \otimes PD) \circ \bar{k}$$

where $\bar{m}\colon \mathrm{Curv}^G \to \mathrm{Curv}^G \otimes (\mathrm{Val}^G)^*$ is the module structure and $\bar{k} := (id \otimes \mathrm{glob})K$ is the so-called semi-local kinematic operator.

4. Complex space forms

Let $\mathbb{C}\mathrm{P}^n_\lambda$ be the n-dimensional simply connected Kähler manifold of constant holomorphic curvature 4λ. For $\lambda > 0$ this is the complex projective space $\mathbb{C}\mathrm{P}^n$ with the (rescaled) Fubini-Study metric. For $\lambda < 0$ this is the complex hyperbolic space. When $\lambda = 0$ it is $\mathbb{C}^n$ again. For $\lambda \neq 0$, let G be the group of isometries of $\mathbb{C}\mathrm{P}^n_\lambda$. In the case $\lambda = 0$, put $G = U(n) \ltimes \mathbb{C}^n$.

By the transfer principle we will identify

$$\mathcal{C}^G(\mathbb{C}\mathrm{P}^n_\lambda) \equiv \mathrm{Curv}^{U(n)} = \mathrm{span}\{B_{k,q}, \Gamma_{k,q}\},$$

where $B_{k,q}, \Gamma_{k,q}$ are the curvature measures defined respectively by $\beta_{k,q}, \gamma_{k,q}$. We consider the globalization map

$$\mathrm{glob}_\lambda \colon \mathrm{Curv}^{U(n)} \longrightarrow \mathcal{V}^n_\lambda := \mathcal{V}^G(\mathbb{C}\mathrm{P}^n_\lambda)$$

Proposition 4.1 ([1]).

$$\ker \mathrm{glob}_\lambda = \mathrm{span}\{\Gamma_{k,q} - B_{k,q} + \lambda \frac{(2n-k)(q+1)}{2\pi(n-k+q)} B_{k+2,q+1}\}, \qquad k \neq 2q, n+q.$$

Proof. Use Stokes and

$$\begin{aligned} d\alpha &= -\theta_s, & d\theta_0 &= -\lambda(\alpha\wedge\theta_1 + \beta\wedge\theta_s),\\ d\beta &= \theta_1, & d\theta_1 &= 0,\\ d\gamma &= 2\theta_0 - 2\lambda\theta_2 - 2\lambda\alpha\wedge\beta, & d\theta_2 &= 0. \end{aligned}$$

□

We define

$$\mu^\lambda_{k,q} := \mathrm{glob}_\lambda(B_{k,q}), \quad k \neq 2q$$

When $k = 2q$ we make the recursive definition

$$\mu^\lambda_{2q,q} := \mathrm{glob}_\lambda(\Gamma_{2q,q}) + \lambda \frac{(2n-2q)(q+1)}{2\pi(n-q)} \mu^\lambda_{2q+2,q+1}$$

This yields a basis of $\mathcal{V}^n_\lambda$ with very nice properties. Even better is the following basis:

$$\tau^\lambda_{kq} := \sum_{i=q}^{\lfloor \frac{k}{2} \rfloor} \binom{i}{q} \mu^\lambda_{ki}.$$

4.1. **The Lipschitz-Killing algebra.** Let $\iota : M^n \to \mathbb{R}^N$ be an isometric immersion. The Lipschitz-Killing curvature measures of M are defined by pull-back of the $\overline{SO(N)}-$invariant curvature measures of $\mathbb{R}^N$:

$$\Lambda^M_k(A, U) = \iota^*\Delta_k(A, U) = \Delta_k(\iota A, \iota U).$$

H. Weyl discovered the remarkable fact that these curvature measures do not depend on ι but only on the metric of M. This is a consequence of the following description of Λ_k in terms of a pair of differential forms $(\Psi_k, \Phi_k) \in \Omega^n(M) \times \Omega^{n-1}(SM)$. Let $e_1(\xi), \ldots, e_n(\xi) = \xi$ be a local moving frame on M defined for $\xi \in SM$. Let θ_i be the associated coframe, and $\omega_{i,j}$ the connection forms. Finally, let Ω_{ij} denote the curvature forms. Then (cf. [20])

$$\Psi_k = \frac{2}{\omega_{n-k+1} k!(n-k+1)!} \sum_\epsilon \mathrm{sgn}(\epsilon) \Omega_{\epsilon_1\epsilon_2} \wedge \cdots \wedge \Omega_{\epsilon_{n-k-1}\epsilon_{n-k}} \wedge \theta_{\epsilon_{n-k+1}} \wedge \cdots \wedge \theta_{\epsilon_n} \tag{33}$$

when $n-k$ is even, and $\Psi_k = 0$ if $n-k$ is odd. Similarly

$$\Phi_k = \sum_{2i \leq n-k-1} \frac{2}{\omega_{2i+1}\omega_{n-2i-k}(2i+1)!(n-2i-k)!} \Phi_{k,i} \tag{34}$$

where

$$\Phi_{k,i} := \sum_\epsilon \mathrm{sgn}(\epsilon) \Omega_{\epsilon_1\epsilon_2} \wedge \cdots \wedge \Omega_{\epsilon_{2i-1}\epsilon_{2i}} \wedge \omega_{n,\epsilon_{2i+1}} \wedge \cdots \wedge \omega_{n,\epsilon_{n-k-1}} \wedge \theta_{\epsilon_{n-k}} \wedge \cdots \wedge \theta_{\epsilon_{n-1}} \tag{35}$$

Both Ψ_k and Φ_k, and in fact all of the $\Phi_{k,i}$, are independent of the moving frame, and hence globally defined differential forms on SM. Moreover Ψ_k is the pullback of a differential form defined on M, which we denote again by Ψ_k.

In particular, the globalization of the Lipschitz-Killing curvature measures give a canonical family of valuations $\iota^*\mu_i$ on every riemannian manifold. It turns out that **the product of valuations commutes with the pullback through immersions**. Hence

$$LK(M) := \operatorname{span}\{\operatorname{glob}\Lambda\} = \operatorname{span}\{\iota^*\mu_i\} = \frac{\mathbb{R}[t]}{(t^{n+1})} \qquad \text{where } t = \frac{2}{\pi}\iota^*\mu_1$$

is a subalgebra of $\mathcal{V}(M)$ called the Lipschitz-Killing algebra of M.

Let now $M = \mathbb{C}\mathrm{P}^n_\lambda$. Of course $\Lambda_k \in \operatorname{Curv}^{U(n)} = \operatorname{span}\{B_{k,q}, \Gamma_{k,q}\}$. To find them explicitly a key observation is the following. Let us say that a translation invariant curvature measure Φ is *angular* if there is a function c_Φ on Gr_k such that

$$\Phi(P, F) = c_\Phi(F) \cdot \angle(P, F),$$

for any polytope P and every k-dimensional face F of P. It is not hard to see that the Lipschitz-Killing curvature measures are angular and that the space of $\overline{U(n)}$-invariant angular curvature measures is precisely

$$\operatorname{span}\{\Delta_{k,q} := \frac{1}{2n-k}(2(n-k+q)\Gamma_{k,q} + (k-2q)B_{k,q}\}$$

Hence, each Λ_r can be expanded in terms of $\Delta_{k,q}$. To do this explicitly, we evaluate Λ_r on $\exp_o(E^{k,q})$ where $E^{k,q} \in \mathrm{Gr}_{k,q}(T_o\,\mathbb{C}\mathrm{P}^n_\lambda)$. The density Pf^r_{kq} of this measure at o yields essentially the coefficient of $\Delta_{k,q}$ in the expansion of Λ_r. By using the explicit description of the curvature tensor of $\mathbb{C}\mathrm{P}^n_\lambda$ one finds a recurrence relation between the coefficients Pf^r_{kq}. From this recurrence, using exponential generating functions, one gets the following.

Theorem 4.2. *Define*

$$g_i(\xi,\eta) := \xi^i\,(1-\xi)^{-i-\frac{1}{2}}\,(1-\eta)^{-\frac{1}{2}}$$
$$h_i(\xi,\eta) := \xi^i\,(1-\xi)^{-i-\frac{3}{2}}\,(1-\eta)^{-\frac{1}{2}}\,.$$

Then

$$t^{2i} = \binom{2i}{i}\lambda^{-i}\sum_{k,p=0}^{\infty}\left(\frac{\lambda}{\pi}\right)^{k+p}\frac{\partial^{k+p}g_i}{\partial^k\xi\partial^p\eta}\Big|_{\xi=\eta=0}\tau^\lambda_{2k+2p,p} \tag{36}$$

$$t^{2i+1} = \frac{2^{2i+1}}{\pi}\lambda^{-i}\sum_{k,p=0}^{\infty}\left(\frac{\lambda}{\pi}\right)^{k+p}\frac{\partial^{k+p}h_i}{\partial^k\xi\partial^p\eta}\Big|_{\xi=\eta=0}\tau^\lambda_{2k+2p+1,p}. \tag{37}$$

4.2. **About s.** We define

$$s = s_\lambda := \frac{n}{\pi}(\operatorname{vol}(\mathbb{C}\mathrm{P}^{n-1}_\lambda))^{-1}\int_G \chi(\cdot\cap gH)dg \in \mathcal{V}^n_\lambda$$

where $H \subset \mathbb{C}\mathrm{P}^n_\lambda$ is a totally geodesic copy of $\mathbb{C}\mathrm{P}^{n-1}_\lambda$.

Proposition 4.3. *The action of $s_\lambda \in \mathcal{V}^n_\lambda$ on $\operatorname{Curv}^{U(n)}$ is independent of λ.*

Proof. Given $\Phi \in \mathrm{Curv}^{U(n)}$,

$$s \cdot \Phi = \frac{n}{\pi}(\mathrm{vol}(\mathbb{C}\mathrm{P}^{n-1}_\lambda))^{-1} \int_G \Phi(\cdot \cap gH, \cdot) dg \qquad (38)$$
$$= \frac{n}{\pi}(\mathrm{vol}(\mathbb{C}\mathrm{P}^{n-1}_\lambda))^{-1} K(\Phi)(H, \mathbb{C}\mathrm{P}^n_\lambda, \cdot, \cdot).$$

The result follows by the transfer principle and explicit evaluation of $B_{k,q}, \Gamma_{k,q}$ on $(H, \mathbb{C}\mathrm{P}^n_\lambda)$: they all vanish except $\Gamma_{2n-2,n-1}$ which gives $\mathrm{vol}(\mathbb{C}\mathrm{P}^{n-1}_\lambda)$. □

Proposition 4.4.

$$s \cdot B_{k,q} \in \mathrm{span}\{B_{k+2,p}\}$$

Proof. The key fact is that $\mathrm{span}\{B_{k,p}\}$ is exactly the space of curvature measures Φ with the following property: if $A \subset \mathbb{C}^n$ is a smooth submanifold then the measure $\Phi(A, \cdot)$ has density 0 at the points where T_pA is complex. This property is clearly invariant under restriction to complex submanifolds. Hence, by (38), the property is invariant under multiplication by s. □

From (30) we deduce

$$s \cdot B_{k,q} = \frac{(k-2q+1)(k-2q+2)}{2\pi(k+2)} B_{k+2,q} + \frac{2(q+1)(k-q+1)}{\pi(k+2)} B_{k+2,q+1}. \qquad (39)$$

Hence, for $k \neq 2q$

$$s \cdot \mu^\lambda_{k,q} = \frac{(k-2q+1)(k-2q+2)}{2\pi(k+2)} \mu^\lambda_{k+2,q} + \frac{2(q+1)(k-q+1)}{\pi(k+2)} \mu^\lambda_{k+2,q+1}. \qquad (40)$$

Actually, the same is true for $k = 2q$. The proof in this case needs an extra bit of information: the explicit expression of s^k in terms of differential forms given in [1].

4.3. **Algebra and coalgebra isomorphism.** The two last subsections give enough information to write any given polynomial $p(s,t)$ in terms of $\tau_{k,q}$. Actually, we can do the opposite.

Proposition 4.5. *Denoting*

$$v := t^2(1 - \lambda s)$$
$$u := 4s - v$$

we have

$$\tau^\lambda_{k,q} = (1 - \lambda s) \frac{\pi^k}{\omega_k (k-2q)!(2q)!} v^{\frac{k}{2}-q} u^q \qquad (41)$$

Proof. First one can use Theorem 4.2 and equation (40) to show that s, t generate $\mathcal{V}^n_\lambda$. Then, one checks (41) on the Lipschitk-Killing algebra $\mathbb{R}[t]$ using Theorem 4.2 again. Finally, it remains only to show the compatibility with (40), which is straightforward. □

Theorem 4.6. *There exists an algebra isomorphism* $I_\lambda : \mathcal{V}^n_0 \to \mathcal{V}^n_\lambda$ *such that*

$$I_\lambda(s) = s, \qquad I_\lambda(t) = t\sqrt{1 - \lambda s}.$$

Proof. Let $\tilde{I}_\lambda : \mathbb{R}[s,t] \to \mathbb{R}[s,t]$ be the algebra isomorphism defined by $\tilde{I}_\lambda(s) = s, \tilde{I}_\lambda(t) = t\sqrt{1-\lambda s}$. We must show the existence of the algebra morphism I_λ in the following diagram, where the vertical maps are restrictions:

$$\begin{array}{ccc} \mathbb{R}[s,t] & \xrightarrow{\tilde{I}_\lambda} & \mathbb{R}[s,t] \\ \downarrow & & \downarrow \\ \mathcal{V}_0^n & \xrightarrow{I_\lambda} & \mathcal{V}_\lambda^n \end{array}$$

Recall that $\mathcal{V}_0^n \cong \mathbb{R}[s,t]/(\mu_{n+1,0}, \mu_{n+2,0})$, and by Proposition 4.5

$$\begin{aligned} \tilde{I}_\lambda(\mu_{k,0}) &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{\pi^k}{\omega_k (k-2i)!(2i)!} (1-\lambda s)^{\frac{k}{2}-i} t^{k-2i} (4s - t^2(1-\lambda s))^i \\ &= \frac{1}{1-\lambda s} \mu_{k,0}^\lambda \end{aligned} \tag{42}$$

whose image in $\mathcal{V}_\lambda^n$ vanishes if $n < k$. Hence I_λ is well-defined. Since it is surjective, by comparing dimensions it follows that I_λ is bijective. □

By the Fundamental Theorem 3.3, the following diagram commutes

$$\begin{array}{ccc} \mathcal{V}_\lambda^n & \xrightarrow{k_\lambda} & \mathcal{V}_\lambda^n \otimes \mathcal{V}_\lambda^n \\ \downarrow \mathrm{pd}_\lambda & & \downarrow \mathrm{pd}_\lambda \otimes \mathrm{pd}_\lambda \\ \mathcal{V}_\lambda^{n*} & \xrightarrow{m^*} & \mathcal{V}_\lambda^{n*} \otimes \mathcal{V}_\lambda^{n*} \\ \downarrow I_\lambda^* & & \downarrow I_\lambda^* \otimes I_\lambda^* \\ \mathcal{V}_0^{n*} & \xrightarrow{m^*} & \mathcal{V}_0^{n*} \otimes \mathcal{V}_0^{n*} \\ \uparrow \mathrm{pd} & & \uparrow \mathrm{pd} \otimes \mathrm{pd} \\ \mathcal{V}_0^n & \xrightarrow{k} & \mathcal{V}_0^n \otimes \mathcal{V}_0^n \end{array}$$

Hence the map $J_\lambda := \mathrm{pd}_\lambda^{-1} \circ (I_\lambda^{-1})^* \circ \mathrm{pd}$ is a co-algebra isomorphism from $\mathcal{V}_0^n$ to $\mathcal{V}_\lambda^n$, i.e.

$$k_\lambda \circ J_\lambda = (J_\lambda \otimes J_\lambda) \circ k. \tag{43}$$

Lemma 4.7 ([24])**.**

$$t^{2i} s^j (\mathbb{C}\mathrm{P}_\lambda^n) = \frac{1}{\lambda^{i+j}} \binom{2i}{i} \binom{n-j+1}{i+1}.$$

Proposition 4.8. $J_\lambda = (1-\lambda s)^2 I_\lambda$.

Proof. First, it is not hard to see that $J_\lambda(\varphi \cdot \phi) = I_\lambda(\varphi) \cdot J_\lambda(\phi)$. Then, it is enough to show $J_\lambda(\chi) = (1-\lambda s)^2$. In turn, this boils down to

$$\mathrm{pd}(\chi, I_\lambda^{-1}(t^{2i}s^j)) = \mathrm{pd}_\lambda(t^{2i}s^j, (1-\lambda s)^2) \qquad \forall i,j$$

which follows from the lemma above. □

From Propositions 4.5 and 4.8 it follows that

$$J_\lambda(\tau_{k,q}) = (1-\lambda s)\tau_{k,q}^\lambda. \tag{44}$$

Let $F_\lambda\colon \mathcal{V}_0^n \to \mathcal{V}_\lambda^n$ be given by $F_\lambda = (1-\lambda s)^{-1} J_\lambda$. Equivalently

$$F_\lambda(\tau_{kq}) = \tau_{kq}^\lambda.$$

Theorem 4.9. *The principal kinematic formula in* $\mathbb{C}\mathrm{P}_\lambda^n$ *is given by*

$$k_\lambda(\chi) = (F_\lambda \otimes F_\lambda) \circ k(\chi).$$

Proof.

$$\begin{aligned} k_\lambda(\chi) &= ((1-\lambda s)^{-1} \otimes (1-\lambda s)^{-1}) k_\lambda((1-\lambda s)^2) \\ &= ((1-\lambda s)^{-1} \otimes (1-\lambda s)^{-1}) k_\lambda(J_\lambda(\chi)) \\ &= (1-\lambda s)^{-1} \otimes (1-\lambda s)^{-1})(J_\lambda \otimes J_\lambda)(k(\chi)) \end{aligned}$$

which with (44) gives the desired relation. □

One can show similarly that

$$k_\lambda(\tau_{k,q}^\lambda) = (F_\lambda \otimes F_\lambda) \circ k((1-\lambda s)\tau_{k,q}). \tag{45}$$

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