Torsion points of abelian varieties over p-adic fields

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Preliminary Version

Abstract

We prove that for every abelian variety over a finite extension K of the p-adic field $\mathbb{Q}_p$ with reduction not containing the multiplicative group $\mathbb{G}_m$ there exists an upper bound for the number of torsion points depending on the dimension of the variety, the absolute ramification index and the decomposition degree of K and p.

Introduction. Let K be a number field and A an abelian variety over K. A well-known theorem of Mordell and Weil implies that A has a finite number of K-rational torsion points. It is natural to ask if there exists only a finite number of finite abelian groups appearing as torsion subgroups of abelian varieties with fixed dimension g over a fixed number field K. The following conjecture was stated by Cassels (for elliptic curves) as "part of the folklore".

Conjecture 1 (Uniform Boundedness Conjecture) *If A is an abelian variety of dimension g over a number field K of degree d, then $|A(K)_{tors}|$ is bounded by a constant depending only on g and d.*

This conjecture has been proved for $g = 1$ by Merel in 1994 [12]. For $g > 1$, there is no number field for which the Uniform Boundedness Conjecture is known to hold. For more information the reader can see the paper of Silverberg [16] and references therein.

In this note, we study a similar question on local number fields (finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers). This problem make sense since the number of torsion points of an abelian variety on such a field is finite.

Flexor and Oesterlé dealt with the problem of bounding the number of torsion points of certain families of elliptic curves over local fields [6]. If we consider the class of elliptic curves of good reduction over local number field K, then the number of torsion points of any curve of the family can be bounded in terms of p and the degree of K (in fact in terms of p, of the absolute ramification index and the decomposition degree of K). On the other hand, elliptic curves with additive reduction has prime-to-p torsion bounded by 4 (independently of K),

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and the p-torsion can be bounded in terms of p and the ramification index e of K, and also only in terms of e (see [6]). On the other hand, it is possible to construct elliptic curves with split multiplicative reduction and torsion points with order as large as desired. This paper is devoted to showing that the situation is quite similar for a general abelian variety. In section 1 we study the torsion of connected smooth group schemes over p-adic rings. We apply these results in section 2 to the abelian varieties with potentially good reduction over a local number field and in section 3 to algebraic tori. These results are undertaken in section 4, where we study the general case using the rigid uniformization of abelian varieties. This allow us to reduce the problem to those of earlier sections.

Some results in this direction have been obtained earlier. For example, Lenstra and Oort have proved in [8] that the prime-to-p part of the group of torsion points of an abelian variety having (totally)additive reduction has order bounded in terms only of the dimension (see also [11, Corollary 1.4]). Silverberg and Zarhin have studied also the prime-to-p part of this group in the general case ([18]).

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Notation. For any abelian group X, any natural number n and any prime number p, we will use the notation X_{tors}, $X[n]$, X_p and $X^{(p)}$ to denote the torsion subgroup of X, the subgroup of elements of order less than or equal to n, the subgroup of elements in X_{tors} with order a power of p and the subgroup of elements in X_{tors} with order prime to p respectively.

1 Torsion points of Formal Lie groups

Let R be complete discrete valuation ring of characteristic zero with maximal ideal $\mathbf{m}$ and finite residue field $k = R/\mathbf{m}$. We denote by p^f the number of elements of k, by v the (normalized) valuation of R and by $e = v(p)$ the absolute ramification index of R. Let $\mathcal{X}$ be a geometrically connected smooth commutative group scheme of finite type over R of dimension g, and let $\mathcal{X}_k$ be its fiber over k. Unless explicit mention, all the group schemes are assumed to be commutative.

The aim of this section is to study the group of torsion points of $\mathcal{X}(R)$, and, in particular, to find a bound for $|\mathcal{X}(R)_{tors}|$ in terms of e, f, g, and p.

Let $red : \mathcal{X}(R) \to \mathcal{X}_k(k)$ be the reduction map; it is surjective since $\mathcal{X}$ is smooth and R is henselian. First, observe that we need only bound the number of torsion points in the kernel of the reduction map red, since the points of a connected smooth group scheme over a finite field are bounded uniformly in terms of k and the dimension, essentially due to a well-known theorem of Weil.

Theorem 1 *Let $\mathcal{X}_k$ be a geometrically connected smooth (commutative) group scheme of finite type with dimension g over a perfect field k. Then, there exists* *an abelian variety B_k, a torus T_k and an unipotent group scheme U_k defined over k and an exact sequence as group schemes*

$$0 \to T_k \times U_k \to \mathcal{X}_k \to B_k \to 0.$$

Now, if k is a finite field of order p^f, the cardinality of the group of points over k of every of this groups is bonded by constants depending only on its dimension, p and f. Explicitly, if we denote by a, t and u the dimensions of A_k, T_k and U_k respectively, we have

$$|B_k(k)| \leq (1+p^{\frac{f}{2}})^{2a}, \ \ |T_k(k)| \leq (1+p^f)^t \ \textit{and} \ |U_k(k)| = p^{fu}.$$

Therefore, since $g = a+t+u$, we have $|\mathcal{X}_k(k)| \leq (1+p^{\frac{f}{2}})^{2g}$.

Proof. All the results are well-known. The decomposition of $\mathcal{X}_k$ is the Chevalley's Theorem and can be found for example in [2, 9.2.1 and 9.2.2]. The subgroup scheme $T_k \times U_k$ is just the maximal affine subgroup scheme and T_k is the maximal torus (unique since $\mathcal{X}_k$ is commutative).

The bound for the cardinal of the group of points of an abelian varieties due to Weil. The bound for tori can be found in [19, Theorem 6.2, section 1, chapter VI] (see also the proof of Theorem 5.3 in [18]). The bound for the unipotent groups is deduced easily from the case of the additive group $\mathbb{G}_a$ by using that any connected smooth unipotent group has an epimorphism to $\mathbb{G}_a$ with kernel a connected smooth group and that $\mathbb{G}_a$ has trivial cohomology. $\square$

For every smooth group scheme $\mathcal{X}_k$ over k, we will say a, t and u the abelian rank, toric rank and unipotent rank respectively. The following Proposition is well-know (see for example Lemma 7.3.2 in [2]).

Proposition 2 *Let R be a complete discrete valuation ring and k its residue field. Let n be a positive integer such that the residue characteristic p of R does not divide n. Then, for any smooth commutative group scheme $\mathcal{X}$ of finite type over R, the multiplication-by-n map is étale. Therefore, the reduction map gives a bijection $\mathcal{X}(R)_{tors}^{(p)} \cong \mathcal{X}_k(k)^{(p)}$.*

Thus, the prime-to-p part of the torsion subgroup of $\mathcal{X}(R)$ is totally determined by its special fiber and the kernel of the reduction map *red* can have only torsion points of order a power of p.

The following Theorem shows that the kernel of the reduction map *red* is always equal to the group of points of the formal completion of $\mathcal{X}$ along the zero section. Recall that a g-dimensional commutative formal Lie group over a ring R is an g-tuple of power series $F(X,Y) := (F_1(X,Y),\ldots,F_g(X,Y)$ in $2g$ indeterminates $X = (X_1,\ldots,X_g)$ and $Y = (Y_1,\ldots,Y_g)$ and with coefficients in R, such that

1. $F(X,Y) \cong X+Y \pmod{\deg 2}$,
2. $F(F(X,Y),Z) = F(X,F(Y,Z))$

3. $F(X,Y) = F(Y,X)$.

Now, if R is a complete discrete valuation ring with maximal ideal $\mathbf{m}$, the group $F(R)$ of points over R of this formal group is, by definition, the set $\mathbf{m}^g$ with the operation

$$(x_1, \ldots, x_g) * (y_1, \ldots, y_g) := F(x_1, \ldots, x_g; y_1, \ldots, y_g).$$

This operation is well-defined since the power series converges, and it gives an structure of abelian group on $F(R)$. The following result is well-known, but we include a proof for lack of a suitable reference.

Theorem 3 *Let $\mathcal{X}$ be a smooth commutative group scheme of finite type over a complete discrete valuation ring R. Then, the kernel of the reduction map $\mathcal{X}(R) \to \mathcal{X}_k(k)$ is isomorphic to the group of points of the formal group completion along the zero section of $\mathcal{X}$, which is a commutative formal Lie group over R of dimension equal to the dimension of $\mathcal{X}$.*

Proof. The result is deduced easily from the following more general result. Let $\mathcal{X}$ be an equidimensional smooth scheme of finite type over $S := \mathrm{Spec}\ R$, and let s a section of the structural morphism. Then the formal completion along the section s is isomorphic as a formal scheme to the formal spectrum of $R[[X_1, \ldots, X_n]]$ for n equal to the dimension of $\mathcal{X}$, and the sections of this formal completion correspond bijectively with the sections of $\mathcal{X}$ that coincide with s over the closed point of S. The first result can be deduced from [15, II.4.17], but we will show both results directly in the following.

Since $\mathcal{X}$ is smooth, and if e denotes the closed point in the image of the section s, there exists an open neighbourhood U of e and morphisms $g : U \to \mathbb{A}^n_S$ étale and $f : \mathbb{A}^n_S \to S$ such that $f \circ g$ is the structural map of U over S, where we denote as usual by $\mathbb{A}^n_S$ the affine space of dimension n over S.

Now, observe that sections on $\mathcal{X}$ that restrict to the section s on $\mathcal{X}_k$ give us sections on U, since for every of this sections e is in the image. On the other hand, the formal completion along the section s of $\mathcal{X}$ and of U are isomorphic as a formal schemes. Thus we can restrict to proving the result for U. Since g is étale it induces an isomorphism for the formal completions . Moreover, we have a bijection between points of U that restrict to the section s and points of $\mathbb{A}^n_S$ that are congruent to e modulo the maximal ideal $\mathbf{m}$ of R. Thus we are reduced to the case that $\mathcal{X}$ is $\mathbb{A}^n_S$, and this case is clear. $\square$

This result allow us to reduce our problem to the problem of bounding the number of torsion points of a formal Lie group F of dimension g over a discrete valuation ring R. For every positive integer m, we denote as usual $[m]_F(X)$ the g-tuple of power series on the g indeterminates $X_1, \ldots, X_g$ defined inductively on m by

$$[m]_F(X) = F(X, [m-1](X)) \text{ if } m \geq 2, \text{ and } [1]_F(X) = X.$$

The following Lemma is an easy consequence of the $\mathbb{Q}$-theorem (see for example [7, 11.1.7]). For every prime number p we denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ with respect to the prime ideal $p\mathbb{Z}$.

Lemma 4 *Let F be a g-dimensional commutative formal Lie group over a $\mathbb{Z}_{(p)}$-algebra R, p a prime number. Then, there exists power series $\beta_0(X)$ and $\beta_1(X) \in R[[X_1, \ldots, X_g]]$, with $\beta_0(X) \cong X \pmod{\deg 2}$ and $\beta_1(0) = 0$ such that*

$$[p]_F(X) = p\beta_0(X) + \beta_1(X^p)$$

Using this result it is easy to bound the p-torsion of a formal Lie group.

Proposition 5 *Let F be a g-dimensional commutative formal Lie group over a complete discrete valuation ring R of characteristic zero with finite residue field k of characteristic $p > 0$, and let v be the (normalized) valuation on R. Suppose that $x = (x_1, \ldots, x_g) \in F(R)$ has exact order p^n for some $n \geq 1$. Then, there exists an $i \in \{1, \ldots, g\}$ such that*

$$v(x_i) \leq \frac{v(p)}{p^n - p^{n-1}}$$

Proof. First of all, observe that for any power series $f(X) \in R[[X]]$ with $f(0) = 0$ and every element $x = (x_1, \ldots, x_g) \in \mathbf{m}^g$, we have $v(f(x)) \geq \inf\{v(x_i)\}$. We will show the proposition by induction on n.

Suppose that $x \neq 0$ and $[p]_F(x) = 0$. Suppose also for simplicity that $v(x_1) \leq v(x_i)$ for all i. By Lemma 1.4 we have $0 = [p]_F(x) = p\beta_0(x) + \beta_1(x^p)$. Considering the first component of this identity, we have

$$0 = p \cdot (x_1 + \gamma_0(x_1, \ldots, x_g)) + \gamma_1(x_1^p, \ldots, x_g^p),$$

where the power series $\gamma_0(X)$ is zero modulo degree 2 and $\gamma_1(0) = 0$. Using our assumption on x we have $v(x_1) < v(\gamma_0(x_1, \ldots, x_g))$, and hence

$$v(\gamma_1(x_1^p, \ldots, x_g^p)) = v(p(x_1 + \gamma_0(x_1, \ldots, x_g))) = v(px_1) = v(p) + v(x_1).$$

On the other hand, we have

$$v(\gamma_1(x_1^p, \ldots, x_g^p)) \geq \inf\{v(x_i^p)\} = v(x_1^p) = pv(x_1),$$

using again our assumption on x. Hence $v(p) \geq (p-1)v(x_1)$ which proves the Proposition for $n = 1$.

Now assume that the theorem is true for n, and let $x \in F(R)$ have exact order p^{n+1}. Then $[p]_F(x) = (y_1, \ldots, y_g)$ has exact order p^n. We assume for simplicity that $v(y_1) \leq v(y_i)$ for all i, so, by the induction hypothesis, $v(p)/(p^n - p^{n-1}) \geq v(y_1)$. But, as in the preceding argument we have that

$$\begin{aligned} v(y_1) \geq \inf\{v(px_1 + & p\gamma_0(x_1, \ldots, x_g)), v(\gamma_1(x_1^p, \ldots, x_g^p))\} \\ \geq & \quad \inf_{i=1,\ldots,g}\{\inf\{v(px_i), v(x_i^p)\}\}. \end{aligned}$$

Since $v(x_i) > 0$ and $n \geq 1$, it certainly is not possible to have $v(p)/(p^n - p^{n-1}) \geq v(px_i)$. We conclude that, for some i, $v(p)/(p^n - p^{n-1}) \geq v(x_i^p)$, which is exactly the desired result. $\square$

Since every element in the maximal ideal of R has valuation greater or equal than 1, the proposition shows that it is not possible to obtain elements in $F(R)$ of arbitrary large order.

Corollary 6 *With the same hypothesis as in Proposition 1.5, we have that for every n such that $p^{n-1} > v(p)/(p-1)$, there are no elements in $F(R)$ with exact order p^n.*

We will show that the group of p^n-torsion points on a smooth connected group scheme of finite type can be always generated by $2g$ elements or less, where g denotes the dimension. Thus, in the case of the formal Lie group obtained as the completion along the zero section of a smooth connected group scheme of finite type, we can bound the order of the group of torsion points of this formal Lie group in terms of the dimension and the absolute ramification index of R.

Notation 7 *Let p be a prime number and $m \geq 1$ an integer. We define*

$$\gamma_p(m) := \left[\log_p(\frac{pm}{p-1})\right],$$

where $\log_p$ denotes the real logarithm with base p and $[x]$ denotes the greater integer less than or equal to x. We define $ord_p(m)$ to be the greatest integer r such that p^r divides m.

Corollary 8 *Let $\mathcal{X}$ be a geometrically connected smooth group scheme of dimension g over a complete discrete valuation ring R of characteristic zero and with finite residue field of characteristic p, and let F be the formal Lie group associated to the formal completion of $\mathcal{X}$ along the zero section. Then, if e denotes the absolute ramification index of R, the torsion subgroup of $F(R)$ is killed by $p^{\gamma_p(e)}$ and can be generated by $2g$ elements. Hence, we have $ord_p(|F(R)_{tors}|) \leq 2g\gamma_p(e)$.*

Proof. The first result is a direct application of Corollary 1.6. To check the second result we have to show that, for any natural number n, the group $\mathcal{X}(R)[p^n]$ can always be generated by $2g$ elements or less. Since $\mathcal{X}$ is separated (because it is smooth), we have that the natural morphism $\mathcal{X}(R) \to \mathcal{X}_K(K)$, where K denotes the quotient field of R, is injective. Now, applying the first part of Theorem 1.1, we can deduce the result from the cases that $\mathcal{X}_K$ is an abelian variety, a torus or a unipotent group, which are well-known. □

If $e < p-1$, we have $\gamma_p(e) = 0$, and therefore there are not non trivial torsion points on $F(R)$. This result is well-known and can also be obtained using the theory of finite group schemes.

Observe that

$$p^{n\gamma_p(e)} \leq \left(\frac{p}{p-1}\right)^n e^n \leq (2e)^n$$

for every integer $n \geq 0$, where the last inequality holds since $p/(p-1)$ is always less than or equal to 2 for any prime p. Thus, we can find a bound independent of p.

Corollary 9 *With the same hypothesis as in Corollary 1.8, we have the bound* $|F(R)_{tors}| \leq (2e)^{2g}$.

Finally, combining these results with the Theorem 1.1, we obtain a bound for the number of torsion points of any geometrically connected smooth group scheme of finite type.

Corollary 10 *With the same hypothesis as in Corollary 1.8, the cardinality of* $\mathcal{X}(R)_{tors}$ *is bounded by a constant depending on* a, t, u, e, f *and* p. *Explicitly, we have*

$$|\mathcal{X}(R)_{tors}| \leq (1+p^{\frac{f}{2}})^{2a}(1+p^f)^t p^{fu} p^{2(a+t+u)\gamma_p(e)}.$$

Therefore, $|\mathcal{X}(R)_{tors}|$ *can be bounded by a constant* $C(g,e,f,p)$ *depending on* g, e, f *and* p. *Explicitly, we can take*

$$C(g,e,f,p) := \begin{cases} (1+p^{\frac{f}{2}})^{2g} e^{2g} (\frac{p}{p-1})^{2g} & \text{if } e \geq p-1 \\ (1+p^{\frac{f}{2}})^{2g} & \text{if } e < p-1. \end{cases}$$

Proof. To get the first bound consider the inequality

$$|\mathcal{X}(R)_{tors}| \leq |\mathcal{X}(k)||F(R)_{tors}|,$$

where F denotes the formal group scheme completion with respect to the zero section of $\mathcal{X}$. By Theorem 1.3 this formal group scheme is isomorphic to a formal Lie group of dimension g, and by Corollary 1.8 we have that $|F(R)_{tors}| \leq p^{2g\gamma_p(e)}$. The bound is now obtained using that g is equal to $a+t+u$.

To obtain the second bound use that $\gamma_p(e) = 0$ if $e < p-1$, and that $p^{2g\gamma_p(e)} \leq e^{2g}(p/p-1)^{2g}$ for every e. □

2 Abelian varieties with potentially good reduction

In this section, we use the results in the last section to bound uniformly the cardinality of the group of torsion points of an abelian variety A with potentially good reduction. Unless explicit mention, let K be a finite extension of $\mathbb{Q}_p$. Let e and f denote as before the absolute ramification index and decomposition index of K, respectively.

The main idea is to use the Néron model $\mathcal{A}$ of A and apply the results in the section 1. Recall that the Néron model is a smooth group scheme of finite type over R with generic fiber A and verifying the following universal property: for every smooth scheme $\mathcal{U}$ over R and every morphism $g_K : \mathcal{U}_K \to A$, there exists a unique morphism $g : \mathcal{U} \to \mathcal{A}$ with generic fiber g_K. This property implies in particular that $A(K) \cong \mathcal{A}(R)$.

The Néron model of an abelian variety is not in general geometrically connected. We denote by $\mathcal{A}^0$ the connected component of the identity and by ϕ_A its group of components. Recall that ϕ_A is a finite étale group scheme over the residual field k of R isomorphic as a sheaf for the étale topology to the quotient $\mathcal{A}/\mathcal{A}^0$. We have thus an exact sequence of abelian groups

$$0 \to \mathcal{A}^0(R) \to A(K) \to \phi_A(k) \to \dots.$$

Since we know by Corollary 1.10 that the torsion subgroup of $\mathcal{A}^0(R)$ has cardinality bounded in terms of its dimension, e, f and p, our problem is reduced to bound uniformly the cardinal of ϕ_A. But, as we have said in the introduction, if $\mathcal{A}$ has toric rank non-trivial, the order of this group is not bounded in terms of g, e, f and p.

We restrict ourselves now to the case that A has potentially good reduction, that is that there exists a finite extension L of K such that the $A_L := A \otimes_K L$ has Néron model with toric rank zero and unipotent rank zero. One can easily show that this implies that the toric rank of the reduction is trivial.In this case, the group ϕ_A is conjecturally bounded by a constant depending only on g. This result is known for the prime-to-p part of ϕ_A, but, unfortunately, it is still not proved for the whole group.

Notation 11 *Given a finite abelian group H, define*

$$\delta(H) := \sum_{\ell \ prime} ord_\ell(|H|)(\ell - 1),$$

where the sum is over all the prime numbers. Observe that if $\delta(H) \leq n$ for some n, then $|H| \leq 2^n$. Moreover, if $p \geq \delta(H) + 1$, then p does not divides $|H|$.

In the next Theorem we will collect all the known results we need on ϕ_A in the potentially good reduction case. More results in this subject can be find in [8], in [9] and [10] in the case of jacobians of curves, and in the general case in [11, 4, 5, 3].

Theorem 12 *Let K be a discrete valuation field with ring of integers R and perfect residue field k of characteristic $p \geq 0$. Let A be an abelian variety with potentially good reduction over K, and let L/K a finite extension of fields where A attains good reduction. Let ϕ_A the group of components of the Néron model of A over R, and denote by u the unipotent rank of the reduction of A. Then $\delta(\phi_A^{(p)}) \leq 2u$ and $[L : K]\phi_A = 0$. In particular, $|\phi_A^{(p)}| \leq 2^{2u}$ and every prime number ℓ different from p dividing $|\phi_A|$ is less than or equal to $2u + 1$. Moreover, if L/K is Galois and I denotes the inertia subgroup of $Gal(L/K)$, then $|I|\phi_A = 0$.*

Proof. The bound for $\delta(\phi_A^{(p)})$ is deduced directly from the theorem 2.15 in [11], using that in the potentially good reduction case we have $t = t_L = 0$. Observe that the δ we use here is not the same δ_L used by Lorenzini, but we have that, for any finite abelian group H, $\delta(H) \leq \delta_L(H)$.

The fact that $[L : K]\phi_A = 0$ is proved in [5], Theorem 1. Since the Néron model is stable by étale base change over R and by completions, we have that

$\phi_A = \phi_{\bar{A}}$, where we denote by $\bar{A}$ the extension of scalars of A into $\bar{K}$, a non-ramified closure of the completion of A with respect to the valuation. But then I is isomorphic to the whole Galois group $\mathrm{Gal}(\bar{L}/\bar{K})$, and hence $|I| = [\bar{L} : \bar{K}]$ kills the group ϕ_A. $\square$

The following proposition was originally proved in [14] in the case that the residue field is finite; a proof in the general case can be find in [13] (see also [17]).

Proposition 13 (Serre-Tate, Oort) *Let A be an abelian variety over a discrete valuation field K with potentially good reduction. Then, if $\ell > 2g+1$ is prime different from p, A attains good reduction over $K(A[\ell])$ (the field of rationality of the points of $A[\ell] := A(\bar{K})[\ell]$).*

Proposition 14 (Serre-Tate) *Let A be an abelian variety with potentially good reduction over a field K complete with respect to a discrete valuation and with finite residue field. Let $\ell \geq 3$ be a prime number different from p. Then, the inertia subgroup I of the extension $K(A[\ell])/K$ is independent of ℓ.*

Using this last proposition, we can find a bound for the order of the inertia subgroup of $K(A[\ell])/K$.

Notation 15 *Let g be an integer greater or equal than 1. We denote by $\eta(g) = \prod_{q\ prime} q^{\mu(g,q)}$, where*

$$\mu(g,2) := \left[\frac{g}{2}\right] + \sum_{j=0}^{\infty} \left[\frac{g}{2^j}\right] \text{ , and } \mu(g,q) := \sum_{i=0}^{\infty} \left[\frac{g}{q^i(q-1)}\right] \text{ if } q \text{ is odd.}$$

The following result is classical and essentially due to Minkowsky.

Theorem 16 (Minkowsky) *Let G be a finite group such that there exists constants $g = g(G)$ and $c(G)$, both integers, and monomorphisms of groups*

$$\phi_\ell : G \to GL_g(\mathbb{F}_\ell)$$

for every prime $\ell \geq c(G)$. Then, the order of G divides $\eta(g)$. Thus, if p is a prime number that divides the order of G, then $p \leq g+1$.

Corollary 17 *Let A be an abelian variety with potentially good reduction over a field K complete with respect to a discrete valuation and with finite residue field. Let $\ell \geq 3$ be a prime number different from p. Then, the inertia group I of the extension $K(A[\ell])/K$ has order dividing $\eta(2g)$. Moreover, if u is the dimension of the unipotent part of the reduction of the Néron model of A, then I has order dividing $\eta(2u)$.*

Proof. The first fact is consequence of the Proposition 2.4 and the Theorem 2.4, since the Galois group of $K(A[\ell])$ over K is a subgroup of $GL_{2g}(\mathbb{F}_\ell)$. To check the second fact, we replace K by a unramified closure K^{nr}. Then I

corresponds to the Galois group of $K^{nr}(A[\ell])/K^{nr}$. Now, consider the subgroup of fixed elements in $A[\ell]$ with respect to the action of I, that is the points $A(K^{nr})[\ell]$. Applying Proposition 1.2, if ℓ is different from p, this points are in bijection with the points of order ℓ in the reduction of the Néron model of $A_{K^{nr}}$. By Theorem 2.2, if $\ell > 2g+1$ and it is different from p, the group of components $\phi(k)$ has no non-trivial elements of order ℓ. Therefore, the group $A(K^{nr})[\ell]$ became isomorphic to the group of ℓ-torsion points in the abelian part of the reduction of the Néron model of $A_{K^{nr}}$. Since the residual field of K^{nr} is algebraically closed, there are as much as ℓ^{2a} of such a points, where a denotes the dimension of this abelian part. Now, the Galois group of $K^{nr}(A[\ell])/K^{nr}$ is a subgroup of the group of automorphisms of the quotient $A[\ell]/A(K^{nr})[\ell]$, which, for $\ell > \max\{2g+1, p\}$, is an $\mathbb{F}_\ell$-vector space of rank $2g-2a=2u$. The result is now a consequence of the Theorem 2.6. $\square$

Notation 18 *Let A be an abelian variety over K, and let $\mathcal{A}$ and $\mathcal{A}^0$ the Néron model and its connected component respectively. We denote by $A_0(K)$ the subgroup of $A(K)$ corresponding to $\mathcal{A}^0(R) \subset \mathcal{A}(R)$ via the natural isomorphism $A(K) \cong \mathcal{A}(R)$. We denote also by $A_1(K)$ the subgroup of $A_0(K)$ corresponding via the same isomorphism to the kernel of the reduction map $\mathcal{A}^0(R) \to \mathcal{A}^0_k(k)$ composed with the morphism $\mathcal{A}^0_k(k) \to B_k(k)$, where B_k denote the abelian part of $\mathcal{A}^0_k$. Finally, we will denote by $A_2(K)$ the subgroup of $A_1(K)$ corresponding to the kernel of the reduction map.*

Observe that $A(K)/A_0(K)$ injects into the group of components $\phi(k)$ of $\mathcal{A}(R)$, that $A_0(K)/A_1(K)$ injects into $B_k(k)$, that $A_1(K)/A_2(K) \cong U_k(k)$ and that, by Theorem 1.3, $A_2(K)$ is isomorphic to the group of points over R of the formal completion F_A of $\mathcal{A}^0$ along the zero section.

This subgroups are functorial in the variable K. That is, if we consider a finite extension L/K, there exists natural morphisms from $A_i(K)$ to $A_i(L)$ for $i=0,1,2$. This fact is an easy consequence of the fact that there are no non-trivial morphisms from a connected group scheme to an étale group scheme, or from a unipotent group scheme to an abelian variety.

Theorem 19 *Let A be an abelian variety of dimension g with potentially good reduction over K. Denote by a and u the dimension of the abelian part and the unipotent part of the reduction of A respectively. Then, there exists a constant $C(a,u,e,f,p)$, depending on a, u, e, f and p, such that $|A(K)_{tors}| \leq C(a,u,e,f,p)$. Explicitly, we can take*

$$C(a,u,e,f,p) = (1+p^{\frac{f}{2}})^{2a} p^{fu} 2^{2u} p^{2(a+u)\gamma'_p(u,e)},$$

where we define $\gamma'_p(u,e) = \max\{\gamma_p(e), \mu(2u,p))\}$, or we can take

$$C(a,u,e,f,p) = (1+p^{\frac{f}{2}})^{2a} 2^{2u} 2^{2(a+u)} e^{2(a+u)} \eta(2u)^{2(a+u)}.$$

Proof. To prove the first bound, consider the inequality

$$\begin{aligned} |A(K)_{tors}| \leq \quad & |A_2(K)^{(p)}_{tors}| \cdot |A_1(K)/A_2(K)| \cdot |A_0(K)/A_1(K)| \cdot \\ \cdot |\,(A(K)/A_0(K))^{(p)}\,| \quad & p^{2g \max\{\exp_p(A_2(K)_p), exp_p((A(K)/A_0(K))_p)\}}, \end{aligned}$$

where, for every abelian p-group X, we denote by $\exp_p(X)$ the smallest integer r such that p^r kills X. We have that

$$|A_0(K)/A_1(K)| \leq |B_k(k)| \leq (1+p^{\frac{f}{2}})^{2a} \text{ and } |A_1(K)/A_2(K)| \leq |U_k(k)| \leq p^{fu}$$

by the Theorem 1.1. By Proposition 1.2, the group $A_2(K)_{tors}$ has order a power of p, hence the first factor on the inequality is trivial. On the other hand, by Corollary 1.8,

$$\exp_p(A_2(K)_p) = \exp_p(F_A(R)_p) \leq \gamma_p(e).$$

Let $\ell > 2g+1$ be a prime number different from p and consider $L = K(A[\ell])$. By Theorem 2.2,

$$|\,(A(K)/A_0(K))^{(p)}\,| \leq |\phi_A^{(p)}| \leq 2^{2u},$$

and, by Proposition 2.3, $(\phi_A)_p$ is killed by the order of the inertia group of the extension $[L:K]$. Corollary 2.7 implies then that

$$\exp_p((A(K)/A_0(K))_p) \leq \mathrm{ord}_p(\eta(2u)) = \mu(2u,p),$$

which finishes the prove of the first bound.

To show the second bound, consider the inequality

$$\begin{aligned} |A(K)_{tors}| \leq \quad & |A_1(K)_{tors}^{(p)}| \cdot |A_0(K)/A_1(K)| \cdot |\,(A(K)/A_0(K))^{(p)}\,| \\ & p^{2g\max\left\{\exp_p(A_1(K)_p), exp_p((A(K)/A_0(K))_p)\right\}}. \end{aligned}$$

Observe that the torsion points of $A_1(K)$ have all order a power of p, hence the first factor of the inequality is 1.

The abelian variety A_L over L has good reduction, thus $A_2(L) = A_1(L)$ and hence it is isomorphic to the points of a g-dimensional commutative formal Lie group over the ring of integers of L. Now, applying Corollary 1.8, we have that $p^{\gamma_p(e_L)}$ annihilates the torsion part of $A_1(L)_{tors}$, where e_L denotes the absolute ramification index of L. Using the Corollary 2.7 we can deduce that e_L divides $e\eta(2u)$. Since the natural map from $A_1(K)$ to $A_1(L)$ is injective, we get that $p^{\gamma_p(e\eta(2u))}$ annihilates $A_1(K)_{tors}$.

To end the proof, observe that

$$\begin{aligned} \mu(2u,p) = \mathrm{ord}_p(\eta(2u)) \leq \quad & [\log_p(\eta(2u))] \leq [\log_p(\tfrac{p\eta(2u)}{p-1})] \\ & \leq \gamma_p(\eta(2u)) \leq \gamma_p(e\eta(2u)), \end{aligned}$$

therefore we have

$$\max\left\{\exp_p(A_1(K)_p), exp_p((A(K)/A_0(K))_p)\right\} \leq \gamma_g(e\eta(2u)).$$

Now, applying the bound $p^{2g\gamma_p(e\eta(2u))} \leq 2^{2g}e^{2g}\eta(2u)^{2g}$ we are done. $\square$

Considering the first constant, we can find a bound depending on the dimension of A.

Corollary 20 *With the same hypothesis as in the Theorem 2.9, we have*

$$|A(K)_{tors}| \leq 2^{2g} p^{fg} p^{2g\gamma'_p(g,e)}.$$

Proof. We need only to observe that $g = a + u$ since A has potentially good reduction and that $1 + p^{\frac{f}{2}} \leq 2p^{\frac{f}{2}}$. $\square$

Remark 21 *Its seems that the same bound for the prime-to-p part of the ϕ in the potentially good reduction case could be true for the whole group. In this case, the same bounds that we obtain in the results above would remain true if we replace the function γ'_p by the function γ_p.*

Considering the second constant in the Theorem 2.9, we show that in the case of totally unipotent reduction there exists a bound independent of p.

Corollary 22 *With the same hypothesis as in Theorem 2.9, suppose, in addition, that A has totally unipotent reduction, i.e. $a = 0$. Then $|A(K)_{tors}|$ can be bounded by a constant depending only on e and g. Explicitly, we have*

$$|A(K)_{tors}| \leq 2^{4g} e^{2g} \eta(2g)^{2g}.$$

Remark 23 *In dimension 1 this latter bound give us the number $2^{12}3^2e^2$. It is shown in [6, Proposition 3] that we can find as an upper bound in dimension 1 the number $48e$. In view of this result, the bound computed in the last corollary can be improve. In fact, it seems that for every g there should exists a bound depending on e^g, and not on e^{2g}.*

Finally, if p is sufficiently large, we can show a bound independent of e. In fact, if $e < p - 1$, then $\gamma_p(e) = 0$, and if $2u < p - 1$, then $\mu(2u, p) = 0$. Thus, under the two conditions we have $\gamma'_p(u, e) = 0$.

Corollary 24 *With the same hypothesis as in Theorem 2.9, if* $\max\{e, 2u\} <$ $p - 1$*, then*

$$|A(K)_{tors}| \leq (1 + p^{\frac{f}{2}})^{2a} 2^{2u} p^{fu} \leq 2^{2g} p^{fg}$$

3 Torsion points of algebraic tori

In this section we bound the order of the subgroup of torsion points in $T(K)$, where T is an algebraic torus of dimension g and K is a finite extension of $\mathbb{Q}_p$. Let the notations be as in the preceding section. We denote also by G the absolute Galois group of K, and by X the character group of T, which is a torsion-free G-module of rank g as abelian group.

The following result is an easy consequence of the Corollary 2.12 in [20], where is showed that the torsion part of the group of components of the Néron model of an algebraic torus T is naturally isomorphic to the Pontrjagin dual of the first cohomology group of X by the action of the inertia.

Proposition 25 *Let T be a torus over K of dimension g. Then the group of torsion points fits into an exact sequence*

$$0 \to \mathcal{T}^0(R)_{tors} \to T(K)_{tors} \to (H^1(I,X)^*) \to \cdots$$

where $\mathcal{T}^0$ is the connected component of the Néron model of T, I is the inertia subgroup of G and the star means Pontrjagin dual.

Thus, we only need to find a bound for the number of torsion points in $\mathcal{T}^0(R)$ and the order of $H^1(I,X)$. Since $\mathcal{T}^0$ is geometrically connected and smooth of dimension g we can apply the Corollary 1.10 which gives us a bound for the number of elements in $\mathcal{T}^0(R)$ in terms of g, e, f and p. We will show in the next Lemma that $H^1(I,X)$ has cardinal bounded in terms only of g.

Lemma 26 *If G is a solvable profinite group and M a G module, which is torsion free and of finite rank g as abelian group. We have then that*

$$\delta(H^1(G,M)) \le g - rank(M^G) =: r.$$

Proof. Let ρ be the morphism from G into $GL_g(\mathbb{Z})$ given by the action on M. Let H be the kernel of ρ. Since G is profinite, the image G/H of ρ is finite, and, since H acts trivially on M, we have $H^1(H,M) \cong \mathrm{Hom}(H,M) = 0$ because M has no non-trivial profinite subgroups. Thus we have $H^1(G,M) = H^1(G/H,M)$ and we can suppose that G is finite and solvable. Moreover, observe that we can suppose that M has no non-trivial G-invariants, since we have a natural injection $H^1(G,M) \hookrightarrow H^1(G,M/M^G)$, and M/M^G is torsion free of rank r.

Suppose first that G is of order prime equal to p. Let σ be a generator of G. Now, since $\sigma^p = id$ and there are no σ-invariants, the minimal polynomial of σ (as automorphism of $M \otimes \mathbb{Q}$) needs to be the cyclotomic polynomial $\Phi_p(x)$. Therefore the characteristic polynomial of σ will be $\Phi_p(x)^k$ and the rank of M will be $k(p-1)$ for some $k \ge 1$. We will show that $H^1(G,M) \cong (\mathbb{Z}/p\mathbb{Z}^k$. Since p annihilates $H^1(G,M)$, it is sufficient to show that the order of this group is p^k. On the other hand, since G is cyclic, $H^1(G,M)$ is isomorphic to the coinvariants group $M/(\sigma - Id)M$. Thus, the order of this group is the determinant of $\sigma - Id$, which is equal to the characteristic polynomial evaluated at 1. The result is hence deduced from the fact $\Phi_p(1) = p$.

Finally, suppose that G is any finite solvable group. Since G is solvable, there always exists a non-trivial normal subgroup H of G. We can easily see that the function δ is additive, that is, for every exact sequence of finite abelian groups $0 \to A \to B \to C \to 0$, we have that $\delta(B) = \delta(A) + \delta(C)$. Consider the following exact sequence

$$0 \to H^1(G/H, M^H) \to H^1(G,M) \to H^1(H,M) \to \dots.$$

By induction on the number of prime factors in the order of G,

$$\begin{aligned}\delta(H^1(G,M)) \le \quad & \delta(H^1(G/H,M^H)) + \delta(H^1(H,M)) \le \\ \le \quad & \mathrm{rank}(M^H) - \mathrm{rank}(M^G) + \mathrm{rank}(M) - \mathrm{rank}(M^H),\end{aligned}$$

hence the result follows. $\square$

Using this results with the results of section 1 we have that the group of torsion points on $T(K)$ is uniformly bounded. Recall that the Néron model of a torus is always affine since T is, therefore the abelian rank of its reduction is always trivial.

Theorem 27 *Let K be a finite extension of $\mathbb{Q}_p$ with ramification index e and decomposition degree f. Let T be a torus over K of dimension g. Then*

$$|T(K)_{tors}| \leq (p^f + 1)^t p^{fu} 2^u p^{g\gamma_p(e)}$$

where t and u are the dimension of the toric part and unipotent part of the reduction of T respectively. Moreover, we have also that

$$|T(K)_{tors}| \leq (p^f + 1)^t 2^u e^g \eta(u)^g.$$

Proof. First of all, observe that X^I, the group of characters fixed by the inertia subgroup, has rank equal to t. In fact, the character group $\bar{X}$ of the toric part of $\mathcal{T}_k^0$, the reduction of the connected component of the Néron model of T, is equal to the maximal free quotient of X with trivial action of I. This group is isomorphic to the (linear) dual of the subgroup of fixed elements by I in the linear dual of X, and the last one is isogenous (as a G-module) to X. Thus, X^I is isogenous to $\bar{X}$, hence they have the same rank.

Now, by Lemma 3.2, and since I is pro-solvable, the group $H^1(I, X)$ has order less or equal to $2^{g-t} = 2^u$. By using Proposition 3.1, the first bound is now a direct application of Corollary 1.10.

To check the second bound, we first find a bound for the ramification index of the splitting field L of T. This field is in fact the fixed field with respect to the kernel of the integral representation of G associated with the character group X. If we denote by J the inertia subgroup of the Galois group of L/K, then J is naturally a subgroup of the group of automorphisms as abelian group of X/X^J, which is a torsion-free abelian group of rank u. Therefore, J can be viewed as a finite subgroup of $GL_u(\mathbb{Z})$. By Theorem 2.6, since the kernel of the natural morphism $GL_u(\mathbb{Z}) \to GL_u(\mathbb{F}_\ell)$ is torsion-free for every $\ell \geq 3$, we have that J has order dividing $\eta(u)$. Hence, the absolute ramification index of L divides $e\eta(u)$. Now, let $T_1(K)$ be the subgroup of $T(K)_{tors}$ corresponding via the isomorphism $T(K) \cong \mathcal{T}(R)$ to $\mathcal{T}^0(R)_p$. We have that

$$|T(K)_{tors}/T_1(K)| \leq |\mathcal{T}^0(R)^{(p)}||H^1(I, X)| \leq (p^f + 1)^t 2^u.$$

by using Theorem 1.1 and Proposition 1.2. On the other hand, $T(K)_1 \hookrightarrow T(L)_1$, and, since T_L has multiplicative reduction, the subgroup $T_1(L)$ is isomorphic to the kernel of the reduction map in the Néron model of T_L. Now, by Theorem 1.3 this is isomorphic to the points of a formal Lie group over the ring of integers of L, which by Corollary 1.8 is killed by $p^{\gamma_p(e\eta(u))}$. Since the torsion subgroup of any torus has always g or less generators, we have that $T_1(L) \leq e^g \eta(u)^g$ using the same argument that in Corollary 1.9. $\square$

4 Torsion points of abelian varieties

The aim of this section is to find uniform bounds for the number of torsion points on any abelian variety A over a local number field K, under the condition that its reduction does not contains $\mathbb{G}_m$, or, equivalently, that the split toric part of its reduction is trivial. The main idea is to apply the theory of rigid uniformization of abelian varieties to reduce this problem to those of early sections.

The following theorem is a consequence of the rigid uniformization theory (cf. [3]).

Theorem 28 *Let A be an abelian variety over a field K complete with respect to a discrete valuation. Denote by G the absolute Galois group of K. Then, there exists a semi-abelian variety E of dimension g defined over K, which has abelian part with potentially good reduction, a G-module M such that as abelian group is torsion-free and of finite rank equal to the dimension of the toric part of E, and an exact sequence of G-modules*

$$0 \to M \to E(K^s) \to A(K^s) \to 0,$$

where K^s denotes a separable closure of K. Moreover, the abelian group M^G is torsion free and of finite rank m equal to the dimension of the split toric part of $\mathcal{A}_k^0$. Moreover, for every extension L/K the special fiber of identity component of the Néron models of A_L and E_L are isomorphic.

Proof. The exact sequence is a consequence of the Theorem 1.2. in [3]; see also the Remark 4.6 in [3]. In this results it is proved that an exact sequence as in the Theorem exists if we consider M, E and A as sheaves for the rigid smooth topology. As M is étale as rigid group, the sequence remain exact when restricted to the rigid étale topology. But a sequence of rigid groups over a field K is exact for the étale topology if and only if the sequence of points over a separable closure of K is exact.

The same argument used in the proof of Proposition 5.1 in [3] shows that the rank of M^G is equal to the rank of X^G, where X denotes the character group of the toric part of E. Now, it is easy to see, using the fact that the abelian part of E has potentially good reduction, that the rank of X^G is equal to the dimension of the split toric part of the reduction of Néron model of E.

The proof is finished using the fact that the reduction of the connected component of the Néron model of E and A are isomorphic ([3, Theorem 2.3]), and that the uniformization is stable with respect to base change by construction. $\square$

We will call E the uniformization space of A, and the complex $M \to E$ the uniformization of A. They are unique modulo isomorphisms (see for example [1, Proposition 6.10] in the case that A has semi-stable reduction and then apply a Galois descent argument).

By using this Theorem we can relate the torsion points of abelian varieties to the torsion points of tori and of abelian varieties potentially of good reduction, and to elements in $H^1(G,M)$, just taking the cohomology sequence associated to the exact sequence in the Theorem 4.1 to obtain the sequence

$$0 \to M^G \to E(K) \xrightarrow{\rho} A(K) \to H^1(G,M) \to \ldots.$$

We define

$$A(K)^{(1)}_{tors} := \mathrm{Im}\ (\rho) \cap A(K)_{tors} \text{ and } A(K)^{(2)}_{tors} := \rho(E(K)_{tors}).$$

Since the uniformization of A is functorial, that is any morphism between two abelian varieties A and A' lifts to a morphism between its uniformization spaces E and E', this groups are functorial in the variables A and K.

Since we have found uniform bounds for the number of torsion points in E, and since by Lemma 3.2 we know that $H^1(G,M)$ is bounded in terms of the rank of M, the only torsion points of A which can have order not uniformly bounded are those in the image of ρ but not in $A(K)^{(2)}_{tors}$.

Theorem 29 *Let A be an abelian variety of dimension g over a local number field K with absolute ramification index e and absolute decomposition index f. Denote by a, u, t and m the dimensions of the abelian part, unipotent part, toric part and split toric part of the reduction of the Néron model of A, respectively. Denote also by a' and t' the dimensions of the abelian part and toric part of the Néron model of A_L respectively, where L/K is any extension where A attains semiabelian reduction. Then, there exists a canonical filtration:*

$$0 \subset A(K)^{(2)}_{tors} \subset A(K)^{(1)}_{tors} \subset A(K)_{tors},$$

of finite abelian groups, functorial on the variables A and K, such that:

1. *The subgroup $A(K)^{(2)}_{tors}$ has cardinality bounded above by*

$$(1+p^{\frac{f}{2}})^{2a}(p^f+1)^t(2p^f)^u 2^{(a'-a)} p^{2a'\gamma'_p(a'-a,e)} p^{t'\gamma_p(e)}.$$

2. *The quotient $\Gamma_K(A) := A(K)^{(1)}_{tors}/A(K)^{(2)}_{tors}$ can be generated by at most m elements.*

3. *The quotient $A(K)_{tors}/A(K)^{(1)}_{tors}$ has order bounded above by $2^{(t'-m)}$.*

Proof. Since the uniformization space E is an extension of an abelian variety B of potentially good reduction of dimension a' by a torus T of dimension t', by using Theorem 2.9 and Theorem 3.3 we have

$$\begin{aligned} |E(K)_{tors}| \leq & \quad |B(K)_{tors}||T(K)_{tors}| \leq \\ \leq & \quad (1+p^{\frac{f}{2}})^{2a} 2^{2v} p^{fv} p^{2(a+v)\gamma'_p(v,e)} (p^f+1)^t 2^{v'} p^{fv'} p^{t'\gamma_p(e)}, \end{aligned}$$

where we denote by v and v' the dimensions of the unipotent part of the reduction of the Néron model of B and of T respectively, and we have used the last statement of Theorem 4.1. Now, since $a' = a + v$, $t' = t + v'$ and $u = v + v'$, we obtain the first result.

Denote by $E(K)^{(1)}$ the subgroup of elements x in $E(K)$ such that there exists an integer $n \neq 0$ with $nx \in M^G$. Then

$$A(K)^{(1)}_{tors} \cong E(K)^{(1)}/M^G \text{ and } \Gamma_K(A) \cong \left(E(K)^{(1)}/E(K)_{tors}\right)/M^G.$$

Since $\Gamma_K(A)$ is finite and $E(K)^{(1)}/E(K)_{tors}$ is torsion free, we have that the rank of this last group is equal to the rank of M^G, which by Theorem 4.1 is equal to m. Thus $\Gamma_K(A)$ is a quotient of a torsion free rank m abelian group, hence it has m generators or less.

The last statement is deduced from Lemma 3.2. $\square$

This result implies that if $m = 0$, then the number of torsion points in $A(K)$ is bounded, depending on the dimension of A, the absolute degree of K and the prime p.

Corollary 30 *Let A be an abelian variety over a local number field K. If the reduction of A does not contain $\mathbb{G}_m$, then $|A(K)_{tors}|$ is bounded above by a constant depending only on $g := \dim(A)$, e, f and p. Explicitly, we can take*

$$B(g,e,f,p) = 2^{2g} p^{fg} p^{2g\gamma'_p(g,e)}.$$

Proof. To deduce the bounds from the theorem above, use the following inequalities

$$p^{2a'\gamma'_p(a'-a,e)+t'\gamma_p(e)} 2^t \leq p^{2g\gamma'_p(g,e)},$$

and

$$(1 + p^{\frac{f}{2}})^{2a}(p+1)^t p^{fu} 2^{2u} \leq 2^{2g} p^{fg}. \qquad \square$$

In the case that A has reduction totally unipotent, we find a bound depending only on e and g, just applying the Corollary 2.12, the second formula of the Theorem 3.3 with $t = 0$ and the Lemma 3.2 as in the proof of Theorem 4.2.

Proposition 31 *Let A be an abelian variety over a local number field which has totally unipotent reduction. Denote by a' and t' the dimension of the abelian part and toric part of the semiabelian reduction of A. Then*

$$|A(K)_{tors}| \leq 2^{4a'+2t'} e^{2a'+t'} \eta(2a')^{2a'} \eta(t')^{t'},$$

and, therefore, if we denote by $g' = 2a' + t'$, then

$$|A(K)_{tors}| \leq 2^{2g'} e^{g'} \eta(g')^{g'} \leq 2^{2g} e^g \eta(g)^g.$$

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