A CONTINUOUS RATING METHOD FOR PREFERENTIAL VOTING

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Abstract

A method is given for quantitatively rating the social acceptance of different options which are the matter of a preferential vote. The proposed method is proved to satisfy certain desirable conditions, among which there is a majority principle, a property of clone consistency, and the continuity of the rates with respect to the data. One can view this method as a quantitative complement for a qualitative method introduced in 1997 by Markus Schulze. It is also related to certain methods of one-dimensional scaling or cluster analysis.

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The outcome of a vote is often expected to entail a quantitative rating of the candidate options according to their social acceptance. Some voting methods are directly based upon such a rating. This is the case when each voter is asked to choose one option and each option is rated by the fraction of the vote in its favour. The resulting rates can be used for filling a single seat (first past the post) or for distributing a number of them (proportional representation). A more elaborate voting method based upon quantitative rates was introduced in 1433 by Nikolaus von Kues [*22* : § 1.4.3, § 4] and again in 1770–1784 by Jean-Charles de Borda [*22* : § 1.5.2, § 5]. Here, each voter is asked to rank the different options in order of preference and each option is rated by the average of its ranks, i. e. the ordinal numbers that give its position in these different rankings (this formulation differs from the traditional one by a linear function). For future reference in this paper, these two rating methods will be called respectively the method of **first-choice fractions** and the method of **average ranks**.

However, both of these methods have important drawbacks, which leads to the point of view of **paired comparisons**, where each option is confronted with every other by counting how many voters prefer the former to the latter and vice versa. From this point of view it is quite natural to abide by the so-called Condorcet principle: an option should be deemed the winner whenever it defeats every other one in this sort of tournament. This approach was introduced as early as in the thirteenth century by Ramon Llull [*22* : § 1.4.2, § 3], and later on it was propounded again by the marquis of Condorcet in 1785–1794 [*22* : § 1.5.4, § 7], and by Charles Dodgson, alias Lewis Carroll, in 1873–1876 [*2*; *22* : § 12]. Its development gives rise to a variety of methods, some of them with remarkably good properties. This is particularly the case of the method of **ranked pairs**, proposed in 1986/87 by Thomas M. Zavist and T. Nicolaus Tideman [*36, 39*], and the method introduced in 1997 by Markus Schulze [*33, 34*], which we will refer to as the method of **paths**. In spite of the fact that generally speaking they can produce different results, both of them comply with the Condorcet principle and they share the remarkable property of clone consistency [*37, 33*].

Nevertheless, these methods do not immediately give a quantitative rating of the candidate options. Instead, they are defined only as algorithms for determining a winner or at most a purely ordinal ranking. On the other hand, they are still based upon the quantitative information provided by the table of paired-comparison scores, which raises the question of whether their qualitative results can be consistently converted into quantitative ratings.

In [*25* : § 10] a quantitative rating algorithm was devised with the aim of complementing the method of ranked pairs. Although a strong evidence was given for its fulfilling certain desirable conditions —like the ones stated below—, it was also pointed out that it fails a most natural one, namely that the output, i. e. the rating, be a continuous function of the input, i. e. the frequency of each possible content of an individual vote. In fact, such a lack of continuity seems unavoidable when the method of ranked pairs is considered and those other conditions are imposed. In contrast, in this paper we will see that the method of paths does admit such a continuous rating procedure.

Our method can be viewed as a projection of the matrix of paired-comparison scores onto a special set of such matrices. This projection is combined with a subsequent application of two standard rating methods, one of which the method of average ranks. The overall idea has some points in common with [*31*].

We will refer to the method described in this paper as the **CLC rating method**, where the capital letters stand for “Continuous Llull Condorcet”.

The paper is organized as follows: In section 1 we state the problem which is to be solved and we make some general remarks. Section 2 presents an heuristic outline of the proposed method. Section 3 gives a summary of the procedure, after which certain variants are introduced. Section 4 presents some illustrative examples. Finally, sections 5–18 give detailed mathematical proofs of the claimed properties for the main variant.

The reader interested to try the CLC method can make use of the tool which is available at `http://mat.uab.cat/~xmora/CLC_calculator/`.

1 Statement of the problem and general remarks

1.1 We consider a set of N options which are the matter of a vote. Although more general cases will be included later on (§ 3.3), for the moment we assume that each voter expresses his preferences in the form of a ranking; by it we mean an ordering of the options in question by decreasing degree of preference, with the possibility of ties and/or truncation (i. e. expressing a top segment only). We want to aggregate these individual preferences into a *social rating*, where each option is assigned a rate that quantifies its social acceptance.

In some places we will restrict our attention to the case of complete votes. For ranking votes, we are in such a situation whenever we are dealing with non-truncated rankings. As we will see, the incomplete case will give us much more work than the complete one.

We will consider two kinds of ratings, which will be referred to respectively as rank-like ratings and fraction-like ones. As it is suggested by these names, a rank-like rating will be reminiscent of a ranking, whereas a fraction-like one will evoke the notion of proportional representation. Our method will produce both a rank-like rating and a fraction-like one. They will agree with each other in the ordering of the candidate options, except that the ordering implied by the fraction-like rating may be restricted to a top segment of the other one. Quantitatively speaking, the two ratings have different meanings. In particular, the fraction-like rates can be viewed as an estimate of the first-choice fractions based not only on the first choices of the voters, but also on the whole set of preferences expressed by them. In contrast, the rank-like rates are not focused on choosing, but they aim simply at positioning all the candidate options on a certain scale.

More specifically, the two ratings are asked to satisfy the following conditions:

A *Scale invariance.* The rates depend only on the relative frequency of each possible content of an individual vote. In other words, if every individual vote is replaced by a fixed number of copies of it, the rates remain exactly the same.

B *Permutation equivariance.* Applying a certain permutation of the options to all of the individual votes has no other effect than getting the same permutation in the social rating.

C *Continuity.* The rates depend continuously on the relative frequency of each possible content of an individual vote.

The next conditions apply to the rank-like rating:

D *Rank-like range.* Each rank-like rate is a number, integer or fractional, between 1 and N. The best possible value is 1 and the worst possible one is N.

E *Rank-like decomposition.* Let us restrict the attention to the complete case. Consider a splitting of the options into a 'top class' X plus a 'low class' Y. Assume that all of the voters have put each member of X above every member of Y. In that case, and only in that case, the rank-like rates can be obtained separately for each of these two classes according to the corresponding restriction of the ranking votes (with the proviso that the unassembled low-class rates differ from the assembled ones by the number of top-class members).

In its turn, the fraction-like rating is required to satisfy the following conditions:

F *Fraction-like character.* Each fraction-like rate is a number greater than or equal to 0. Their sum is equal to a fixed value. More specifically, we will take this value to be the participation fraction, i. e. the fraction of non-empty votes.

G *Fraction-like decomposition.* Consider the same situation as in E with the additional assumption that there is no proper subset of X with the same splitting property as X (namely, that all voters have put each option from that set above every one outside it). In that case, and only in that case, the top-class fraction-like rates are all of them positive and they can be obtained according to the corresponding restriction of the ranking votes, whereas the low-class fraction-like rates are all of them equal to 0.

H *Case of plumping votes.* Assume that each voter plumps for a single option. In that case, the fraction-like rates coincide with the fractions of the vote obtained by each option.

Furthermore, we ask for some properties that concern only the concomitant social ranking, i. e. the purely ordinal information contained in the social rating:

I *Majority principle.* Consider a splitting of the options into a 'top class' X plus a 'low class' Y. Assume that for each member of X and every member of Y there are more than half of the individual votes where the former is preferred to the latter. In that case, the social ranking also prefers each member of X to every member of Y.

J *Clone consistency.* A set C of options is said to be a *cluster* (of *clones*) for a given ranking when each element from outside C compares with all elements of C in the same way (i. e. either it lies above all of them, or it lies below all of them, or it ties with all of them). In this connection, it is required that if a set of options is a cluster for each of the individual votes, then: (a) it is a cluster for the social ranking; and (b) contracting it to a single option in all of the individual votes has no other effect in the social ranking than getting the same contraction.

1.2 Let us emphasize that the individual votes that we are dealing with do not have a quantitative character (at least for the moment): each voter is allowed to express a preference for x rather than y, or vice versa, or maybe a tie between them, but he is not allowed to quantify such a preference.

This contrasts with 'range voting' methods, where each individual vote is already a quantitative rating [*35, 1*]. Such methods are free from many of the difficulties that lurk behind the present setting. However, they make sense only as long as all voters mean the same by each possible value of the rating variable. This hypothesis may be reasonable in some cases, but quite often it is hardly applicable (a typical symptom of its not being appropriate is a concentration of the rates in a small set independently of which particular options are under consideration). In such cases, it is quite natural that the individual votes express only qualitative comparisons between pairs of options. If the issue is not too complicated, one can expect these comparisons to form a ranking. In the own words of [*1a*], "When there is no common language, a judge's only meaningful input is the order of his grades". Certainly, the judges will agree upon the qualitative comparison between two options much more often than they will agree upon their respective rates in a certain scale. Such a lack of quantitative agreement may be due to truly different opinions; but quite often it is rather meaningless. Of course, the rates will coincide more easily if a discrete scale of few grades is used. But then it may happen that the judges rate equally two options about which they all share a definite preference for one over the other, in which case these discrete rates are throwing away genuine information. Anyway, voting is often used in connection with moral, psychological or aesthetic qualities, whose appreciation may be as little quantifiable, but also as much "comparable", as, for instance, the feelings of pleasure or pain.

So, in our case the quantitative character of the output is not present in the individual votes (unless we adopt the general setting considered at the end of § 3.3), but it derives from the fact of having a number of them. The larger this number, the more meaningful is the quantitative character of the social rating. This is especially applicable to the continuity property C, according to which a small variation in the proportion of votes with a given content produces only small variations in the rates. In fact, if all individual votes have the same weight, a few votes will be a small proportion only in the measure that the total number of votes is large enough.

In this connection, it should be noticed that property C differs from the continuity property adopted in [*1*] (axiom 6), which does not refer to small variations in the proportion of votes with a given content, but to small variations in the quantitative content of each individual vote. In the general setting considered at the end of § 3.3, the CLC method satisfies not only the continuity property C, but also the axiom 6 of [*1*]; in contrast, the "majority-grade" method considered in [*1*] satisfies the latter but not the former.

1.3 One can easily see that the method of average ranks satisfies conditions A–E. In principle that method assumes that all of the individual votes are complete rankings; however, one can extend it to the general case of rankings with ties and/or truncation while keeping those conditions (it suffices to use formula (6) of § 2.5). In their turn, the first-choice fractions are easily seen to satisfy conditions A–C and F–H. However, neither of these two methods satisfies conditions I and J. In fact, these conditions were introduced precisely as particularly desirable properties that are not satisfied by those methods [*22, 2, 37*].

Of course, one can go for a particular ranking method that satisfies conditions I and J and then look for an appropriate algorithm to convert the ranking result into the desired rating according to the quantitative information coming from the vote. But this should be done in such a way that the final rating be always in agreement with the ranking method as well as in compliance with conditions A–H, which is not so easy to achieve. From this point of view, our proposal can be viewed as providing such a complement for one of the variants of the method of paths [*33*, *34*].

1.4 When the set X consists of a single option, the majority principle I takes the following form:

I1 *Majority principle, winner form.* If an option x has the property that for every $y \neq x$ there are more than half of the individual votes where x is preferred to y, then x is the social winner.

In the complete case the preceding condition is equivalent to the following one:

I1′ *Condorcet principle.* If an option x has the property that for every $y \neq x$ there are more individual votes where x is preferred to y than vice versa, then x is the social winner.

However, we want to admit the possibility of individual votes where no information is given about certain pairs of options. For instance, in the case of a truncated ranking it makes sense to interpret that there is no information about two particular options which are not present in the list. In that case condition I1 is weaker than I1′, and the CLC method will satisfy only the weaker version.

This lack of compliance with the Condorcet principle and its being replaced by a weaker condition may be considered undesirable. However, other authors have already remarked that such a weakening of the Condorcet principle is necessary in order to be able to keep other properties [*38*] (see also [*17*]). In our case, Condorcet principle seems to conflict with the continuity property C (see § 3.3). On the other hand, the Condorcet principle was originally proposed in connection with the complete case [*22*], its generalization in the form I1′ instead of I1 being due to later authors. Even so, nowadays it is a common practice to refer to I1′ by the name of "Condorcet principle'.

1.5 As we mentioned in the preceding subsection, we want to admit the possibility of individual votes where no information is given about certain pairs of options. In this connection, the CLC method will carefully distinguish a *definite indifference* about two or more options from a *lack of information* about them (see [*12*]). For instance, if all of the individual votes are complete rankings but they balance into an exact social indifference —in particular if each individual vote expresses such a complete indifference—, the resulting rank-like rates will be all of them equal to $(N{+}1)/2$ and the corresponding fraction-like rates will be equal to $1/N$. In contrast, in the case of a full abstention, i. e. where no voter has expressed any opinion, the rank-like rates will be all of them equal to N and the corresponding fraction-like rates will be equal to 0.

Although the decomposition conditions E and G have been stated only for the complete case, some partial results of that sort will hold under more general conditions. In particular, the following condition will be satisfied for general, possibly incomplete, ranking votes: The winner will be rated exactly 1 (in both the rank-like rating and the fraction-like one) if and *only if* all of the voters have put that option into first place.

Conditions E and G, as well as the preceding property, refer to cases where "all of the voters" proceed in a certain way. Of course, it should be clear whether we mean all of the "actual" voters or maybe all of the "potential" ones (i. e. actual voters plus abstainers). We assume that one has made a choice in that connection, thus defining a total number of voters V. Considering all potential voters instead of only the actual ones has no other effect than contracting the final rating towards the point where all rates take the minimal value (namely, N for rank-like rates and 0 for fraction-like ones).

1.6 It is interesting to look at the results of the CLC rating method when it is applied to the approval voting situation, i. e. the case where each voter gives only a list of approved options, without any expression of preference between them. In such a situation it is quite natural to rate each option by the number of received approvals; the resulting method has pretty good properties, not the least of which is its eminent simplicity [5].

Now, an individual vote of approval type can be viewed as a truncated ranking which ties up all of the options that appear in it. So it makes sense to apply the CLC rating method. Quite remarkably, one of its variants turns out to order the options in exactly the same way as the number of received approvals (see § 17). More specifically, the variant in question corresponds to interpreting that the non-approved options of an individual vote are tied to each other. However, the main variant, which acknowledges a lack of comparison between non-approved options, can lead to different results.

2 Heuristic outline

This section presents our proposal as the result of a quest for the desired properties. Hopefully, this will communicate the main ideas that lie behind the formulas.

2.1 The aim of complying with conditions I and J calls for the point of view of **paired comparisons**. In accordance with it, our procedure will be based upon considering every pair of options and counting how many voters prefer one to the other or vice versa. To that effect, we must adopt some rules for translating the ranking votes (possibly truncated or with ties) into binary preferences. In principle, these rules will be the following:

(a) When x and y are both in the list and x is ranked above y (without a tie), we certainly interpret that x is preferred to y.

(b) When x and y are both in the list and x is ranked as good as y, we interpret it as being equivalent to half a vote preferring x to y plus another half a vote preferring y to x.

(c) When x is in the list and y is not in it, we interpret that x is preferred to y.

(d) When neither x nor y are in the list, we interpret nothing about the preference of the voter between x and y.

Later on (§ 3.2, 3.3) we will consider certain alternatives to rules (d) and (c).

The preceding rules allow us to count how many voters support a given binary preference, i. e. a particular statement of the form "x is preferable to y". By doing so for each possible pair of options x and y, the whole vote gets summarized into a set of $N(N-1)$ numbers (since x and y must be different from each other). We will denote these numbers by V_{xy} and we will call them the binary **scores** of the vote. The collection of these numbers will be called the **Llull matrix** of the vote. Since we look for scale invariance, it makes sense to divide all of these numbers by the total number of votes V, which normalizes them to range from 0 to 1. In the following we will work mostly with these normalized scores, which will be denoted by v_{xy}. In practice, however, the absolute scores V_{xy} have the advantage that they are integer numbers, so we will use them in the examples.

In general, the numbers V_{xy} are bound to satisfy $V_{xy}+V_{yx} \leq V$, or equivalently $v_{xy} + v_{yx} \leq 1$. The special case where the ranking votes are all of them complete, i. e. without truncation, is characterized by the condition that $V_{xy}+V_{yx} = V$, or equivalently $v_{xy}+v_{yx} = 1$. From now on we will refer to such a situation as the case of **complete votes**.

Besides the scores v_{xy}, in the sequel we will often deal with the **margins** m_{xy} and the **turnovers** t_{xy}, which are defined respectively by

$$m_{xy} = v_{xy} - v_{yx}, \qquad t_{xy} = v_{xy} + v_{yx}. \tag{1}$$

Obviously, their dependence on the pair xy is respectively antisymmetric and symmetric, that is

$$m_{yx} = -m_{xy}, \qquad t_{yx} = t_{xy}. \tag{2}$$

It is clear also that the scores v_{xy} and v_{yx} can be recovered from m_{xy} and t_{xy} by means of the formulas

$$v_{xy} = (t_{xy}+m_{xy})/2, \qquad v_{yx} = (t_{xy}-m_{xy})/2. \tag{3}$$

2.2 A natural candidate for defining the social preference is the following: x is socially preferred to y whenever $v_{xy} > v_{yx}$. Of course, it can happen that $v_{xy} = v_{yx}$, in which case one would consider that x is socially equivalent to y. The binary relation that includes all pairs xy for which $v_{xy} > v_{yx}$ will be denoted by $\mu(v)$ and will be called the **comparison relation**; together with it, we will consider also the **adjoint comparison relation** $\hat{\mu}(v)$ which is defined by the condition $v_{xy} \geq v_{yx}$.

As it is well-known, the main problem with paired comparisons is that the comparison relations $\mu(v)$ and $\hat{\mu}(v)$ may lack transitivity even if the individual preferences are all of them transitive [*22, 2*]. More specifically, $\mu(v)$ can contain a 'Condorcet cycle', i. e. a sequence $x_0 x_1 \ldots x_n$ such that $x_n = x_0$ and $x_i x_{i+1} \in \mu(v)$ for all i.

A most natural reaction to it is going for the transitive closure of $\hat{\mu}(v)$, which we will denote by $\hat{\mu}^*(v)$. By definition, $\hat{\mu}^*(v)$ includes all (ordered) pairs xy for which there is a path $x_0 x_1 \ldots x_n$ from $x_0 = x$ to $x_n = y$ whose links $x_i x_{i+1}$ are all of them in $\hat{\mu}(v)$. In other words, we can say that $\hat{\mu}^*(v)$ includes all pairs that are "indirectly related" through $\hat{\mu}(v)$. However, this operation replaces each cycle of intransivity by an equivalence between its members. Instead of that, we would rather break these equivalences according to the quantitative information provided by the scores v_{xy}. This is what is done in such methods as ranked pairs or paths. However, these methods use that quantitative information to reach only qualitative results. In contrast, our results will keep a quantitative character until the end.

2.3 The next developments rely upon an operation $(v_{xy}) \rightarrow (v^*_{xy})$ that transforms the original binary scores into a new one. This operation is defined in the following way: for every pair xy, one considers all possible paths $x_0 x_1 \ldots x_n$ going from $x_0 = x$ to $x_n = y$; every such path is associated with the score of its weakest link, i. e. the smallest value of $v_{x_i x_{i+1}}$; finally, v^*_{xy} is defined as the maximum value of this associated score over all paths from x

to y. In other words,

$$v^*_{xy} \quad = \quad \max_{\substack{x_0 = x \\ x_n = y}} \quad \min_{\substack{i \geq 0 \\ i < n}} \quad v_{x_i x_{i+1}}, \tag{4}$$

where the max operator considers all possible paths from x to y, and the min operator considers all the links of a particular path. The scores v^*_{xy} will be called the **indirect scores** associated with the (direct) scores v_{xy}.

If (v_{xy}) is the table of 0's and 1's associated with a binary relation ρ (by putting $v_{xy} = 1$ if and only if $xy \in \rho$), then (v^*_{xy}) is exactly the table associated with ρ^*, the transitive closure of ρ. So, the operation $(v_{xy}) \mapsto (v^*_{xy})$ can be viewed as a quantitative analog of the notion of transitive closure (see [*7*: Ch. 25]).

The main point, remarked in 1998 by Markus Schulze [*33 b*], is that the comparison relation associated with a table of indirect scores is always transitive (Theorem 6.3). So, $\mu(v^*)$ is always transitive, no matter what the case is for $\mu(v)$. This is true in spite of the fact that $\mu(v^*)$ can easily differ from $\mu^*(v)$. In the following we will refer to $\mu(v^*)$ as the **indirect comparison relation**.

Remark

Somewhat surprisingly, in the case of incomplete votes the transitive relation $\mu(v^*)$ may differ from $\mu(v)$ even when the latter is already transitive. An example is given by the following profile, where each indicated preference is preceded by the number of people who voted in that way: 17 a, 24 c, 16 $a \succ b \succ c$, 16 $b \succ a \succ c$, 8 $b \succ c \succ a$, 8 $c \succ b \succ a$; in this case the direct comparison relation $\mu(v)$ is the ranking $a \succ b \succ c$, whereas the indirect comparison relation $\mu(v^*)$ is the ranking $b \succ a \succ c$. More specifically, we have $V_{ab} = 33 > 32 = V_{ba}$ but $V^*_{ab} = 33 < 40 = V^*_{ba}$.

The agreement with $\mu(v)$ can be forced by suitably redefining the indirect scores; more specifically, formula (4) can be replaced by an analogous one where the max operator is not concerned with all possible paths from x to y but only those contained in $\mu(v)$. This idea is put forward in [*34*]. Generally speaking, however, such a method cannot be made into a continuous rating procedure since one does quite different things depending on whether $v_{xy} > v_{yx}$ or $v_{xy} < v_{yx}$. On the other hand, we will see that in the complete case the indirect comparison relation does not change when the paths are restricted to be contained in $\mu(v)$ (§ 7).

2.4 In the following we put

$$\nu = \mu(v^*), \qquad \hat{\nu} = \hat{\mu}(v^*), \qquad m^\nu_{xy} = v^*_{xy} - v^*_{yx}. \tag{5}$$

So, $xy \in \nu$ if and only if $v^*_{xy} > v^*_{yx}$, i.e. $m^\nu_{xy} > 0$, and $xy \in \hat{\nu}$ if and only if $v^*_{xy} \geq v^*_{yx}$, i.e. $m^\nu_{xy} \geq 0$. From now on we will refer to m^ν_{xy} as the **indirect margin** associated with the pair xy.

As it has been stated above, the relation ν is transitive. Besides that, it is clearly antisymmetric (one cannot have both $v^*_{xy} > v^*_{yx}$ and vice versa). On the other hand, it may be not complete (one can have $v^*_{xy} = v^*_{yx}$). When it differs from ν, the complete relation $\hat{\nu}$ is not antisymmetric and —somewhat surprisingly— it may be not transitive either. For instance, consider the profile given by $4\ b \succ a \succ c$, $3\ a \succ c \succ b$, $2\ c \succ b \succ a$, $1\ c \succ a \succ b$; in this case the indirect comparison relation $\nu = \mu(v^*)$ contains only the pair ac; as a consequence, $\hat{\nu}$ contains cb and ba but not ca. However, one can always find a total order ξ which satisfies $\nu \subseteq \xi \subseteq \hat{\nu}$ (Theorem 8.2). From now on, any total order ξ that satisfies this condition will be called an **admissible order**.

The rating that we are looking for will be based on such an order ξ. More specifically, it will be compatible with ξ in the sense that the rates r_x will satisfy the inequality $r_x \leq r_y$ whenever $xy \in \xi$. If ν is already a total order, so that $\xi = \nu$, the preceding inequality will be satisfied in the strict form $r_x < r_y$, and this will happen if and only if $xy \in \nu$ (Theorem 10.2).

If there is more than one admissible order then some options will have equal rates. In fact, we will have $r_x = r_y$ whenever $xy \in \xi_1$ and $yx \in \xi_2$, where ξ_1, ξ_2 are two admissible orders. This will be so because we want the rating to be independent of the choice of ξ. This independence with respect to ξ seems essential for achieving the continuity property C; in fact, each possible choice of ξ for a given profile of vote frequencies may easily become the only one for a slight perturbation of that profile (but not necessarily, as it is illustrated by example 10 of [*34*: § 4.6]).

The following steps assume that one has fixed an admissible order ξ. From now on the situation $xy \in \xi$ will be expressed also by $x \overset{\xi}{\succcurlyeq} y$. According to the definitions, the inclusions $\nu \subseteq \xi \subseteq \hat{\nu}$ are equivalent to saying that $v^*_{xy} > v^*_{yx}$ implies $x \overset{\xi}{\succcurlyeq} y$ and that the latter implies $v^*_{xy} \geq v^*_{yx}$. In other words, if the different options are ordered according to $\overset{\xi}{\succcurlyeq}$, the matrix v^*_{xy} has then the property that each element above the diagonal is larger than or equal to its symmetric over the diagonal.

2.5 Rating the different options means positioning them on a line. Besides complying with the qualitative restriction of being compatible with ξ in the sense above, we want that the distances between items reflect the quantitative information provided by the binary scores. However, a rating is expressed by N numbers, whereas the binary scores are $N(N-1)$ numbers. So we are bound to do some sort of projection. Problems of this kind have a certain tradition in combinatorial data analysis and cluster analysis [*14, 23, 13*]. In fact, some of the operations that will be used below can be viewed from that point of view.

Let us assume for a while that we are dealing with complete ranking votes, so that it makes sense to talk about the average ranks. It is well-known [*2*: Ch. 9] that their values, which we will denote by $\bar{r}_x$, can be obtained from the Llull matrix by means of the following formula:

$$\bar{r}_x \;=\; N - \sum_{y\neq x} v_{xy}. \tag{6}$$

Equivalently, we can write

$$\bar{r}_x \;=\; (N + 1 - \sum_{y\neq x} m_{xy}\,)\,/\,2, \tag{7}$$

where the m_{xy} are the margins of the original scores v_{xy}, i. e. $m_{xy} = v_{xy} - v_{yx}$. In fact, the hypothesis of complete votes means that $v_{xy} + v_{yx} \;=\; 1$, so that $m_{xy} \;=\; 2v_{xy} - 1$, which gives the equivalence between (6) and (7).

Let us look at the meaning of the margins m_{xy} in connection with the idea of projecting the Llull matrix into a rating: If there are no other items than x and y, we can certainly view the sign and magnitude of m_{xy} as giving respectively the qualitative and quantitative aspects of the relative positions of x and y on the rating line, that is, the order and the distance between them. When there are more than two items, however, we have several pieces of information of this kind, one for every pair, and these different pieces may be incompatible with each other, quantitatively or even qualitatively, which motivates indeed the problem that we are dealing with. In particular, the average ranks often violate the desired compatibility with the relation ξ.

In order to construct a rating compatible with ξ, we will use a formula analogous to (6) where the scores v_{xy} are replaced by certain **projected scores** v^π_{xy} to be defined in the following paragraphs. Together with them, we will make use of the corresponding **projected margins** $m^\pi_{xy} = v^\pi_{xy} - v^\pi_{yx}$ and the corresponding **projected turnovers** $t^\pi_{xy} = v^\pi_{xy} + v^\pi_{yx}$. So, the rank-like rates that we are looking for will be obtained in the following way:

$$r_x \;=\; N - \sum_{y\neq x} v^\pi_{xy}. \tag{8}$$

This formula will be used not only in the case of complete ranking votes, but also in the general case where the votes are allowed to be incomplete and/or intransitive binary relations.

2.6 Let us begin by the case of complete votes, i. e. $t_{xy} = v_{xy} + v_{yx} = 1$. In this case, we put also $t^{\pi}_{xy} = 1$. Analogously to (6) and (7), formula (8) is then equivalent to the following one:

$$r_x \;=\; (N + 1 - \sum_{y \neq x} m^{\pi}_{xy}\,)\,/\,2. \tag{9}$$

We want to define the projected margins m^{π}_{xy} so that the rating defined by (9) be compatible with the ranking ξ, i. e. $xy \in \xi$ implies $r_x \leq r_y$. Now, this ranking derives from the relation ν, which is concerned with the sign of $m^{\nu}_{xy} = v^*_{xy} - v^*_{yx}$. This clearly points towards taking $m^{\pi}_{xy} = m^{\nu}_{xy}$. However, this is still not enough for ensuring the compatibility with ξ. In order to ensure this property, it suffices that the projected margins, which we assume antisymmetric, behave in the following way:

$$x \overset{\xi}{\succ} y \implies m^{\pi}_{xy} \geq 0 \;\text{ and }\; m^{\pi}_{xz} \geq m^{\pi}_{yz} \;\text{ for any } z \notin \{x, y\}. \tag{10}$$

On the other hand, we also want the rates to be independent of ξ when there are several possibilities for it. To this effect, we will require the projected margins to have already such an independence.

The next operation will transform the indirect margins so as to satisfy these conditions. It is defined in the following way, where we assume $x \overset{\xi}{\succ} y$ and x' denotes the item that immediately follows x in the total order ξ:

$$m^{\nu}_{xy} \;=\; v^*_{xy} - v^*_{yx}, \tag{11}$$

$$m^{\sigma}_{xy} \;=\; \min\{\, m^{\nu}_{pq} \mid p \overset{\xi}{\succeq} x,\; y \overset{\xi}{\succeq} q \,\}, \tag{12}$$

$$m^{\pi}_{xy} \;=\; \max\{\, m^{\sigma}_{pp'} \mid x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y \,\}, \tag{13}$$

$$m^{\pi}_{yx} \;=\; -m^{\pi}_{xy}. \tag{14}$$

One can easily see that the m^{σ}_{xy} obtained in (12) already satisfy a condition analogous to (10). However, the independence of ξ is not be ensured until steps (13–14). This property is a consequence of the fact that the projected margins given by the preceding formulas satisfy not only condition (10) but also the following one:

$$m^{\pi}_{xy} = 0 \implies m^{\pi}_{xz} = m^{\pi}_{yz} \;\text{ for any } z \notin \{x, y\}. \tag{15}$$

In particular, this will happen whenever $m^{\nu}_{xy} = 0$ (since this implies $m^{\sigma}_{pp'} = 0$ for all p such that $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$). More particularly, in the event of having two admissible orders that interchange two consecutive elements p and p' we will have $m^{\pi}_{pp'} = m^{\sigma}_{pp'} = m^{\nu}_{pp'} = 0$ and consequently $m^{\pi}_{pz} = m^{\pi}_{p'z}$ for any $z \notin \{p, p'\}$, as it is required by the desired independence of ξ.

Anyway, the projected margins are finally introduced in (9), which determines the rank-like rates r_x. The corresponding fraction-like rates will be introduced in § 2.9.

Remarks

1. Condition (10) gives the pattern of growth of the projected margins m^π_{pq} when p and q vary according to an admissible order ξ. This pattern is illustrated in figure 1 below, where the square represents the matrix (m^π_{pq}) with p and q ordered according to ξ, from better to worse. As usual, the first index labels the rows, and the second one labels the columns. The diagonal corresponds to the case $p = q$, which we systematically leave out of consideration. Having said that, here it would be appropriate to put $m^\pi_{pp} = 0$. Anyway, the projected margins are greater than or equal to zero above the diagonal and smaller than or equal to zero below it, and they increase or remain the same as one moves along the indicated arrows. The right-hand side of the figure follows from the left-hand one because the projected margins are antisymmetric.

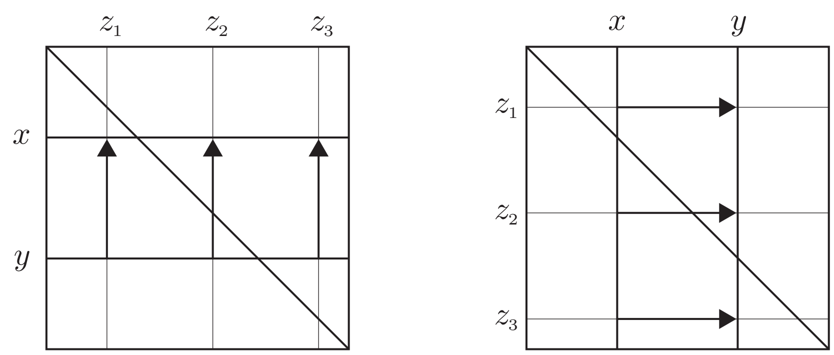

Figure 1. Directions of growth of the projected margins.

Of course, the absolute values $d_{xy} = |m^\pi_{xy}|$ keep this pattern in the upper triangle but they behave in the reverse way in the lower one. Such a behaviour is often considered in combinatorial data analysis, where it is associated with the name of W. S. Robinson, a statistician who in 1951 introduced a condition of this kind as the cornerstone of a method for seriating archaeological deposits (i. e. placing them in chronological order) [*30*; *13*: § 4.1.1, 4.1.2, 4.1.4; *32*].

Condition (15), more precisely its expression in terms of the d_{xy}, is also considered in cluster analysis, where it is referred to by saying that the 'dissimilarities' d_{xy} are 'even' [*14*: § 9.1] ('semidefinite' according to other authors).

In the present case of complete votes, the projected margins m^π_{xy} defined by (11–14) satisfy not only (10) and (15), but also the stronger condition

$$m^\pi_{xz} = \max(m^\pi_{xy}, m^\pi_{yz}), \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z. \tag{16}$$

Besides (10) and (15), this property implies also that the d_{xy} satisfy the following inequality, which makes no reference to the relation ξ:

$$d_{xz} \le \max(d_{xy}, d_{yz}), \qquad \text{for any } x, y, z. \tag{17}$$

This condition, called the ultrametric inequality, is also well known in cluster analysis, where it appears as a necessary and sufficient condition for the dissimilarities d_{xy} to define a hyerarchical classification of the set under consideration [*13*: § 3.2.1 ; *28*].

Our problem differs from the standard one of combinatorial data analysis in that our dissimilarities, namely the margins, are antisymmetric, whereas the standard problem considers symmetric dissimilarities. In other words, our dissimilarities have both magnitude and direction, whereas the standard ones have magnitude only. This makes an important difference in connection with the seriation problem, i. e. positioning the items on a line. Let us remark that the case of directed dissimilarities is considered in [*13*: § 4.1.2].

2. The operation $(m^\nu_{xy}) \to (m^\pi_{xy})$ defined by (12–13) is akin to the single-link method of cluster analysis, which can be viewed as a continuous method for projecting a matrix of dissimilarities onto the set of ultrametric distances; such a continuous projection is achieved by taking the maximal ultrametric distance which is bounded by the given matrix of dissimilarities [*14*: § 7.3, 7.4, 8.3, 9.3]. The operation $(m^\nu_{xy}) \to (m^\pi_{xy})$ does the same kind of job under the constraint that the clusters —in the sense of cluster analysis— be intervals of the total order ξ.

2.7 In order to get more insight into the case of incomplete votes, it is interesting to look at the case of plumping votes, i. e. the case where each vote plumps for a single option. In this case, and assuming interpretation (d), the binary scores of the vote have the form $v_{xy} = f_x$ for every $y \neq x$, where f_x is the fraction of voters who choose x.

In the spirit of condition H, in this case we expect the projected scores v^π_{xy} to coincide with the original ones $v_{xy} = f_x$. So, both the projected margins m^π_{xy} and the projected turnovers t^π_{xy} should also coincide with the original ones, namely $f_x - f_y$ and $f_x + f_y$. In this connection, one easily sees that the indirect scores v^*_{xy} coincide with v_{xy} (see Proposition 9.4). As a consequence, ξ is any total order for which the f_x are non-increasing.

If we apply formulas (11–14), we first get $m^\nu_{xy} = v^*_{xy} - v^*_{yx} = v_{xy} - v_{yx} = f_x - f_y$, and then $m^\sigma_{xx'} = f_x - f_{x'}$, but the projected margins resulting from (13) cease to coincide with the original ones. Most interestingly, such a coincidence would hold if the max operator of formula (13) was replaced by a sum.

Now, these two apparently different operations —maximum and addition— can be viewed as particular cases of a general procedure which involves taking the union of certain intervals, namely $\gamma_{xx'} = [\,(t_{xx'}-m^\sigma_{xx'})/2\,,\,(t_{xx'}+m^\sigma_{xx'})/2\,]$. In fact, in the case of complete votes, all the turnovers are equal to 1, so these intervals are all of them centred at $1/2$ and the union operation is equivalent to looking for the maximum of the widths. In the case of plumping votes, we know that $t_{xx'} = f_x + f_{x'}$ and we have just seen that $m^\sigma_{xx'} = f_x - f_{x'}$, which implies that $\gamma_{xx'} = [f_{x'}, f_x]$; so, the intervals $\gamma_{xx'}$ and $\gamma_{x'x''}$ are then adjacent to each other (the right end of the latter coincides with the left end of the former) and their union involves adding up the widths.

This remark strongly suggests that the general method should rely on such intervals. In the following we will refer to them as **score intervals**. A score interval can be viewed as giving a pair of scores about two options, the two scores being respectively in favour and against a specified preference relation about the two options. Alternatively, it can be viewed as giving a certain margin together with a certain turnover.

More specifically, one is immediately tempted to replace the minimum and maximum operations of (12–13) by the intersection and union of score intervals. The starting point would be the score intervals that combine the original turnovers t_{xy} with the indirect margins m^ν_{xy}. Such a procedure works as desired both in the case of complete votes and that of plumping ones. Unfortunately, however, it breaks down in other cases of incomplete votes which produce empty intersections or disjoint unions. So, a more elaborate method is required.

2.8 In this subsection we will finally describe a rank-like rating procedure which is able to cope with the general case. This procedure will use score intervals. However, these intervals will not be based directly on the original turnovers, but on certain transformed ones. This prior transformation of the turnovers will have the virtue of avoiding the problems pointed out at the end of the preceding paragraph.

So, we are given as input from one side the indirect margins m^ν_{xy}, and from the other side the original turnovers t_{xy}. The output to be produced is a set of projected scores v^π_{xy}. They should have the virtue that the associated rank-like rating given by (8) has the following properties: (a) it is the exactly the same for all admissible orders ξ; and (b) it is compatible with any such order ξ, i.e. $xy \in \xi$ implies $r_x \le r_y$.

As we did in the complete case, we will require the projected scores v^π_{xy} to satisfy the condition of independence with respect to ξ.

On the other hand, in order to ensure the compatibility condition (b), it suffices that the projected scores behave in the following way:

$$x \overset{\xi}{\succ} y \implies v^\pi_{xy} \ge v^\pi_{yx} \;\text{ and }\; v^\pi_{xz} \ge v^\pi_{yz} \;\text{ for any } z \notin \{x, y\}. \tag{18}$$

If we think in terms of the associated margins m^π_{xy} and turnovers t^π_{xy} —which add up to $2v^\pi_{xy}$— it suffices that both of them satisfy conditions analogous to (18). More, specifically, it suffices that the projected margins be antisymmetric and satisfy condition (10) of § 2.6 and that the projected turnovers be symmetric and satisfy

$$x \overset{\xi}{\succ} y \implies t^\pi_{xz} \ge t^\pi_{yz} \;\text{ for any } z \notin \{x, y\}. \tag{19}$$

So, we want the projected scores to be independent of ξ, and their associated margins and turnovers to satisfy conditions (10) and (19). These requirements are fulfilled by the procedure formulated in (20–26) below. These formulas use the following notations: Ψ is an operator to be described in a while; $[a, b]$ means the closed interval $\{\, x \in \mathbb{R} \mid a \le x \le b \,\}$; $|\gamma|$ means the length of such an interval $\gamma = [a, b]$, i.e. the number $b-a$; and $\dot\gamma$ means its barycentre, or centroid, i.e. the number $(a+b)/2$. As in (11–14), the following formulas assume that $x \overset{\xi}{\succ} y$, and x' denotes the option that immediately follows x in the total order ξ.

$$m^\nu_{xy} = v^*_{xy} - v^*_{yx}, \qquad t_{xy} = v_{xy} + v_{yx}, \tag{20}$$

$$m^\sigma_{xy} = \min\{\, m^\nu_{pq} \mid p \overset{\xi}{\succeq} x,\ y \overset{\xi}{\succeq} q \,\}, \qquad t^\sigma_{xy} = \Psi[(t_{pq}), (m^\sigma_{pp'})]_{xy}, \tag{21}$$

$$\gamma_{xx'} = [\,(t^\sigma_{xx'} - m^\sigma_{xx'})/2\,,\,(t^\sigma_{xx'} + m^\sigma_{xx'})/2\,], \tag{22}$$

$$\gamma_{xy} = \bigcup\{\, \gamma_{pp'} \mid x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y \,\}, \tag{23}$$

$$m^\pi_{xy} = |\gamma_{xy}|, \qquad t^\pi_{xy} = 2\,\dot\gamma_{xy}, \tag{24}$$

$$m^\pi_{yx} = -m^\pi_{xy}, \qquad t^\pi_{yx} = t^\pi_{xy}, \tag{25}$$

$$v^\pi_{xy} = \max\gamma_{xy} = (t^\pi_{xy} + m^\pi_{xy})/2, \qquad v^\pi_{yx} = \min\gamma_{xy} = (t^\pi_{xy} - m^\pi_{xy})/2. \tag{26}$$

Like (11–14), the preceding procedure can be viewed as a two-step transformation. The first step is given by (21) and it transforms the input margins and turnovers (m^ν_{xy}, t_{xy}) into certain intermediate projections $(m^\sigma_{xy}, t^\sigma_{xy})$

which already satisfy conditions analogous to (10) and (19) but are not independent of ξ. The condition of independence requires a second step which is described by (22–26). As in § 2.6, the superdiagonal final projections coincide with the intermediate ones, i. e. $m^\pi_{xx'} = m^\sigma_{xx'}$ and $t^\pi_{xx'} = t^\sigma_{xx'}$. Notice also that the intermediate margins m^σ_{xy} are constructed exactly as in (12).

The main difficulty lies in constructing the intermediate turnovers t^σ_{xy} so that they do not depend on ξ. The reason is that this condition involves the admissible orders, which depend on the relation ν associated with the indirect margins m^ν_{xy}. So, that construction must take into account not only the original turnovers but also the indirect margins. This connection with the m^ν_{xy} will be controlled indirectly through the $m^\sigma_{xx'}$. In fact, we will look for the t^σ_{xy} so as to satisfy the following conditions:

$$m^\sigma_{xx'} \leq t^\sigma_{xx'} \leq 1, \tag{27}$$

$$0 \leq t^\sigma_{py} - t^\sigma_{p'y} \leq m^\sigma_{pp'}, \tag{28}$$

$$0 \leq t^\sigma_{xq} - t^\sigma_{xq'} \leq m^\sigma_{qq'}. \tag{29}$$

Notice that (28) ensures that t^σ_{py} and $t^\sigma_{p'y}$ will coincide with each other whenever $m^\sigma_{pp'} = 0$. Since $m^\sigma_{pp'} = m^\nu_{pp'}$, we are in the case of having two admissible orders that interchange p with p'. The fact that this implies $t^\sigma_{py} = t^\sigma_{p'y}$ eventually ensures the independence of ξ (Theorem 9.2; we say 'eventually' because the full proof is quite long).

In the case of complete votes we will have $t^\sigma_{xy} = 1$, so that condition (27) will be satisfied with an equality sign in the right-hand inequality, whereas (28) and (29) will be satisfied with an equality sign in the left-hand inequality. In the case of plumping votes, where we know that $m^\sigma_{xx'} = m_{xx'} = f_x - f_{x'}$ (§ 2.7), we will have $t^\sigma_{xy} = t_{xy} = f_x + f_y$, so that (28) and (29) will be satisfied with an equality sign in the right-hand inequalities (equation (27) is satisfied too, but in this case the inequalities can be strict).

Notice also that conditions (28–29) imply the following one:

$$0 \leq t^\sigma_{xx'} - t^\sigma_{x'x''} \leq m^\sigma_{xx'} + m^\sigma_{x'x''}. \tag{30}$$

In geometrical terms, the inequalities in (27) mean that (a) the interval $\gamma_{xx'}$ is contained in $[0,1]$. On the other hand, the inequalities in (30) mean that the intervals $\gamma_{xx'}$ and $\gamma_{x'x''}$ are related to each other in the following way: (b) the barycentre of the first one lies to the right of that of the second one; (c) the two intervals overlap each other.

Conditions (27–29) can be easily achieved by taking simply $t^\sigma_{xy} = 1$. However, this choice goes against our aim of distinguishing between definite indifference and lack of information; in particular, condition H requires that in the case of plumping votes the projected turnovers should coincide with the original ones (which are then less than 1). Now, conditions (27–29) are convex with respect to the t^σ_{xy} (the whole set of them), i. e. if they are satisfied by two different choices of these numbers, they are satisfied also by any convex combination of them. This implies that for any given set of original turnovers t_{xy} there is a unique set of values t^σ_{xy} which minimizes the euclidean distance to the given one while satisfying those conditions.

So, the operator Ψ can defined in the following way: t^σ_{xy} is the set of turnovers which is determined by conditions (27–29) together with that of minimizing the following measure of deviation with respect to the t_{xy}:

$$\Phi = \sum_x \sum_y (t^\sigma_{xy} - t_{xy})^2. \tag{31}$$

The actual computation of the t^σ_{xy} can be carried out in a finite number of steps by means of a quadratic programming algorithm [*21* : § 14.1].

Anyway, the preceding operations have the virtue of ensuring the desired properties.

Remarks

1. Condition (19) is illustrated in figure 2, where the arrows indicate the directions of growth of the projected turnovers. The right-hand side of the figure follows from the left-hand one because the projected turnovers are symmetric.

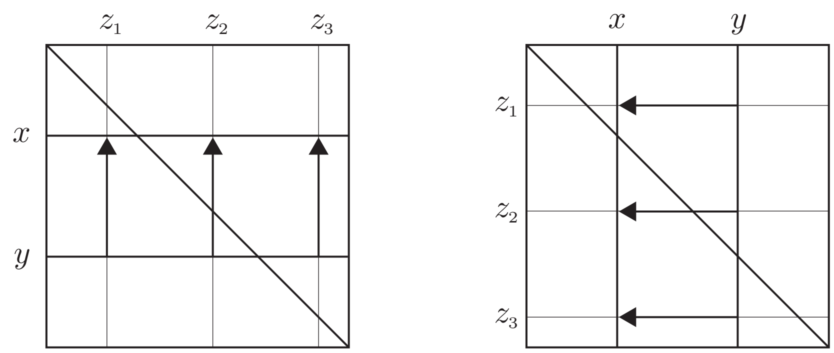

Figure 2. Directions of growth of the projected turnovers.

This condition can be associated with the name of Marshall G. Greenberg, a mathematical psychologist who in 1965 considered a condition of this form —at the suggestion of Clyde H. Coombs— in connection with the problem of producing a rating after paired-comparison data, specially in the incomplete case [*11*; *13*: § 4.1.2]. Having said that, we strongly differ from that author in that he applies a property like (19) to the scores, whereas we consider more appropriate to apply it to the turnovers.

In fact, under the general assumption that each vote is a ranking, possibly incomplete, and that each ranking is translated into a set of binary preferences according to rules (a–d) of § 2.1, it is fairly reasonable to expect that the turnover for a pair xy, i. e. the number of voters who expressed an opinion about x in comparison with y, should increase as x and/or y are higher in the social ranking. In practice, the original turnovers can deviate to a certain extent from this ideal behaviour. In contrast, our projected turnovers are always in agreement with it (with respect to the total order ξ).

2. The projected scores turn out to satisfy not only (18), but also the following stronger property:

$$\begin{aligned} \text{if} \quad x \succeq^{\xi} y \quad \text{then} \quad & v^{\pi}_{xy} \geq v^{\pi}_{yx} \\ \text{and} \quad & v^{\pi}_{xz} \geq v^{\pi}_{yz}, \; v^{\pi}_{zx} \leq v^{\pi}_{zy} \quad \text{for any } z \notin \{x, y\}. \end{aligned} \tag{32}$$

So, the projected scores increase or remain constant in the directions shown in figure 1. Furthermore, we will see that the quotients $m^{\pi}_{xy}/t^{\pi}_{xy}$ have also the same property.

3. In contrast to the case of complete votes, in this case the projected margins do not satisfy (16) but only

$$m^{\pi}_{xz} \leq m^{\pi}_{xy} + m^{\pi}_{yz}, \qquad \text{whenever } x \succeq^{\xi} y \succeq^{\xi} z. \tag{33}$$

As a consequence, the absolute values $d_{xy} = |m^{\pi}_{xy}|$ satisfy the triangular inequality:

$$d_{xz} \leq d_{xy} + d_{yz}, \qquad \text{for any } x, y, z. \tag{34}$$

4. Under the assumption of ranking votes (but not necessarily in a more general setting) one can see that the original turnovers already satisfy (27). In this case, the preceding definition of Ψ turns out to be equivalent to an analogous one where conditions (27–29) are replaced simply by (28–29). From this it follows that the intermediate turnovers have then the same sum as the original ones:

$$\sum_x \sum_y t^{\sigma}_{xy} = \sum_x \sum_y t_{xy}. \tag{35}$$

2.9 Finally, let us see how shall we define the fraction-like rates φ_x. As in the case of the rank-like rates, we will use a classical method which would usually be applied to the original scores, but here we will apply it to the projected scores. This method was introduced in 1929 by Ernst Zermelo [*40*] and it was rediscovered by other authors in the 1950s [*3*, *10*] (see also [*15*]). Zermelo's work was motivated by chess tournaments, whereas the other authors were considering comparative judgments. Anyway, all of them were especially interested in the incomplete case, i. e. the case where turnovers may depend on the pair xy.

More specifically, the fraction-like rates φ_x will be determined by the following system of equations (together with the condition that $\varphi_x \geq 0$ for every x):

$$\sum_{y\neq x} t^{\pi}_{xy}\,\varphi_x/(\varphi_x+\varphi_y) \;=\; \sum_{y\neq x} v^{\pi}_{xy} \quad (= N - r_x), \tag{36}$$

$$\sum_{x} \varphi_x \;=\; f, \tag{37}$$

where (36) contains one equation for every x, and f stands for the fraction of non-empty votes (i. e. $f = F/V$ where F is the number of non-empty votes and V is the total number of votes). In spite of having $N+1$ equations, the N equations contained in (36) are not independent, since their sum results in the identity $(\sum_x \sum_{y\neq x} t^{\pi}_{xy})/2 = \sum_x \sum_{y\neq x} v^{\pi}_{xy}$. On the other hand, it is clear that (36) is insensitive to all of the φ_x being multiplied by a constant factor. This indeterminacy disappears once (36) is supplemented with equation (37).

In the case of plumping votes, where we know that $v^{\pi}_{xy} = f_x$ and $t^{\pi}_{xy} = f_x + f_y$, the solution of (36–37) is easily seen to be $\varphi_x = f_x$, as required by condition H.

The problem of solving the system (36–37) is well posed when the projected Llull matrix (v^{π}_{xy}) is irreducible. This means that there is no splitting of the options into a 'top class' X plus a 'low class' Y so that $v^{\pi}_{yx} = 0$ for any $x \in X$ and $y \in Y$. When such a splitting exists, one is forced to put $\varphi_y = 0$ for all $y \in Y$. For more details, the reader is referred to section 11.

Zermelo (and the other authors) dealed also with the problem of numerically solving a non-linear system of the form (36). In this connection, he showed that in the irreducible case its solution (up to a multiplicative constant) can be approximated to an arbitrary degree of accuracy by means of an iterative scheme of the form

$$\varphi^{n+1}_x \;=\; \Big(\sum_{y\neq x} v^{\pi}_{xy}\Big)\Big/\Big(\sum_{y\neq x} t^{\pi}_{xy}/(\varphi^n_x+\varphi^n_y)\Big), \tag{38}$$

starting from an arbitrary set of values $\varphi^0_x > 0$.

The fraction-like rates φ_x determined by (36–37) can be viewed as an estimate of the first-choice fractions using not only the first choices but the whole rankings. Properly speaking, Zermelo's method (with the original scores and turnovers) corresponds to a maximum likelihood estimate of the parameters of a certain probabilistic model for the outcomes of a tournament between several players, or, more in the lines of our applications, for the outcomes of comparative judgments. This model will be briefly described in section 11. Although we are far from its hypotheses, we will see that Zermelo's method is quite suitable for translating our rank-like rates into fraction-like ones.

3 Summary of the method. Variants. General forms of vote

3.1 Let us summarize the whole procedure. In the *general case*, where the votes are not necessarily complete, it consists of the following steps:

1. Form the Llull matrix (v_{xy}) (§ 2.1). Work out the turnovers $t_{xy} = v_{xy} + v_{yx}$.
2. Compute the indirect scores v^*_{xy} defined by (4). An efficient way to do it is the Floyd-Warshall algorithm [*7*: § 25.2]. Work out the indirect margins $m^\nu_{xy} = v^*_{xy} - v^*_{yx}$ and the associated indirect comparison relation $\nu = \{ xy \mid m^\nu_{xy} > 0 \}$.
3. Find an admissible order ξ (§ 2.4) and arrange the options according to it. For instance, it suffices to arrange the options by non-decreasing values of the 'tie-splitting' Copeland scores $\kappa_x = 1 + |\{ y \mid y \neq x,\ m^\nu_{yx} > 0 \}| + \frac{1}{2} |\{ y \mid y \neq x,\ m^\nu_{yx} = 0 \}|$ (Proposition 8.5).
4. Starting from the indirect margins m^ν_{xy}, work out the superdiagonal intermediate projected margins $m^\sigma_{xx'}$ as defined in (21.1).
5. Starting from the original turnovers t_{xy}, and taking into account the superdiagonal intermediate projected margins $m^\sigma_{xx'}$, determine the intermediate projected turnovers t^σ_{xy} so as to minimize (31) under the constraints (27–29). This can be carried out in a finite number of steps by means of a quadratic programming algorithm [*21*: § 14.1].
6. Form the intervals $\gamma_{xx'}$ defined by (22), derive their unions γ_{xy} as defined by (23), and read off the projected scores v^π_{xy} (26).
7. Compute the rank-like ranks r_x according to (8).
8. Determine the fraction-like rates φ_x by solving the system (36–37). This can be done numerically by means of the iterative scheme (38).

In the *complete case*, the scores v_{xy} and the margins m_{xy} are related to each other by the monotone increasing transformation $v_{xy} = (1 + m_{xy})/2$. Because of this fact, the preceding procedure can then be simplified in the following way:

- Step 2 computes m^*_{xy} instead of v^*_{xy} and takes $m^\nu_{xy} = (m^*_{xy} - m^*_{yx})/2$.
- Step 5 is not needed.
- Step 6 reduces to (13–14).
- Step 7 makes use of formula (9).

3.2 The preceding procedure admits of certain variants which might be appropriate to some special situations. Next we will distinguish four of them, namely

1. Main
2. Dual
3. Balanced
4. Margin-based

The above-described procedure is included in this list as the main variant. The four variants are exactly equivalent to each other in the complete case, but in the incomplete case they can produce different results. In spite of this, they all share the main properties.

The dual variant is analogous to the main one except that the max-min indirect scores v^*_{xy} are replaced by the following min-max ones:

$$ {}^*v_{xy} \;=\; \min_{\substack{x_0 = x \\ x_n = y}} \;\; \max_{\substack{i \geq 0 \\ i < n}} \;\; v_{x_i x_{i+1}}. \tag{39} $$

Equivalently, ${}^*v_{xy} = 1 - \hat{v}^*_{yx}$ where $\hat{v}_{xy} = 1 - v_{yx}$. In the complete case one has ${}^*v_{xy} = 1 - v^*_{yx}$, so that ${}^*v_{xy} - {}^*v_{yx} = v^*_{xy} - v^*_{yx}$ and $\mu({}^*v) = \mu(v^*)$; as a consequence, the dual variant is then equivalent to the main one.

The balanced variant takes $\nu = \mu(v^*) \cap \mu({}^*v)$ together with

$$ m^\nu_{xy} = \begin{cases} \min\,(v^*_{xy} - v^*_{yx}, {}^*v_{xy} - {}^*v_{yx}), & \text{if } xy \in \mu(v^*) \cap \mu({}^*v), \\ -m^\nu_{yx}, & \text{if } yx \in \mu(v^*) \cap \mu({}^*v), \\ 0, & \text{otherwise.} \end{cases} \tag{40} $$

The remarks made in connection with the dual variant show that in the complete case the balanced variant is also equivalent to the preceding ones.

The margin-based variant follows the simplified procedure of the end of § 3.1 even if one is not originally in the complete case. Equivalently, it corresponds to replacing the original scores v_{xy} by the following ones: $v'_{xy} = (1+m_{xy})/2$. This amounts to replacing any lack of information about a pair of options by a definite indifference between them, which brings the problem into the complete case. So, the specific character of this variant lies only in its interpretation of incomplete votes. Although this interpretation goes against the general principle stated in § 1.5, it may be suitable to certain situations where the voters are well acquainted with all of the options. In the case of ranking votes, it amounts to replace rule (d) of § 2.1 by the following one:

(d′) When neither x nor y are in the list, we interpret that they are considered equally good (or equally bad), so we proceed as in (b).

In other words, each truncated vote is completed by appending to it all the missing options tied to each other.

Remark

Other variants —in the incomplete case— arise when equation (8) is replaced by the following one:

$$r_x \;=\; 1 + \sum_{y\neq x} v^{\pi}_{yx}. \tag{41}$$

3.3 Most of our results will hold if the “votes” are not required to be rankings, but they are allowed to be general binary relations. In particular, this allows to deal with certain situations where it makes sense to replace rule (c) of § 2.1 by the following one:

(c′) When x is in the list and y is not in it, we interpret nothing about the preference of the voter between x and y.

One could even allow the votes to be non-transitive binary relations; such a lack of transitivity in the individual preferences may arise when individuals are aggregating a variety of criteria [*12*].

A vote in the form of a binary relation ρ contributes to the binary scores with the following amounts:

$$v_{xy} = \begin{cases} 1, & \text{if } xy \in \rho \text{ and } yx \notin \rho \\ 1/2, & \text{if } xy \in \rho \text{ and } yx \in \rho \\ 0, & \text{if } xy \notin \rho. \end{cases} \tag{42}$$

Even more generally, a vote could be any set of normalized binary scores, i. e. an element of the set $\Omega = \{\, v \in [0,1]^{\mathit{\Pi}} \mid v_{xy} + v_{yx} \leq 1 \,\}$, where $\mathit{\Pi}$ denotes the set of pairs $xy \in A \times A$ with $x \neq y$.

Anyway, the collective Llull matrix is simply the center of gravity of a distribution of individual votes:

$$v_{xy} = \sum_k \alpha_k \, v^k_{xy}, \tag{43}$$

where α_k are the relative frequencies or weights of the individual votes v^k.

4 Examples

4.1 As a first example of a vote which involved truncated rankings, we look at an election which took place the 16th of February of 1652 in the Spanish royal household. This election is quoted in [*29*], but we use the slightly different data which are given in [*27*: vol. 2, p. 263–264]. The office under election was that of "aposentador mayor de palacio", and the king was assessed by six noblemen, who expressed the following preferences:

Marqués de Ariça .... $\mathsf{b} \succ \mathsf{e} \succ \mathsf{d} \succ \mathsf{a}$
Conde de Barajas .... $\mathsf{b} \succ \mathsf{a} \succ \mathsf{f}$
Conde de Montalbán .... $\mathsf{a} \succ \mathsf{f} \succ \mathsf{b} \succ \mathsf{d}$
Marqués de Povar .... $\mathsf{e} \succ \mathsf{b} \succ \mathsf{f} \succ \mathsf{c}$
Conde de Puñonrostro .... $\mathsf{e} \succ \mathsf{a} \succ \mathsf{b} \succ \mathsf{f}$
Conde de Ysinguién .... $\mathsf{b} \succ \mathsf{d} \succ \mathsf{a} \succ \mathsf{f}$

The candidates a – f nominated in these preferences were:

a Alonso Carbonel (architect, 1583–1660)
b Gaspar de Fuensalida (died 1664)
c Joseph Nieto
d Simón Rodríguez
e Francisco de Rojas (1583–1659)
f Diego Velázquez (painter, 1599–1660)

The CLC computations are as follows:

x	V_{xy} a	b	c	d	e	f
a	$*$	2	5	3	3	5
b	4	$*$	6	6	4	5
c	1	0	$*$	1	0	0
d	2	0	3	$*$	2	2
e	3	2	3	3	$*$	3
f	1	1	5	4	3	$*$

x	V^*_{xy} a	b	c	d	e	f	κ
a	$*$	2	**5**	**4**	3	**5**	$2\frac{1}{2}$
b	**4**	$*$	**6**	**6**	**4**	**5**	1
c	1	1	$*$	1	1	1	6
d	2	2	**3**	$*$	2	2	5
e	3	2	**3**	**3**	$*$	3	3
f	3	2	**5**	**4**	3	$*$	$3\frac{1}{2}$

x	M^{ν}_{xy} b	a	e	f	d	c
b	$*$	2	2	3	4	5
a	$*$	$*$	0	2	2	4
e	$*$	$*$	$*$	0	1	2
f	$*$	$*$	$*$	$*$	2	4
d	$*$	$*$	$*$	$*$	$*$	2
c	$*$	$*$	$*$	$*$	$*$	$*$

x	T_{xy} b	a	e	f	d	c
b	$*$	6	6	6	6	6
a	$*$	$*$	6	6	5	6
e	$*$	$*$	$*$	6	5	3
f	$*$	$*$	$*$	$*$	6	5
d	$*$	$*$	$*$	$*$	$*$	4
c	$*$	$*$	$*$	$*$	$*$	$*$

x	M^{σ}_{xy} b	a	e	f	d	c
b	$*$	2	2	3	4	5
a	$*$	$*$	0	2	2	4
e	$*$	$*$	$*$	0	1	2
f	$*$	$*$	$*$	$*$	1	2
d	$*$	$*$	$*$	$*$	$*$	2
c	$*$	$*$	$*$	$*$	$*$	$*$

x	T^{σ}_{xy} b	a	e	f	d	c
b	$*$	6	6	6	6	6
a	$*$	$*$	6	6	$5\frac{1}{3}$	$4\frac{2}{3}$
e	$*$	$*$	$*$	6	$5\frac{1}{3}$	$4\frac{2}{3}$
f	$*$	$*$	$*$	$*$	$5\frac{1}{3}$	$4\frac{2}{3}$
d	$*$	$*$	$*$	$*$	$*$	4
c	$*$	$*$	$*$	$*$	$*$	$*$

x	M^{π}_{xy} b	a	e	f	d	c
b	$*$	2	2	2	2	3
a	$*$	$*$	0	0	1	$2\frac{1}{6}$
e	$*$	$*$	$*$	0	1	$2\frac{1}{6}$
f	$*$	$*$	$*$	$*$	1	$2\frac{1}{6}$
d	$*$	$*$	$*$	$*$	$*$	2
c	$*$	$*$	$*$	$*$	$*$	$*$

x	T^{π}_{xy} b	a	e	f	d	c
b	$*$	6	6	6	6	5
a	$*$	$*$	6	6	$5\frac{1}{3}$	$4\frac{1}{6}$
e	$*$	$*$	$*$	6	$5\frac{1}{3}$	$4\frac{1}{6}$
f	$*$	$*$	$*$	$*$	$5\frac{1}{3}$	$4\frac{1}{6}$
d	$*$	$*$	$*$	$*$	$*$	4
c	$*$	$*$	$*$	$*$	$*$	$*$

x	V^{π}_{xy} b	a	e	f	d	c
b	$*$	4	4	4	4	4
a	2	$*$	3	3	$3\frac{1}{6}$	$3\frac{1}{6}$
e	2	3	$*$	3	$3\frac{1}{6}$	$3\frac{1}{6}$
f	2	3	3	$*$	$3\frac{1}{6}$	$3\frac{1}{6}$
d	2	$2\frac{1}{6}$	$2\frac{1}{6}$	$2\frac{1}{6}$	$*$	3
c	1	1	1	1	1	$*$

x	r_x	φ_x
b	2.6667	0.3049
a	3.6111	0.1703
e	3.6111	0.1703
f	3.6111	0.1703
d	4.0833	0.1293
c	5.1667	0.0549

According to these results, the office should have been given to candidate b, who is also the winner by most other methods. In the CLC method, this candidate is followed by three runners-up tied to each other, namely candidates a, e and f. In spite of the clear advantage of candidate b, the king appointed candidate f, namely, the celebrated painter Diego Velázquez.

4.2 As an example where the votes are complete strict rankings, we will consider the final round of a dancesport competition. Specifically, we will take the final round of the Professional Latin Rising Star section of the 2007 Blackpool Dance Festival (Blackpool, England, 25th May 2007). The data were taken from `http://www.scrutelle.info/results/estelle/2007/blackpool_2007/`.

As usual, the final was contested by six couples, whose numbers were 3, 4, 31, 122, 264, 238. Eleven adjudicators ranked their simultaneous performances in four equivalent dances.

The all-round official result was $3 \succ 122 \succ 264 \succ 4 \succ 31 \succ 238$. This result comes from the so-called "Skating System", whose name reflects a prior use in figure-skating. The Skating System has a first part which produces a separate result for each dance. This is done mainly on the basis of the median rank obtained by each couple, a criterion which Condorcet proposed as a "practical" method in 1792/93 [*22*: ch. 8]. However, the fine properties of this criterion are lost in the second part of the Skating System, where the all-round result is obtained by adding the up the final ranks obtained in the different dances.

From the point of view of paired comparisons, it makes sense to base the all-round result on the Llull matrix which collects the 44 rankings produced by the 11 adjudicators over the 4 dances [*25*: § 11]. As one can see below, in the present case this matrix exhibits several Condorcet cycles, like for instance $3 \succ 4 \succ 264 \succ 3$ and $3 \succ 122 \succ 264 \succ 3$, which means that the competition was closely contested. In such close contests, the Skating System often has to resort to certain tie-breaking rules which are virtually equivalent to throwing the dice. In contrast, the all-round Llull matrix has the virtue of being a more accurate quantitative aggregate over the different dances. On the basis of this more accurate aggregate, in this case the indirect scores reveal an all-round ranking which is quite different from the one produced by the Skating System (but it coincides with the one produced by other paired-comparison methods, like ranked pairs). In consonance with all this, the CLC rates obtained below are quite close to each other, particularly for the couples 3, 4, 122 and 264.

Since we are dealing with complete votes, in this case the CLC computations can be carried out entirely in terms of the margins. In the following we have chosen to pass to margins after computing the indirect scores, but we could have done it before that step.

x	V_{xy} 3	4	31	122	238	264
3	$*$	23	28	23	28	20
4	21	$*$	23	20	30	24
31	16	21	$*$	15	25	18
122	21	24	29	$*$	28	23
238	16	14	19	16	$*$	19
264	24	20	26	21	25	$*$

x	V^*_{xy} 3	4	31	122	238	264	κ
3	$*$	23	**28**	23	**28**	23	4
4	**24**	$*$	**24**	23	**30**	**24**	2
31	21	21	$*$	21	**25**	21	5
122	**24**	**24**	**29**	$*$	**28**	**24**	1
238	19	19	19	19	$*$	19	6
264	**24**	23	**26**	23	**25**	$*$	3

x	M^ν_{xy} 122	4	264	3	31	238
122	$*$	1	1	1	8	9
4	$*$	$*$	1	1	3	11
264	$*$	$*$	$*$	1	5	6
3	$*$	$*$	$*$	$*$	7	9
31	$*$	$*$	$*$	$*$	$*$	6
238	$*$	$*$	$*$	$*$	$*$	$*$

x	M^π_{xy} 122	4	264	3	31	238
122	$*$	1	1	1	3	6
4	$*$	$*$	1	1	3	6
264	$*$	$*$	$*$	1	3	6
3	$*$	$*$	$*$	$*$	3	6
31	$*$	$*$	$*$	$*$	$*$	6
238	$*$	$*$	$*$	$*$	$*$	$*$

x	r_x	φ_x
122	3.3636	0.1815
4	3.3864	0.1788
264	3.4091	0.1761
3	3.4318	0.1734
31	3.5682	0.1583
238	3.8409	0.1318

4.3 As a second example of an election involving truncated rankings we take the Debian Project leader election, which is using the method of paths since 2003. So far, the winners of these elections have been clear enough. However, a quantitative measure of this clearness was lacking. In the following we consider the 2006 election, which had a participation of $V = 421$ actual voters out of a total population of 972 members. The individual votes are available in `http://www.debian.org/vote/2006/vote_002`.

That election resulted in the following Llull matrix:

x	V_{xy} 1	2	3	4	5	6	7	8
1	$*$	**321**	144	$159\frac{1}{2}$	**$193\frac{1}{2}$**	**$347\frac{1}{2}$**	**246**	**320**
2	51	$*$	42	53	50	**262**	65	163
3	**251**	**340**	$*$	$198\frac{1}{2}$	**253**	**362**	**300**	**345**
4	**$245\frac{1}{2}$**	**341**	**$204\frac{1}{2}$**	$*$	**256**	**$371\frac{1}{2}$**	**$291\frac{1}{2}$**	**$339\frac{1}{2}$**
5	**$193\frac{1}{2}$**	**325**	144	149	$*$	**357**	**254**	**$321\frac{1}{2}$**
6	$26\frac{1}{2}$	77	24	$22\frac{1}{2}$	21	$*$	30	$74\frac{1}{2}$
7	137	**292**	90	$109\frac{1}{2}$	131	**330**	$*$	**296**
8	76	**207**	54	$71\frac{1}{2}$	$75\frac{1}{2}$	**$302\frac{1}{2}$**	89	$*$

Notice that candidate 4 is the winner according to the Condorcet principle (but not according to the majority principle, since V_{43} does not reach $V/2$). Notice also that there is no Condorcet cycle. However, candidates 1 and 5 are in a tie for third place: both of them defeat all other candidates except 4 and 3, and V_{15} coincides exactly with V_{51}.

The ensuing CLC computations are as follows:

x	V^*_{xy} 1	2	3	4	5	6	7	8	κ
1	$*$	**321**	$159\frac{1}{2}$	$159\frac{1}{2}$	**$193\frac{1}{2}$**	**$347\frac{1}{2}$**	**246**	**320**	$3\frac{1}{2}$
2	89	$*$	89	89	89	**262**	89	163	7
3	**251**	**340**	$*$	$198\frac{1}{2}$	**253**	**362**	**300**	**345**	2
4	**$245\frac{1}{2}$**	**341**	**$204\frac{1}{2}$**	$*$	**256**	**$371\frac{1}{2}$**	**$291\frac{1}{2}$**	**$339\frac{1}{2}$**	1
5	**$193\frac{1}{2}$**	**325**	$159\frac{1}{2}$	$159\frac{1}{2}$	$*$	**357**	**254**	**$321\frac{1}{2}$**	$3\frac{1}{2}$
6	77	77	77	77	77	$*$	77	77	8
7	137	**292**	137	137	137	**330**	$*$	**296**	5
8	89	**207**	89	89	89	**$302\frac{1}{2}$**	89	$*$	6

x	M^{ν}_{xy} 4	3	1	5	7	8	2	6
4	$*$	6	86	$96\frac{1}{2}$	$154\frac{1}{2}$	$250\frac{1}{2}$	252	$294\frac{1}{2}$
3	$*$	$*$	$91\frac{1}{2}$	$93\frac{1}{2}$	163	256	251	285
1	$*$	$*$	$*$	0	109	231	232	$270\frac{1}{2}$
5	$*$	$*$	$*$	$*$	117	$232\frac{1}{2}$	236	280
7	$*$	$*$	$*$	$*$	$*$	207	203	253
8	$*$	$*$	$*$	$*$	$*$	$*$	44	$225\frac{1}{2}$
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	185
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

x	T_{xy} 4	3	1	5	7	8	2	6
4	$*$	403	405	405	401	411	394	394
3	$*$	$*$	395	397	390	399	382	386
1	$*$	$*$	$*$	387	383	396	372	374
5	$*$	$*$	$*$	$*$	385	397	375	378
7	$*$	$*$	$*$	$*$	$*$	385	357	360
8	$*$	$*$	$*$	$*$	$*$	$*$	370	377
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	339
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

x	M^{σ}_{xy} 4	3	1	5	7	8	2	6
4	$*$	6	86	$96\frac{1}{2}$	$154\frac{1}{2}$	$250\frac{1}{2}$	252	$294\frac{1}{2}$
3	$*$	$*$	86	$93\frac{1}{2}$	$154\frac{1}{2}$	$250\frac{1}{2}$	251	285
1	$*$	$*$	$*$	0	109	231	232	$270\frac{1}{2}$
5	$*$	$*$	$*$	$*$	109	231	232	$270\frac{1}{2}$
7	$*$	$*$	$*$	$*$	$*$	203	203	253
8	$*$	$*$	$*$	$*$	$*$	$*$	44	$225\frac{1}{2}$
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	185
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

x	T^{σ}_{xy} 4	3	1	5	7	8	2	6
4	$*$	403.4	403.4	403.4	403.25	403.25	392	392
3	$*$	$*$	397.4	397.4	397.25	397.25	386	386
1	$*$	$*$	$*$	389.6	389.6	389.6	374.75	374.75
5	$*$	$*$	$*$	$*$	389.6	389.6	374.75	374.75
7	$*$	$*$	$*$	$*$	$*$	385	366	366
8	$*$	$*$	$*$	$*$	$*$	$*$	366	366
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	339
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

x	M^{π}_{xy} 4	3	1	5	7	8	2	6
4	$*$	6	86	86	109	203	203	217
3	$*$	$*$	86	86	109	203	203	217
1	$*$	$*$	$*$	0	109	203	203	217
5	$*$	$*$	$*$	$*$	109	203	203	217
7	$*$	$*$	$*$	$*$	$*$	203	203	217
8	$*$	$*$	$*$	$*$	$*$	$*$	44	185
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	185
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

x	T^{π}_{xy} 4	3	1	5	7	8	2	6
4	$*$	403.4	397.4	397.4	389.6	385	385	371
3	$*$	$*$	397.4	397.4	389.6	385	385	371
1	$*$	$*$	$*$	389.6	389.6	385	385	371
5	$*$	$*$	$*$	$*$	389.6	385	385	371
7	$*$	$*$	$*$	$*$	$*$	385	385	371
8	$*$	$*$	$*$	$*$	$*$	$*$	366	339
2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	339
6	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

x	V^{π}_{xy} 4	3	1	5	7	8	2	6		r_x	φ_x
4	$*$	204.7	241.7	241.7	249.3	294	294	294		3.6784	0.2067
3	198.7	$*$	241.7	241.7	249.3	294	294	294		3.6926	0.2048
1	155.7	155.7	$*$	194.8	249.3	294	294	294		4.1105	0.1596
5	155.7	155.7	194.8	$*$	249.3	294	294	294		4.1105	0.1596
7	140.3	140.3	140.3	140.3	$*$	294	294	294		4.5720	0.1218
8	91	91	91	91	91	$*$	205	262		5.8100	0.0599
2	91	91	91	91	91	161	$*$	262		5.9145	0.0559
6	77	77	77	77	77	77	77	$*$		6.7197	0.0317

As one can see, the CLC results are in full agreement with the Copeland scores of the original Llull matrix. In particular, they still give an exact tie between candidates 1 and 5. Even so, the CLC rates yield a quantitative information which is not present in the Copeland scores. In particular, they show that the victory of candidate 4 over candidate 3 was relatively narrow.

For the computation of the rates we have taken $V = 421$ (the actual number of votes) instead of $V = 972$ (the number of people with the right to vote); in particular, the fraction-like rates φ_x have been computed so that they add up to $f = 1$ instead of the true participation ratio $f = 421/972$. This is especially justified in Debian elections since they systematically include "none of the above" as one of the alternatives, so it is reasonable to interpret that abstention does not have a critical character. In the present case, "none of the above" was alternative 8, which obtained a better result than two of the real candidates.

4.4 Finally, we look at an example of approval voting. Specifically, we consider the 2006 Public Choice Society election [*4*]. Besides an approval vote, here the voters were also asked for a preferential vote "in the spirit of research on public choice". However, here we will limit ourselves to the approval vote, which was the official one. The vote had a participation of $V = 37$ voters, most of which approved more than one candidate.

The actual votes are listed in the following table,[1] where we give not only the approval voting data but also the associated preferential votes. The approved candidates are the ones which lie at the left of the slash.

[1]We are grateful to Prof. Steven J. Brams, who was the president of the Public Choice Society when that election took place, for his kind permission to reproduce these data.

$A \succ B\,/$	$A\,/$	$C\,/\!\succ B \succ D \succ A \succ E$
$A \succ C \succ B\,/$	$A\,/$	$C\,/$
$D\,/\!\succ A \succ B \succ E \succ C$	$D\,/\!\succ A \sim B \sim C \sim E$	$D \sim E\,/\!\succ A \succ B \sim C$
$B \succ A\,/\!\succ D \succ C \succ E$	$A \sim C\,/$	$B\,/\!\succ C \succ A \succ D \succ E$
$D \succ A \succ B \succ C\,/\!\succ E$	$/\,B \succ E \succ A \succ D \succ C$	$D \succ C \succ E\,/$
$C \succ B \succ A\,/$	$A \sim B \sim E\,/$	$C\,/\!\succ A \succ B \sim D \sim E$
$E\,/\!\succ D$	$A \sim B \sim C \sim D \sim E\,/$	$C\,/$
$C \succ A \succ B \succ E\,/$	$D \succ A \succ B\,/$	$B \succ D\,/\!\succ E \succ C \succ A$
$D \succ E\,/\!\succ C \succ A \succ B$	$B \succ D \succ A\,/\!\succ C \succ E$	$B \succ C\,/\!\succ A \succ E \succ D$
$E\,/$	$A\,/\!\succ B \succ E \succ C \succ D$	$D \succ A \succ C \succ B\,/$
$B \succ C\,/$	$D\,/$	$D \succ E\,/\!\succ A \succ B$
$D \succ C\,/\!\succ B \succ E \succ A$	$A \sim C \succ B\,/\!\succ D \succ E$	
$B\,/$	$A\,/\!\succ D \succ B \succ C \succ E$	

The approval voting scores are the following: A: 17, B: 16, C: 17, D: 14, D: 9. So according to approval voting there was a tie between candidates A and C, which were followed at a minimum distance by candidate B.

The CLC computations are as follows:

	V_{xy}				
x	A	B	C	D	E
A	$*$	$12\frac{1}{2}$	11	14	$15\frac{1}{2}$
B	$11\frac{1}{2}$	$*$	12	$13\frac{1}{2}$	$14\frac{1}{2}$
C	11	13	$*$	$14\frac{1}{2}$	$15\frac{1}{2}$
D	11	$11\frac{1}{2}$	$11\frac{1}{2}$	$*$	$11\frac{1}{2}$
E	$7\frac{1}{2}$	$7\frac{1}{2}$	$7\frac{1}{2}$	$6\frac{1}{2}$	$*$

	V^*_{xy}					
x	A	B	C	D	E	κ
A	$*$	$\mathbf{12\frac{1}{2}}$	**12**	**14**	$\mathbf{15\frac{1}{2}}$	1
B	$11\frac{1}{2}$	$*$	12	$\mathbf{13\frac{1}{2}}$	$\mathbf{14\frac{1}{2}}$	3
C	$11\frac{1}{2}$	**13**	$*$	$\mathbf{14\frac{1}{2}}$	$\mathbf{15\frac{1}{2}}$	2
D	$11\frac{1}{2}$	$11\frac{1}{2}$	$11\frac{1}{2}$	$*$	$\mathbf{11\frac{1}{2}}$	4
E	$7\frac{1}{2}$	$7\frac{1}{2}$	$7\frac{1}{2}$	$7\frac{1}{2}$	$*$	5

	M^{ν}_{xy}				
x	A	C	B	D	E
A	$*$	$\frac{1}{2}$	1	$2\frac{1}{2}$	8
C	$*$	$*$	1	3	8
B	$*$	$*$	$*$	2	7
D	$*$	$*$	$*$	$*$	4
E	$*$	$*$	$*$	$*$	$*$

	T_{xy}				
x	A	C	B	D	E
A	$*$	22	24	25	23
C	$*$	$*$	25	26	23
B	$*$	$*$	$*$	25	22
D	$*$	$*$	$*$	$*$	18
E	$*$	$*$	$*$	$*$	$*$

x	M^σ_{xy} A	C	B	D	E
A	$*$	$\frac{1}{2}$	1	$2\frac{1}{2}$	8
C	$*$	$*$	1	$2\frac{1}{2}$	8
B	$*$	$*$	$*$	2	7
D	$*$	$*$	$*$	$*$	4
E	$*$	$*$	$*$	$*$	$*$

x	T^σ_{xy} A	C	B	D	E
A	$*$	$24\frac{1}{2}$	$24\frac{1}{2}$	$24\frac{1}{2}$	$22\frac{7}{8}$
C	$*$	$*$	$24\frac{1}{2}$	$24\frac{1}{2}$	$22\frac{3}{8}$
B	$*$	$*$	$*$	$24\frac{1}{2}$	$21\frac{3}{8}$
D	$*$	$*$	$*$	$*$	$19\frac{3}{8}$
E	$*$	$*$	$*$	$*$	$*$

x	M^π_{xy} A	C	B	D	E
A	$*$	$\frac{1}{2}$	1	2	5.56
C	$*$	$*$	1	2	5.56
B	$*$	$*$	$*$	2	5.56
D	$*$	$*$	$*$	$*$	4
E	$*$	$*$	$*$	$*$	$*$

x	T^π_{xy} A	C	B	D	E
A	$*$	$24\frac{1}{2}$	$24\frac{1}{2}$	$24\frac{1}{2}$	20.94
C	$*$	$*$	$24\frac{1}{2}$	$24\frac{1}{2}$	20.94
B	$*$	$*$	$*$	$24\frac{1}{2}$	20.94
D	$*$	$*$	$*$	$*$	$19\frac{3}{8}$
E	$*$	$*$	$*$	$*$	$*$

x	V^π_{xy} A	C	B	D	E
A	$*$	$12\frac{1}{2}$	$12\frac{3}{4}$	$13\frac{1}{4}$	$13\frac{1}{4}$
C	12	$*$	$12\frac{3}{4}$	$13\frac{1}{4}$	$13\frac{1}{4}$
B	$11\frac{3}{4}$	$11\frac{3}{4}$	$*$	$13\frac{1}{4}$	$13\frac{1}{4}$
D	$11\frac{1}{4}$	$11\frac{1}{4}$	$11\frac{1}{4}$	$*$	11.69
E	7.69	7.69	7.69	7.69	$*$

x	r_x	φ_x
A	3.6014	0.2315
C	3.6149	0.2276
B	3.6486	0.2181
D	3.7720	0.1928
E	4.1689	0.1299

So, the winner by the CLC method is candidate A. However, this is true only for the main variant. For the other three variants (dual, balanced and margin-based) the result is a tie between A and C, in full agreement with the approval voting scores. In § 17 we will see that the margin-based variant always gives such a full agreement.

Remark. In all of the preceding examples, the matrix of the indirect scores has a constant row which corresponds to the loser. However, it is not always so.

5 Some terminology and notation

We consider a finite set A. Its elements represent the options which are the matter of a vote. The number of elements of A will be denoted by N. We will be particularly concerned with (binary) **relations** on A. Stating that two elements a and b are in a certain relation ρ is equivalent to saying that the (ordered) **pair** formed by these two elements is a member of a certain set ρ. The pair formed by a and b, in this order, will be denoted simply as ab.

The pairs that consist of two copies of the same element, i. e. those of the form aa, are not relevant for our purposes. So, we will systematically exclude them from our relations. This can be viewed as a sort of normalization. The set of all proper pairs, i. e. the pairs ab with $a \neq b$, will be denoted as $\mathit{\Pi}$, or if necessary as $\mathit{\Pi}(A)$. So, we will restrict our attention to relations contained in $\mathit{\Pi}$ (such relations are sometimes called "strict", or "irreflexive"). In particular, the relation that includes the whole of $\mathit{\Pi}$ will be called **complete tie**.

A relation $\rho \subseteq \mathit{\Pi}$ will be called:

- **total**, or complete, when at least one of $ab \in \rho$ and $ba \in \rho$ holds for every pair ab.
- **antisymmetric** when $ab \in \rho$ and $ba \in \rho$ cannot occur simultaneously.
- **transitive** when the simultaneous occurrence of $ab \in \rho$ and $bc \in \rho$ implies $ac \in \rho$.
- a **partial order**, when it is at the same time transitive and antisymmetric.
- a **total order**, or **strict ranking**, when it is at the same time transitive, antisymmetric and total.
- a **complete ranking** when it is at the same time transitive and total.
- a **truncated ranking** when it consists of a complete ranking on a subset X of A together with all pairs ab with $a \in X$ and $b \notin X$.

For every relation $\rho \subseteq \mathit{\Pi}$, we will denote by ρ' the relation that consists of all pairs of the form ab where $ba \in \rho$; ρ' will be called the **converse** of ρ. On the other hand, we will denote by $\bar{\rho}$ the relation that consists of all pairs ab for which $ab \notin \rho$; $\bar{\rho}$ will be called the **complement** of ρ. For certain purposes, it will be useful to consider also the relation $\hat{\rho}$ given by the complement of the converse of ρ, or equivalently by the converse of its complement. So, $ab \in \hat{\rho}$ if and only if $ba \notin \rho$. This relation will be called the **adjoint** of ρ. This operation will be used mainly in § 8, in connection with the indirect comparison relation $\nu = \mu(v^*)$. The following proposition collects several properties which are immediate consequences of the definitions:

Lemma 5.1.

(a) $\hat{\hat{\rho}} = \rho$.

(b) $\rho \subset \sigma \iff \hat{\sigma} \subset \hat{\rho}$.

(c) *ρ is antisymmetric* $\iff \rho \subseteq \hat{\rho} \iff$ *$\hat{\rho}$ is total.*

(d) *ρ is total* $\iff \hat{\rho} \subseteq \rho \iff$ *$\hat{\rho}$ is antisymmetric.*

Besides pairs, we will be concerned also with longer sequences $a_0 a_1 \dots a_n$. They will be referred to as **paths**, and in the case $a_n = a_0$ they are called **cycles**. When $a_i a_{i+1} \in \rho$ for every i, we will say that the path $a_0 a_1 \dots a_n$ is **contained in** ρ, and also that a_0 and a_n are **indirectly related through** ρ. When ρ is transitive, the condition "a is indirectly related to b through ρ" implies $ab \in \rho$. In general, however, it defines a new relation, which is called the **transitive closure** of ρ, and will be denoted by ρ^*; this is the minimum transitive relation that contains ρ. The transitive-closure operator is easily seen to have the following properties: $\rho^* \subseteq \sigma^*$ whenever $\rho \subseteq \sigma$; $(\rho \cap \sigma)^* \subseteq (\rho^*) \cap (\sigma^*)$; $(\rho^*) \cup (\sigma^*) \subseteq (\rho \cup \sigma)^*$; $(\rho^*)^* = \rho^*$. On the other hand, one can easily check that

Lemma 5.2. *The transitive closure ρ^* is antisymmetric if and only if ρ contains no cycle. More specifically, $ab, ba \in \rho^*$ if and only if ρ contains a cycle that includes both a and b.*

A subset $C \subseteq A$ will be said to be a **cluster** for a relation ρ when, for any $x \notin C$, having $ax \in \rho$ for some $a \in C$ implies $bx \in \rho$ for any $b \in C$, and similarly, having $xa \in \rho$ for some $a \in C$ implies $xb \in \rho$ for any $b \in C$. On the other hand, $C \subseteq A$ will be said to be an **interval** for a relation ρ when the simultaneous occurrence of $ax \in \rho$ and $xb \in \rho$ with $a, b \in C$ implies $x \in C$. The following facts are easy consequences of the definitions: If ρ is antisymmetric and C is a cluster for ρ then C is also an interval for ρ. If ρ is total and C is an interval for ρ then C is also a cluster for ρ. As a corollary, if ρ is total and antisymmetric, then C is a cluster for ρ if and only if it is an interval for that relation. Later on we will make use of the following fact, which is also an easy consequence of the definitions:

Lemma 5.3. *The following conditions are equivalent to each other:*

(a) *C is a cluster for ρ.*

(b) *C is a cluster for $\hat{\rho}$.*

(c) *The simultaneous occurrence of $ax \in \rho$ and $xb \in \hat{\rho}$ with $a, b \in C$ implies $x \in C$, and similarly, the simultaneous occurrence of $ax \in \hat{\rho}$ and $xb \in \rho$ with $a, b \in C$ implies also $x \in C$.*

When C is a cluster for ρ, it will be useful to consider a new set $\widetilde{A}$ and a new relation $\widetilde{\rho}$ defined in the following way: $\widetilde{A}$ is obtained from A by replacing the set C by a single element $\widetilde{c}$, i.e. $\widetilde{A} = (A \setminus C) \cup \{\widetilde{c}\}$; for $x, y \in A \setminus C$, $x\widetilde{c} \in \widetilde{\rho}$ if and only if there exists $c \in C$ such that $xc \in \rho$, $\widetilde{c}y \in \widetilde{\rho}$ if and only if there exists $c \in C$ such that $cy \in \rho$, and finally, $xy \in \widetilde{\rho}$ if and only if $xy \in \rho$. We will refer to this operation as the **contraction** of ρ by the cluster C. If ρ is a strict ranking (resp. a complete ranking) on A, then $\widetilde{\rho}$ is a strict ranking (resp. a complete ranking) on $\widetilde{A}$.

Given a relation ρ, we will associate every element x with the following sets:

- the **set of predecessors**, P_x, i.e. the set of $y \in A$ such that $yx \in \rho$.
- the **set of successors**, S_x, i.e. the set of $y \in A$ such that $xy \in \rho$.
- the **set of collaterals**, C_x, i.e. the set of $y \in A \setminus \{x\}$ which are neither predecessors nor successors of x in ρ.

The sets P_x, S_x and C_x are especially meaningful when the relation ρ is a partial order. In that case, and it is quite natural to rank the elements of A by their number of predecessors, or by the number of elements which are not their successors. More precisely, it makes sense to define the **rank** of x in ρ as

$$\kappa_x = 1 + |\mathsf{P}_x| + \vartheta\,|\mathsf{C}_x| = 1 + (1-\vartheta)\,|\mathsf{P}_x| + \vartheta\,(N - 1 - |\mathsf{S}_x|), \tag{44}$$

where ϑ is a fixed number in the interval $0 \leq \vartheta \leq 1$. If we do not say otherwise, we will take $\vartheta = 1/2$. The following facts are easy consequences of the definitions:

Lemma 5.4. *Assume that ρ is a partial order. In that case, having $xy \in \rho$ implies the following facts: $\mathsf{P}_x \subset \mathsf{P}_y$, $\mathsf{S}_x \supset \mathsf{S}_y$ (both inclusions are strict), and $\kappa_x < \kappa_y$ (for any ϑ in the interval $0 \leq \vartheta \leq 1$). For $\vartheta = 1/2$, the average of the numbers κ_x is equal to $(N+1)/2$. If ρ is a total order, then κ_x does not depend on ϑ; furthermore, having $xy \in \rho$ is then equivalent to $\kappa_x < \kappa_y$.*

As in § 2.2, given a set of binary scores s_{xy}, we denote by $\mu(s)$ the corresponding **comparison relation**:

$$xy \in \mu(s) \quad \equiv \quad s_{xy} > s_{yx}. \tag{45}$$

For such a relation, the adjoint $\hat{\mu}(s)$ corresponds to replacing the strict inequality by the non-strict one.

6 The indirect scores and its comparison relation

Let us recall that the indirect scores v^*_{xy} are defined in the following way:

$$v^*_{xy} \;=\; \max\{v_\alpha \mid \alpha \text{ is a path } x_0x_1\ldots x_n \text{ from } x_0 = x \text{ to } x_n = y\},$$

where the score v_α of a path $\alpha = x_0x_1\ldots x_n$ is defined as

$$v_\alpha \;=\; \min\{v_{x_ix_{i+1}} \mid 0 \le i < n\}.$$

In the following statements, and the similar ones which appear elsewhere, "any x, y, z" should be understood as meaning "any x, y, z which are pairwise different from each other".

Remark. The matrix of indirect scores v^* can be viewed as a power of v (supplemented with $v_{xx} = 1$) for a matrix product defined in the following way: $(vw)_{xz} = \max_y \min(v_{xy}, w_{yz})$. More precisely, v^* coincides with such a power for any exponent greater than or equal to $N - 1$.

Lemma 6.1. *The indirect scores satisfy the following inequalities:*

$$v^*_{xz} \ge \min(v^*_{xy}, v^*_{yz}) \quad \textit{for any } x, y, z. \tag{46}$$

Proof. Let α be a path from x to y such that $v^*_{xy} = v_\alpha$; let β be a path from y to z such that $v^*_{yz} = v_\beta$. Consider now their concatenation $\alpha\beta$. Since $\alpha\beta$ goes from x to z, one has $v^*_{xz} \ge v_{\alpha\beta}$. On the other hand, the definition of the score of a path ensures that $v_{\alpha\beta} = \min(v_\alpha, v_\beta)$. Putting these things together gives the desired result. □

The following lemma is somehow a converse of the preceding one:

Lemma 6.2. *Assume that the original scores satisfy the following inequalities:*

$$v_{xz} \ge \min(v_{xy}, v_{yz}) \quad \textit{for any } x, y, z. \tag{47}$$

In that case, the indirect scores coincide with the original ones.

Proof. The inequality $v^*_{xz} \ge v_{xz}$ is an immediate consequence of the definition of v^*_{xz}. The converse inequality can be obtained in the following way: Let $\gamma = x_0x_1x_2\ldots x_n$ be a path from x to z such that $v^*_{xz} = v_\gamma$. By virtue of (47), we have

$$\min\left(v_{x_0x_1}, v_{x_1x_2}, v_{x_2x_3}, \ldots, v_{x_{n-1}x_n}\right) \;\le\; \min\left(v_{x_0x_2}, v_{x_2x_3}, \ldots, v_{x_{n-1}x_n}\right).$$

So, $v^*_{xz} \le v_{\gamma'}$ where $\gamma' = x_0x_2\ldots x_n$. By iteration, one eventually gets $v^*_{xz} \le v_{xz}$. □

Theorem 6.3 (Schulze, 1998 [*33 b*])**.** *$\mu(v^*)$ is a transitive relation.*

Proof. We will argue by contradiction. Let us assume that $xy \in \mu(v^*)$ and $yz \in \mu(v^*)$, but $xz \notin \mu(v^*)$. This means respectively that (a) $v^*_{xy} > v^*_{yx}$ and (b) $v^*_{yz} > v^*_{zy}$, but (c) $v^*_{zx} \geq v^*_{xz}$. On the other hand, Lemma 6.1 ensures also that (d) $v^*_{xz} \geq \min(v^*_{xy}, v^*_{yz})$. We will distinguish two cases depending on which of the two last quantities is smaller: (i) $v^*_{yz} \geq v^*_{xy}$; (ii) $v^*_{xy} \geq v^*_{yz}$.

Case (i): $v^*_{yz} \geq v^*_{xy}$. We will see that in this case (c) and (d) entail a contradiction with (a). In fact, we have the following chain of inequalities: $v^*_{yx} \geq \min(v^*_{yz}, v^*_{zx}) \geq \min(v^*_{yz}, v^*_{xz}) \geq \min(v^*_{yz}, v^*_{xy}) = v^*_{xy}$, where we are using successively: Lemma 6.1, (c), (d) and (i).

Case (ii): $v^*_{xy} \geq v^*_{yz}$. An entirely analogous argument shows that in this case (c) and (d) entail a contradiction with (b). In fact, we have $v^*_{zy} \geq \min(v^*_{zx}, v^*_{xy}) \geq \min(v^*_{xz}, v^*_{xy}) \geq \min(v^*_{yz}, v^*_{xy}) = v^*_{yz}$, where we are using successively: Lemma 6.1, (c), (d) and (ii). □

7 Restricted paths

In this section we consider paths restricted to either $\mu(v)$ or $\hat{\mu}(v)$. Such restricted paths allow to achieve not only the majority principle I1, but also the Condorcet principle I1′. In exchange, however, this idea can hardly be made into a continuous rating method, since one is doing quite different things depending on whether $v_{xy} > v_{yx}$ or $v_{xy} < v_{yx}$. Even so, we will see that in the complete case —where I1 is equivalent to I1′— the indirect comparison relations which are obtained under such restrictions coincide with the one which is obtained when arbitrary paths are used. More specifically, we will look at the comparison relations associated with u^*_{xy} and w^*_{xy}, where u_{xy} and w_{xy} are defined as

$$u_{xy} = \begin{cases} v_{xy}, & \text{if } v_{xy} > v_{yx}, \\ 0, & \text{otherwise;} \end{cases} \qquad w_{xy} = \begin{cases} v_{xy}, & \text{if } v_{xy} \geq v_{yx}, \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

Proposition 7.1.
(a) $\mu(u^*) \subseteq \mu^*(v)$.
(b) $\mu(w^*) \subseteq \hat{\mu}^*(v)$.

Proof. Part (a). Let us begin by recalling that $\mu^*(v)$ means the transitive closure of $\mu(v)$. Let us assume that $xy \in \mu(u^*)$, i. e. $u^*_{xy} > u^*_{yx}$. Since we are dealing with non-negative numbers, this ensures that $u^*_{xy} > 0$. By the definition of u^*_{xy}, this implies the existence of a path $x_0x_1\dots x_n$ from $x_0 = x$ to $x_n = y$ such that $u_{x_ix_{i+1}} > 0$ for all i. According to (48.1), this ensures that $v_{x_ix_{i+1}} > v_{x_{i+1}x_i}$, i. e. $x_ix_{i+1} \in \mu(v)$, for all i. Therefore, $xy \in \mu^*(v)$. An entirely analogous argument proves part (b). □

Lemma 7.2.

(a) $u^*_{xy} \le w^*_{xy} \le v^*_{xy}$.

(b) $v^*_{xy} > 1/2 \implies u^*_{xy} = w^*_{xy} = v^*_{xy}$.

(c) $v^*_{xy} = 1/2 \implies w^*_{xy} = v^*_{xy}$.

In the complete case one has:

(d) $v^*_{xy} < 1/2 \implies u^*_{xy} = w^*_{xy} = 0$.

(e) $v^*_{xy} = 1/2 \implies u^*_{xy} = 0$.

Proof. Part (a). It is simply a matter of noticing that $u_{xy} \le w_{xy} \le v_{xy}$ and checking that the inequality $p_{xy} \le q_{xy}$ for all x, y implies $p^*_{xy} \le q^*_{xy}$ for all x, y. As an intermediate result towards this implication, one can see that $p_\gamma \le q_\gamma$ for all paths γ. In fact, if $\gamma = x_0x_1\dots x_n$ and i is such that $q_\gamma = q_{x_ix_{i+1}}$, the definition of p_γ and the inequality between p_{xy} and q_{xy} give $p_\gamma \le p_{x_ix_{i+1}} \le q_{x_ix_{i+1}} = q_\gamma$. The second step of that implication uses an analogous argument: if γ is a path from x to y such that $p^*_{xy} = p_\gamma$, we can write $p^*_{xy} = p_\gamma \le q_\gamma \le q^*_{xy}$, where we are using the intermediate result and the definition of q^*_{xy}.

Part (b). Let $\gamma = x_0x_1\dots x_n$ be a path from x to y such that $v^*_{xy} = v_\gamma$. Since $v^*_{xy} > 1/2$, every link of that path satisfies $v_{x_ix_{i+1}} > 1/2$, which implies that $v_{x_ix_{i+1}} > v_{x_{i+1}x_i}$ (because $v_{x_ix_{i+1}} + v_{x_{i+1}x_i} \le 1$). Now, that inequality entails that $u_{x_ix_{i+1}} = v_{x_ix_{i+1}}$, from which it follows that $u_\gamma = v_\gamma$. Finally, it suffices to combine these facts with the inequality $u_\gamma \le u^*_{xy}$ and the inequalities of part (a):

$$v^*_{xy} = v_\gamma = u_\gamma \le u^*_{xy} \le w^*_{xy} \le v^*_{xy}.$$

Part (c). The proof is similar to that of part (b). Here we deal with the non-strict inequality $v_{x_ix_{i+1}} \ge 1/2$, which entails $v_{x_ix_{i+1}} \ge v_{x_{i+1}x_i}$ and $w_{x_ix_{i+1}} = v_{x_ix_{i+1}}$. These facts allow to conclude that

$$v^*_{xy} = v_\gamma = w_\gamma \le w^*_{xy} \le v^*_{xy}.$$

Part (d). The hypothesis that $v^*_{xy} < 1/2$ means that for every path $\gamma = x_0x_1\dots x_n$ from x to y there exists at least one i such that $v_{x_ix_{i+1}} < 1/2$. By the assumption of completeness, this implies that $v_{x_ix_{i+1}} < v_{x_{i+1}x_i}$, so that

$u_{x_ix_{i+1}} = w_{x_ix_{i+1}} = 0$. This implies that $u_\gamma = w_\gamma = 0$. Since γ is arbitrary, it follows that $u^*_{xy} = w^*_{xy} = 0$.

Part (e). The proof is similar to that of part (d). Here we deal with the non-strict inequality $v_{x_ix_{i+1}} \leq 1/2$, which implies $v_{x_ix_{i+1}} \leq v_{x_{i+1}x_i}$ and $u_{x_ix_{i+1}} = 0$. This holds for at least one link of every path γ from x to y. So, $u^*_{xy} = 0$. □

Theorem 7.3. *In the complete case one has $\mu(u^*) = \mu(w^*) = \mu(v^*)$.*

Proof. It suffices to prove the three following statements:

$$v^*_{xy} > v^*_{yx} \implies u^*_{xy} > u^*_{yx} \quad \text{and} \quad w^*_{xy} > w^*_{yx} \tag{49}$$

$$w^*_{xy} > w^*_{yx} \implies v^*_{xy} > v^*_{yx}, \tag{50}$$

$$u^*_{xy} > u^*_{yx} \implies v^*_{xy} > v^*_{yx}, \tag{51}$$

Proof of (49). By combining the completeness assumption with the hypothesis of (49) we can write $1 = v_{xy} + v_{yx} \leq v^*_{xy} + v^*_{yx} < 2v^*_{xy}$, so that $v^*_{xy} > 1/2$. According to part (b) of Lemma 7.2, this inequality implies that $u^*_{xy} = w^*_{xy} = v^*_{xy}$. On the other hand, part (a) of the same lemma ensures that $u^*_{yx} \leq w^*_{yx} \leq v^*_{yx}$. By combining these facts with the hypothesis of (49) we obtain the right-hand side of it.

Proof of (50). Here we begin by noticing that the left-hand side implies $w^*_{xy} > 0$, which by part (d) of Lemma 7.2 entails $v^*_{xy} \geq 1/2$. If $v^*_{yx} < 1/2$, we are finished. If, on the contrary, $v^*_{yx} \geq 1/2$, then parts (a), (b) and (c) of Lemma 7.2 allow to conclude that $v^*_{yx} = w^*_{yx} < w^*_{xy} \leq v^*_{xy}$.

Proof of (51). Similarly to above, the left-hand side implies $u^*_{xy} > 0$, which by parts (d) and (e) of Lemma 7.2 entails $v^*_{xy} > 1/2$. If $v^*_{yx} \leq 1/2$, we are finished. If, on the contrary, $v^*_{yx} > 1/2$, then parts (a) and (b) of Lemma 7.2 allow to conclude that $v^*_{yx} = u^*_{yx} < u^*_{xy} \leq v^*_{xy}$. □

8 Admissible orders

Let us recall that an admissible order is a total order ξ such that $\nu \subseteq \xi \subseteq \hat{\nu}$. Here ν is the indirect comparison relation $\nu = \mu(v^*)$. So $xy \in \nu$ if and only if $m^\nu_{xy} = v^*_{xy} - v^*_{yx} > 0$, and $xy \in \hat{\nu}$ if and only if $m^\nu_{xy} \geq 0$.

Lemma 8.1. *Assume that ρ is an antisymmetric and transitive relation. If ρ contains neither xy nor yx, then $(\rho \cup \{xy\})^*$ is also antisymmetric.*

Proof. We will proceed by contradiction. According to Lemma 5.2, if $(\rho \cup \{xy\})^*$ were not antisymmetric, $\rho \cup \{xy\}$ would contain a cycle γ. On the other hand, the hypotheses on ρ ensure, by the same lemma, that ρ contains no cycles. Therefore, γ must involve the pair xy. By following this cycle from one ocurrence of the pair xy until the next ocurrence of x, one obtains a path from y to x which is contained in ρ. But, since ρ is transitive, this entails that $yx \in \rho$, which contradicts one of the hypotheses. □

Theorem 8.2. *Given a transitive antisymmetric relation ρ on a finite set A, one can always find a total order ξ such that $\rho \subseteq \xi \subseteq \hat{\rho}$. If ρ contains neither xy nor yx, one can constrain ξ to include the pair xy.*

Proof. If ρ is total, it suffices to take $\xi = \rho$ (notice that $\hat{\rho} = \rho$ because of statements (c) and (d) of Lemma 5.1). Otherwise, let us consider the relation $\rho_1 = (\rho \cup \{xy\})^*$, where xy is any pair such that ρ contains neither xy nor yx. According to Lemma 8.1, ρ_1 is antisymmetric. Furthermore, it is obvious that $\rho \subset \rho_1$. Therefore, the statements (b) and (c) of Lemma 5.1 ensure that $\rho \subset \rho_1 \subseteq \hat{\rho}_1 \subset \hat{\rho}$. From here, one can repeat the same process with ρ_1 substituted for ρ: if ρ_1 is total we take $\xi = \rho_1$; otherwise we consider $\rho_2 = (\rho_1 \cup \{x_1y_1\})^*$, where x_1y_1 is any pair such that ρ_1 contains neither x_1y_1 nor y_1x_1, and so on. This iteration will conclude in a finite number of steps since A is finite. □

Corollary 8.3. *One can always find an admissible order ξ.*

Proof. It follows from Theorem 8.2 because $\nu = \mu(v^*)$ is certainly antisymmetric and Theorem 6.3 ensures that it is transitive. □

Later on we will make use of the following fact:

Theorem 8.4. *Given a transitive antisymmetric relation ρ on a finite set A and a set C which is a cluster for ρ, one can always find a total order ξ such that $\rho \subseteq \xi \subseteq \hat{\rho}$ and such that C is a cluster for ξ.*

Proof. As in the proof of Theorem 8.2, we will progressively extend ρ until we get a total order. Here, we will take care that besides being transitive and antisymmetric, the successive extensions ρ_i keep the property that C be a cluster for ρ_i. To this effect, the successive additions to ρ will follow a certain specific order, and we will make an extensive use of the necessary and sufficient condition given by Lemma 5.3.

In a first phase we will deal with pairs of the form cd with $c, d \in C$. Let us assume that neither cd nor dc is contained in ρ, and let us consider $\rho_1 = (\rho \cup \{cd\})^*$. Besides the properties mentioned in the proof of Theorem 8.2, we claim that this relation has the property that C is a cluster for ρ_1. According to Lemma 5.3, it suffices to check that the simultaneous occurrence of $ax \in \rho_1$ and $xb \in \hat{\rho}_1$ with $a, b \in C$ implies $x \in C$, and similarly, that the simultaneous occurrence of $ax \in \hat{\rho}_1$ and $xb \in \rho_1$ with $a, b \in C$ implies also $x \in C$. So, let us assume first that $ax \in \rho_1$ and $xb \in \hat{\rho}_1$ with $a, b \in C$. Since $\rho \subset \rho_1$, we have $xb \in \hat{\rho}$ (because $\hat{\rho}_1 \subset \hat{\rho}$). If $ax \in \rho$, we immediately get $x \in C$ since C is known to be a cluster for ρ (Lemma 5.3). Otherwise, i. e. if $ax \in \rho_1 \setminus \rho$, we see that ρ_1 contains a path of the form $\gamma = a \ldots cd \ldots x$. But this entails the existence of a path from d to x contained in ρ. So, by transitivity, $dx \in \rho$. Again, this fact together with $xb \in \hat{\rho}$ ensures that $x \in C$ since C is known to be a cluster for ρ. A similar argument takes care of the case where $ax \in \hat{\rho}_1$ and $xb \in \rho_1$ with $a, b \in C$.

By repeating the same process we will eventually get an extension of ρ with the same properties plus the following one: it includes either cd or dc for any $c, d \in C$. In other words, its restriction to C is a total order. In the following, this relation will be denoted by η.

Now we will deal with pairs of the form cq or qc with $c \in C$ and $q \notin C$. Let us assume that neither cq nor qc belong to η. In this case we will proceed by taking $\eta_1 = (\eta \cup \{\ell q\})^*$, where ℓ denotes the last element of C according to the total order determined by η (alternatively, one could take $\eta_1 = (\eta \cup \{qf\})^*$, where f denotes the first element of C by η). By so doing, we make sure that η_1 contains all pairs of the form zq with $z \in C$. As a consequence, C will keep the property of being a cluster for η_1. In fact, let us assume, in the lines of Lemma 5.3, that $ax \in \eta_1$ and $xb \in \hat{\eta}_1$ with $a, b \in C$. The hypothesis that $ax \in \eta_1$ can be divided in two cases, namely either $ax \in \eta$ or $ax \in \eta_1 \setminus \eta$. Let us consider first the case $ax \in \eta_1 \setminus \eta$. By the definition of η_1, this means that η_1 contains a path of the form $\gamma = a \ldots \ell q \ldots x$, whose final part shows that $qx \in \eta_1$. On the other hand, we know that η_1 contains bq (since $b \in C$). By transitivity, this entails $bx \in \eta_1$ and therefore $xb \notin \hat{\eta}_1$, in contradiction with the hypothesis that $xb \in \hat{\eta}_1$. So, the only possibility of having $ax \in \eta_1$ and $xb \in \hat{\eta}_1$ is $ax \in \eta$. Besides, $xb \in \hat{\eta}_1$ implies that $xb \in \hat{\eta}$. So $x \in C$ because C is a cluster for η (Lemma 5.3). Let us assume now that $ax \in \hat{\eta}_1$ and $xb \in \eta_1$. Like before, the former implies $ax \in \hat{\eta}$. Again, the hypothesis that $xb \in \eta_1$ can be divided in two cases, namely either $xb \in \eta$ or $xb \in \eta_1 \setminus \eta$. In the first case we have $ax \in \hat{\eta}$ and $xb \in \eta$. So $x \in C$ because C is a cluster for η. In the second case we can still use the same argument since η_1 contains a path of the form $\gamma = x \dots \ell q \dots b$, which shows that $x\ell \in \eta$.

By repeating the same process we will eventually get an extension of η with the same properties plus the following one: it includes either cq or qc for any $c \in C$ and $q \notin C$.

Finally, it rests to deal with any pairs of the form pq with $p, q \notin C$. However, these pairs do not cause any problems since they do not appear in the definition of C being a cluster. □

In practice, one can easily obtain admissible orders by suitably arranging the elements of A according to their number of victories, ties and defeats against the others according to the indirect comparison relation ν. More precisely, it suffices to arrange the elements of A by non-decreasing values of their rank κ_x in ν as defined in (44). According to the the particular nature of ν and the definitions given in § 5, the sets P_x, S_x and C_x which appear in (44) are given by

$$\mathsf{P}_x = \{\, y \mid y \neq x,\ m^{\nu}_{xy} < 0 \,\}, \tag{52}$$

$$\mathsf{S}_x = \{\, y \mid y \neq x,\ m^{\nu}_{xy} > 0 \,\}, \tag{53}$$

$$\mathsf{C}_x = \{\, y \mid y \neq x,\ m^{\nu}_{xy} = 0 \,\}. \tag{54}$$

So, ranking by κ_x amounts to applying the Copeland rule to the tournament defined by the indirect comparison relation $\nu = \mu(v^*)$ (see for instance [*37*: p. 206–209]).

Proposition 8.5. *Any total ordering of the elements of A by non-decreasing values of $\kappa_x(\nu)$ is an admissible order. This is true for any fixed value of ϑ in the interval $0 \leq \vartheta \leq 1$.*

Proof. Let ξ be a total order of A for which $x \mapsto \kappa_x$ does not decrease. This means that

$$xy \in \xi \implies \kappa_x \leq \kappa_y,$$

or equivalently,

$$\kappa_y < \kappa_x \implies xy \notin \xi.$$

Furthermore, the total character of ξ allows to derive that

$$\kappa_y < \kappa_x \implies yx \in \xi.$$

On the other hand, we know by Theorem 6.3 that $\nu = \mu(v^*)$ is transitive. As a consequence, by Lemma 5.4, $xy \in \nu$ implies $\kappa_x < \kappa_y$. By combining this with the preceding implication (with x and y interchanged with each other), we get that $\nu \subseteq \xi$. In order to complete the proof that ξ is admissible, we must check that $\xi \subseteq \hat{\nu}$, or equivalently, that $xy \notin \hat{\nu}$ implies $xy \notin \xi$. This is true because of the following chain of implications:

$$xy \notin \hat{\nu} \iff yx \in \nu \implies \kappa_y < \kappa_x \implies xy \notin \xi,$$

where we used respectively the definition of $\hat{\nu}$, Lemma 5.4, and the hypothesis that κ_x does not decrease along ξ. □

In the following section we will make use of the following fact:

Lemma 8.6. *Given two admissible orders ξ and $\widetilde{\xi}$, one can find a sequence of admissible orders ξ_i $(i = 0 \dots n)$ such that $\xi_0 = \xi$, $\xi_n = \widetilde{\xi}$, and such that ξ_{i+1} differs from ξ_i only by the transposition of two consecutive elements.*

Proof. Given two total orders ρ and σ, we will denote as $d(\rho, \sigma)$ the number of pairs ab such that $ab \in \rho \setminus \sigma$. Obviously, $\rho = \sigma$ if and only if $d(\rho, \sigma) = 0$. Furthermore, we will say that ab is a *consecutive* pair in ρ whenever $ab \in \rho$ and there is no $x \in A$ such that $ax, xb \in \rho$. If all pairs ab which are consecutive in ξ belong to $\widetilde{\xi}$, the transitivity of $\widetilde{\xi}$ allows to derive that $\xi \subseteq \widetilde{\xi}$; furthermore, the fact that all total orders on the finite set A have the same number of pairs allows to conclude that $\xi = \widetilde{\xi}$. So, if $\widetilde{\xi} \neq \xi$, there must be some pair ab which is consecutive in ξ but it does not belong to $\widetilde{\xi}$. Since ab belongs to the admissible order ξ and ba belongs to the admissible order $\widetilde{\xi}$, it follows that $m^{\nu}_{ab} = 0$. Let us take as ξ_1 the total order which differs from ξ only by the transposition of the two consecutive elements a and b; i.e. $\xi_1 = (\xi \setminus \{ab\}) \cup \{ba\}$. This order is admissible since ξ is so and $m^{\nu}_{ab} = 0$. Obviously, $d(\xi_1, \widetilde{\xi}) = d(\xi, \widetilde{\xi}) - 1$. From here, one can repeat the same process with ξ_1 substituted for ξ: if ξ_1 still differs from $\widetilde{\xi}$ we take $\xi_2 = (\xi_1 \setminus \{a_1 b_1\}) \cup \{b_1 a_1\}$, where $a_1 b_1$ is any pair which is consecutive in ξ_1 but it does not belong to $\widetilde{\xi}$, and so on. This iteration will conclude in a number of steps equal to $d(\xi, \widetilde{\xi})$, since $d(\xi_i, \widetilde{\xi})$ decreases by one unit in each step. □

9 The projection

Let us recall that our rating method is based upon certain projected scores v^{π}_{xy}. These quantities (or equivalently, the projected margins $m^{\pi}_{xy} = v^{\pi}_{xy} - v^{\pi}_{yx}$ and the projected turnovers $t^{\pi}_{xy} = v^{\pi}_{xy} + v^{\pi}_{yx}$) are worked out by means of the procedure (20–26) of page 18. Its starting point are the indirect margins $m^{\nu}_{xy} = v^*_{xy} - v^*_{xy}$ and the original turnovers $t_{xy} = v_{xy} + v_{yx}$. From these quantities, equations (21.1) and (21.2), used in this order, determine what we called the intermediate projected margins and turnovers, m^{σ}_{xy} and t^{σ}_{xy}. After their construction, one becomes interested only in their superdiagonal elements $m^{\sigma}_{xx'}$ and $t^{\sigma}_{xx'}$. In fact, these quantities are combined into certain intervals $\gamma_{xx'}$ whose unions give rise to the whole set of projected scores.

Let us recall in more detail the meaning of the operator Ψ which appears in step (21.2). This operator produces the intermediate projected turnovers (t^{σ}_{xy}) as a function of the original turnovers (t_{xy}) and the superdiagonal intermediate projected margins $(m^{\sigma}_{pp'})$. Here we are using parentheses to emphasize that we are dealing with the whole collection of turnovers and the whole collection of superdiagonal intermediate projected margins. Specifically, (t^{σ}_{xy}) is found by imposing certain conditions, namely (27–29), and minimizing the function (31), which is nothing else than the euclidean distance to (t_{xy}). Equivalently, we can think in the following way (where the pair xy is not restricted to belong to ξ): we consider a candidate (τ_{xy}) which varies over the set T which is determined by the following conditions:

$$\tau_{yx} = \tau_{xy}, \tag{55}$$

$$m^{\sigma}_{xx'} \leq \tau_{xx'} \leq 1; \tag{56}$$

$$0 \leq \tau_{xy} - \tau_{xy'} \leq m^{\sigma}_{yy'}, \tag{57}$$

we associate each candidate (τ_{xy}) with its euclidean distance from (t_{xy}); finally, we define (t^{σ}_{xy}) as the only value of (τ_{xy}) which minimizes such a distance. The minimizer exists and it is unique as a consequence of the fact that T is a closed convex set [*16*: ch. I, § 2]. In this connection, one can say that (t^{σ}_{xy}) is the orthogonal projection of (t_{xy}) onto the convex set T.

The procedure (20–26) produces the projected scores as the end points of the intervals

$$\text{(23)} \qquad \gamma_{xy} = \bigcup \{\gamma_{pp'} \mid x \overset{\xi}{\preceq} p \overset{\xi}{\succ} y\},$$

where

$$\text{(22)} \qquad \gamma_{xx'} = [\,(t^{\sigma}_{xx'} - m^{\sigma}_{xx'})/2\,,\ (t^{\sigma}_{xx'} + m^{\sigma}_{xx'})/2\,].$$

The desired properties of the projected scores and the associated margins and turnovers will be based upon the following properties of the intervals γ_{xy}, where we recall that $|\gamma|$ means the length of an interval, and $\dot{\gamma}$ means its barycentre, or centroid, i. e. the number $(a+b)/2$ if $\gamma = [a, b]$.

Lemma 9.1. *The sets* γ_{xy} *have the following properties for* $x \overset{\xi}{\succeq} y \overset{\xi}{\succeq} z$*:*

(a) γ_{xy} *is a closed interval.*

(b) $\gamma_{xy} \subseteq [0,1]$.

(c) $\gamma_{xz} = \gamma_{xy} \cup \gamma_{yz}$.

(d) $\gamma_{xy} \cap \gamma_{yz} \neq \emptyset$.

(e) $|\gamma_{xz}| \geq \max(|\gamma_{xy}|, |\gamma_{yz}|)$.

(f) $\dot{\gamma}_{xy} \geq \dot{\gamma}_{xz} \geq \dot{\gamma}_{yz}$.

(g) $|\gamma_{xz}|/\dot{\gamma}_{xz} \geq \max(|\gamma_{xy}|/\dot{\gamma}_{xy}, |\gamma_{yz}|/\dot{\gamma}_{yz})$.

Proof. Let us start by recalling that the superdiagonal intermediate turnovers and margins are ensured to satisfy the following inequalities:

$$0 \leq m^\sigma_{xx'} \leq t^\sigma_{xx'} \leq 1 \tag{58}$$

(30)
$$0 \leq t^\sigma_{xx'} - t^\sigma_{x'x''} \leq m^\sigma_{xx'} + m^\sigma_{x'x''}. \tag{59}$$

From (58) it follows that $0 \leq (t^\sigma_{xx'} - m^\sigma_{xx'})/2 \leq (t^\sigma_{xx'} + m^\sigma_{xx'})/2 \leq 1$. So, every $\gamma_{xx'}$ is an interval (possibly reduced to one point) and this interval is contained in $[0,1]$. Also, the inequalities of (59) ensure on the one hand that $\dot{\gamma}_{xx'} \geq \dot{\gamma}_{x'x''}$, and on the other hand that the intervals $\gamma_{xx'}$ and $\gamma_{x'x''}$ overlap each other. In the following we will see that these facts about the elementary intervals $\gamma_{xx'}$ entail the stated properties of the sets γ_{xy} defined by (59).

Part (a). This is an obvious consequence of the fact that $\gamma_{pp'}$ and $\gamma_{p'p''}$ overlap each other.

Part (b). This follows from the fact that $\gamma_{pp'} \subseteq [0,1]$.

Part (c). This is a consequence of the associative property enjoyed by the set-union operation.

Part (d). This is again an obvious consequence of the fact that $\gamma_{pp'}$ and $\gamma_{p'p''}$ overlap each other (take $p' = y$).

Part (e). This follows from (c) because $\gamma \subseteq \eta$ implies $|\gamma| \leq |\eta|$.

Part (f). This follows from the fact that $\dot{\gamma}_{pp'} \geq \dot{\gamma}_{p'p''}$ because of the following general fact: If γ and η are two intervals with $\dot{\gamma} \geq \dot{\eta}$ then $\dot{\gamma} \geq (\gamma \cup \eta)^{\bullet} \geq \dot{\eta}$. This is clear if γ and η are disjoint and also if one of them is contained in the other. Otherwise, $\gamma \setminus \eta$ and $\eta \setminus \gamma$ are nonempty intervals and the preceding disjoint case allows to proceed in the following way:

$$\dot{\gamma} \geq (\gamma \cup (\eta \setminus \gamma))^{\bullet} = (\gamma \cup \eta)^{\bullet} = ((\gamma \setminus \eta) \cup \eta)^{\bullet} \geq \dot{\eta}.$$

Part (g). This follows from (c) and (d) because of the following general fact: If γ and η are two closed intervals with $\gamma \subseteq \eta \subset [0, +\infty)$ then $|\gamma|/\dot{\gamma} \leq$

$|\eta|/\overset{\bullet}{\eta}$. In fact, let $\gamma = [a, b]$ and $\eta = [c, d]$. The hypothesis that $\gamma \subseteq \eta$ takes then the following form : $c \le a$ and $b \le d$. On the other hand, the claim that $|\gamma|/\overset{\bullet}{\gamma} \le |\eta|/\overset{\bullet}{\eta}$ takes the following form: $(b-a)/(b+a) \le (d-c)/(d+c)$. An elementary computation shows that the latter is equivalent to $bc \le ad$, which is a consequence of the preceding inequalities. □

The projection procedure makes use of a particular admissible order ξ. In fact, this order occurs in equations (21–23), as well as in conditions (27–29). In spite of this, the next theorem claims that the final results are independent of ξ. The proof is not difficult, but it is rather long.

Theorem 9.2. *The projected scores do not depend on the admissible order ξ used for their calculation, i. e. the value of v^{π}_{xy} is independent of ξ for every $xy \in \Pi$. On the other hand, the matrix of the projected scores in an admissible order ξ is also independent of ξ; i. e. if x_i denotes the element of rank i in ξ, the value of $v^{\pi}_{x_i x_j}$ is independent of ξ for every pair of indices i, j.*

Remark. The two statements say different things since the identity of x_i and x_j may depend on the admissible order ξ.

Proof. For the purposes of this proof it becomes necessary to change our set-up in a certain way. In fact, until now the intermediate objects m^{σ}_{xy}, t^{σ}_{xy} and γ_{xy} were considered only for $x \overset{\xi}{\succcurlyeq} y$, i. e. $xy \in \xi$. However, since we have to deal with changing the admissible order ξ, here we will allow their argument xy to be any pair (of different elements), no matter whether it belongs to ξ or not. In this connection, we will certainly put $m^{\sigma}_{yx} = -m^{\sigma}_{xy}$ and $t^{\sigma}_{yx} = t^{\sigma}_{xy}$. On the other hand, concerning γ_{xy} and γ_{yx}, we will proceed in the following way: if $\gamma_{xy} = [a, b]$ then $\gamma_{yx} = [b, a]$. So, generally speaking the γ_{xy} are here "oriented intervals", i. e. ordered pairs of real numbers. However, γ_{xy} will always be "positively oriented" when xy belongs to an admissible order (but it will be reduced to a point whenever there is another admissible order which includes yx). In particular, the $\gamma_{pp'}$ which are combined in (23) are always positively oriented intervals; so, the union operation performed in that equation can always be understood in the usual sense. In the following, γ' denotes the oriented interval "reverse" to γ, i. e. $\gamma' = [b, a]$ if $\gamma = [a, b]$.

So, let us consider the effect of replacing ξ by another admissible order $\widetilde{\xi}$. In the following, the tilde is systematically used to distinguish between homologous objects which are associated respectively with ξ and $\widetilde{\xi}$; in particular, such a notation will be used in connection with the labels of the equations which are formulated in terms of the assumed admissible order.

With this terminology, we will prove the two following equalities. First,

$$\gamma_{xy} = \widetilde{\gamma}_{xy}, \qquad \text{for any pair } xy \ (x \neq y), \tag{60}$$

where γ_{xy} are the intervals produced by (21–23) together with the operation $\gamma_{yx} = \gamma'_{xy}$, and $\widetilde{\gamma}_{xy}$ are those produced by ($\widetilde{21}$–$\widetilde{23}$) together with the operation $\widetilde{\gamma}_{yx} = \widetilde{\gamma}'_{xy}$. Secondly, we will see also that

$$\gamma_{x_i x_j} = \widetilde{\gamma}_{\tilde{x}_i \tilde{x}_j}, \qquad \text{for any pair of indices } ij \ (i \neq j), \tag{61}$$

where x_i denotes the element of rank i in ξ, and analogously for $\tilde{x}_i$ in $\widetilde{\xi}$. These equalities contain the statements of the theorem since the projected scores are nothing else than the end points of the γ intervals.

Now, by Lemma 8.6, it suffices to deal with the case of two admissible orders ξ and $\widetilde{\xi}$ which differ from each other by one inversion only. So, we will assume that there are two elements a and b such that the only difference between ξ and $\widetilde{\xi}$ is that ξ contains ab whereas $\widetilde{\xi}$ contains ba. According to the definition of an admissible order, this implies that $m^{\nu}_{ab} = m^{\nu}_{ba} = 0$.

In order to control the effect of the differences between ξ and $\widetilde{\xi}$, we will make use of the following notation: P and p will denote respectively the set of predecessors of a in ξ and its lowest element, i. e. the immediate predecessor of a in ξ; in this connection, any statement about p will be understood to imply the assumption that P is not empty. Similarly, Q and q will denote respectively the set of successors of b in ξ and its top element, i. e. the immediate successor of b in ξ; here too, any statement about q will be understood to imply the assumption that Q is not empty. So, ξ and $\widetilde{\xi}$ contain respectively the paths $pabq$ and $pbaq$.

Let us look first at the superdiagonal intermediate projected margins $m^{\sigma}_{hh'}$. According to (21.1), $m^{\sigma}_{hh'}$ is the minimum of a certain set of values of m^{ν}_{xy}. In a table where x and y are ordered according to ξ, this set is an upper-right rectangle with lower-left vertex at hh'. Using $\widetilde{\xi}$ instead of ξ amounts to interchanging two consecutive columns and the corresponding rows of that table, namely those labeled by a and b. In spite of such a rearrangement, in all cases but one the underlying set from which the minimum is taken is exactly the same, so the mininum is the same. The only case where the underlying set is not the same occurs for $h = a$ in the order ξ, or $h = b$ in the order $\widetilde{\xi}$; but then the minimum is still the same because the underlying set includes $m^{\nu}_{ab} = m^{\nu}_{ba} = 0$. So,

$$m^{\sigma}_{x_i x_{i+1}} = \widetilde{m}^{\sigma}_{\tilde{x}_i \tilde{x}_{i+1}}, \qquad \text{for any } i = 1, 2, \dots N{-}1. \tag{62}$$

In more specific terms, we have

$$m^\sigma_{xx'} = \widetilde{m}^\sigma_{xx'}, \qquad \text{whenever } x \neq p, a, b, \tag{63}$$

$$m^\sigma_{pa} = \widetilde{m}^\sigma_{pb}, \tag{64}$$

$$m^\sigma_{ab} = \widetilde{m}^\sigma_{ba} = 0, \tag{65}$$

$$m^\sigma_{bq} = \widetilde{m}^\sigma_{aq}. \tag{66}$$

In connection with equation (63) it should be clear that *for* $x \neq p, a, b$ *the immediate successor* x' *is the same in both orders* ξ *and* $\widetilde{\xi}$.

Next we will see that the intermediate projected turnovers t^σ_{xy} are invariant with respect to ξ:

$$t^\sigma_{xy} = \widetilde{t}^\sigma_{xy}, \qquad \text{for any pair } xy \ (x \neq y), \tag{67}$$

where t^σ_{xy} are the numbers produced by (21.2) together with the symmetry $t^\sigma_{yx} = t^\sigma_{xy}$, and $\widetilde{t}^\sigma_{xy}$ are those produced by $(\widetilde{21}.2)$ together with the symmetry $\widetilde{t}^\sigma_{yx} = \widetilde{t}^\sigma_{xy}$.

We will prove (67) by seeing that the set T determined by conditions (55,56,57) coincides exactly with the set $\widetilde{T}$ determined by $(55,\widetilde{56},\widetilde{57})$. In other words, conditions (56–57) are exactly equivalent to $(\widetilde{56}–\widetilde{57})$ under condition (55), which does not depend on ξ.

In order to prove this equivalence we begin by noticing that condition (56) coincides exactly with $(\widetilde{56})$ when $x \neq p, a, b$. This is true because, on the one hand, x' is then the same in both orders ξ and $\widetilde{\xi}$, and, on the other hand, (63) ensures that the right-hand sides have the same value. Similarly happens with conditions (57) and $(\widetilde{57})$ when $y \neq p, a, b$. So, it remains to deal with conditions (56) and $(\widetilde{56})$ for $x = p, a, b$, and with conditions (57) and $(\widetilde{57})$ for $y = p, a, b$. Now, on account of the symmetry (55), one easily sees that condition (56) with $x = a$ is equivalent to $(\widetilde{56})$ with $x = b$. In fact, both of them reduce to $0 \leq \tau_{ab} \leq 1$ since $m^\sigma_{ab} = \widetilde{m}^\sigma_{ba} = 0$, as it was obtained in (65). This last equality ensures also the equivalence between condition (57) with $y = a$ and condition $(\widetilde{57})$ with $y = b$. In this case both of them reduce to

$$\tau_{xa} = \tau_{xb}. \tag{68}$$

This common equality plays a central role in the equivalence between the remaining conditions. Thus, its combination with (66) ensures the equivalence between (56) with $x = b$ and $(\widetilde{56})$ with $x = a$, as well as the equivalence between (57) with $y = b$ and $(\widetilde{57})$ with $y = a$ when $x \neq a, b$. On the other hand, its combination with (64) ensures the equivalence between (56) and $(\widetilde{56})$ when $x = p$, as well as the equivalence between (57) and $(\widetilde{57})$ when $y = p$ and $x \neq a, b$. Finally, we have the two following equivalences: (57) with $y = p$ and $x = b$ is equivalent to $(\widetilde{57})$ with $y = p$ and $x = a$ because of the same equality (68) together with (64) and the symmetry (55); and similarly, (57) with $y = b$ and $x = a$ is equivalent to $(\widetilde{57})$ with $y = a$ and $x = b$ because of (68) together with (66) and (55). This completes the proof of (67).

Having seen that condition (68) is included in both (57) and $(\widetilde{57})$, it follows that the intermediate projected turnovers satisfy

$$t^\sigma_{xa} = t^\sigma_{xb}, \qquad \widetilde{t}^\sigma_{xa} = \widetilde{t}^\sigma_{xb}. \tag{69}$$

By taking $x = p, q$ and using also (67), it follows that

$$t^\sigma_{xx'} = \widetilde{t}^\sigma_{xx'}, \qquad \text{whenever } x \neq p, a, b, \tag{70}$$

$$t^\sigma_{pa} = \widetilde{t}^\sigma_{pb}, \tag{71}$$

$$t^\sigma_{ab} = \widetilde{t}^\sigma_{ba}, \tag{72}$$

$$t^\sigma_{bq} = \widetilde{t}^\sigma_{aq}. \tag{73}$$

In other words, the superdiagonal intermediate turnovers satisfy

$$t^\sigma_{x_i x_{i+1}} = \widetilde{t}^\sigma_{\tilde{x}_i \tilde{x}_{i+1}}, \qquad \text{for any } i = 1, 2, \dots N-1. \tag{74}$$

On account of the definition of $\gamma_{x_i x_{i+1}}$ and $\widetilde{\gamma}_{\tilde{x}_i \tilde{x}_{i+1}}$, the combination of (62) and (74) results in

$$\gamma_{x_i x_{i+1}} = \widetilde{\gamma}_{\tilde{x}_i \tilde{x}_{i+1}}, \qquad \text{for any } i = 1, 2, \dots N-1, \tag{75}$$

from which the union operation (23) produces (61).

Finally, let us see that (60) holds too. To this effect, we begin by noticing that (65) together with (72) are saying not only that $\gamma_{ab} = \widetilde{\gamma}_{ba}$ but also that this interval reduces to a point. As a consequence, we have

$$\gamma_{ba} = \gamma_{ab} = \widetilde{\gamma}_{ba} = \widetilde{\gamma}_{ab}. \tag{76}$$

Let us consider now the equation $\gamma_{pa} = \widetilde{\gamma}_{pb}$, which is contained in (75). Since γ_{ab} reduces to a point, the overlapping property $\gamma_{pa} \cap \gamma_{ab} \neq \emptyset$ (part (d) of Lemma 9.1) reduces to $\gamma_{ab} \subseteq \gamma_{pa}$. Therefore, $\gamma_{pb} = \gamma_{pa} \cup \gamma_{ab} = \gamma_{pa}$ (where we used part (c) of Lemma 9.1). Analogously, $\widetilde{\gamma}_{pa} = \widetilde{\gamma}_{pb} \cup \widetilde{\gamma}_{ba} = \widetilde{\gamma}_{pb}$. Altogether, this gives

$$\gamma_{pb} = \gamma_{pa} = \widetilde{\gamma}_{pb} = \widetilde{\gamma}_{pa}. \tag{77}$$

By means of an analogous argument, one obtains also that

$$\gamma_{aq} = \gamma_{bq} = \widetilde{\gamma}_{aq} = \widetilde{\gamma}_{bq}. \tag{78}$$

On the other hand, (75) ensures that

$$\gamma_{xx'} = \widetilde{\gamma}_{xx'}, \qquad \text{whenever } x \neq p, a, b. \tag{79}$$

Finally, part (c) of Lemma 9.1 allows to go from (76–79) to the desired general equality (60). □

Theorem 9.3. *The projected scores and their asssociated margins and turnovers satisfy the following properties with respect to any admissible order* ξ*:*
(a) *The following inequalities hold whenever* $x \succcurlyeq^{\xi} y$*:*

$$v^\pi_{xy} \geq v^\pi_{yx} \qquad m^\pi_{xy} \geq 0, \tag{80}$$

$$v^\pi_{xz} \geq v^\pi_{yz}, \qquad v^\pi_{zx} \leq v^\pi_{zy}, \tag{81}$$

$$m^\pi_{xz} \geq m^\pi_{yz}, \qquad m^\pi_{zx} \leq m^\pi_{zy}, \tag{82}$$

$$t^\pi_{xz} \geq t^\pi_{yz}, \qquad t^\pi_{zx} \geq t^\pi_{zy}, \tag{83}$$

$$m^\pi_{xz}/t^\pi_{xz} \geq m^\pi_{yz}/t^\pi_{yz}, \qquad m^\pi_{zx}/t^\pi_{zx} \leq m^\pi_{zy}/t^\pi_{zy}. \tag{84}$$

(b) *If* $v^\pi_{xy} = v^\pi_{yx}$*, or equivalently* $m^\pi_{xy} = 0$*, then* (81–84) *are satisfied all of them with an equality sign.*

(c) *In the complete case, the projected margins satisfy the following property:*

$$m^\pi_{xz} = \max(m^\pi_{xy}, m^\pi_{yz}), \qquad \textit{whenever } x \succcurlyeq^{\xi} y \succcurlyeq^{\xi} z. \tag{85}$$

Proof. We will see that these properties derive from those satisfied by the γ intervals, which are collected in Lemma 9.1. For the derivation one has to bear in mind that v^π_{xy} and v^π_{yx} are respectively the right and left end points of the interval γ_{xy}, and that $m^\pi_{xy} = -m^\pi_{yx}$ and $t^\pi_{xy} = t^\pi_{yx}$ are respectively the width and twice the barycentre of γ_{xy}.

Part (a). Let us begin by noticing that (82) will be an immediate consequence of (81), since $m^\pi_{xz} = v^\pi_{xz} - v^\pi_{zx}$ and $m^\pi_{yz} = v^\pi_{yz} - v^\pi_{zy}$. On the other hand, (83.2) is equivalent to (83.1) and (84.2) is equivalent to (84.1). These equivalences hold because the turnovers and margins are respectively symmetric and antisymmetric. Now, (80) holds as soon as γ_{xy} is an interval, as it is ensured by part (a) of Lemma 9.1. So, it remains to prove the inequalities (81), (83.1) and (84.1). In order to prove them we will distinguish three cases, namely: (i) $x \succcurlyeq^{\xi} y \succcurlyeq^{\xi} z$; (ii) $z \succcurlyeq^{\xi} x \succcurlyeq^{\xi} y$; (iii) $x \succcurlyeq^{\xi} z \succcurlyeq^{\xi} y$.

Case (i) : By part (c) of Lemma 9.1, in this case we have $\gamma_{xz} \supseteq \gamma_{yz}$. This immediately implies (81) because $[a,b] \supseteq [c,d]$ is equivalent to saying that $b \geq d$ and $a \leq c$. On the other hand, the inequalities (83.1) and (84.1) are contained in parts (f) and (g) of Lemma 9.1. Case (ii) is analogous to case (i).

Case (iii) : In this case, (81) follows from part (d) of Lemma 9.1 since $[a,b] \cap [c,d] \neq \emptyset$ is equivalent to saying that $b \geq c$ and $a \leq d$. On the other hand, (83.1) is still contained in part (f) of Lemma 9.1 (because of the symmetric character of the turnovers), and (84.1) holds since $m^{\pi}_{xz} \geq 0 \geq m^{\pi}_{yz}$.

Part (b). The hypothesis that $v^{\pi}_{xy} = v^{\pi}_{yx}$ is equivalent to saying that γ_{xy} reduces to a point, i. e. $\gamma_{xy} = [v,v]$ for some v. The claimed equalities will be obtained by showing that in these circumstances one has $\gamma_{xz} = \gamma_{yz}$. We will distinguish the same three cases as in part (a).

Case (i) : On account of the overlapping property $\gamma_{xy} \cap \gamma_{yz} \neq \emptyset$ (part (d) of Lemma 9.1), the one-point interval $\gamma_{xy} = [v,v]$ must be contained in γ_{yz}. So, $\gamma_{xz} = \gamma_{xy} \cup \gamma_{yz} = \gamma_{yz}$ (where we used part (c) of Lemma 9.1). Case (ii) is again analogous to case (i).

Case (iii) : By part (c) of Lemma 9.1 (with y and z interchanged with each other), the fact that γ_{xy} reduces to the one-point interval $[v,v]$ implies that both γ_{xz} and γ_{zy} reduce also to this one-point interval

Part (c). In the complete case the intermediate projected turnovers are all of them equal to 1, so the intervals $\gamma_{pp'}$ and γ_{xy} are all of the centred at $1/2$. In these circumstances, (85) is exactly equivalent to part (c) of Lemma 9.1. □

The following propositions identify certain situations where the preceding projection reduces to the identity.

Proposition 9.4. *In the case of plumping votes the projected scores coincide with the original ones.*

Proof. Let us begin by recalling that in the case of plumping votes the binary scores have the form $v_{xy} = f_x$ for every $y \neq x$, where f_x is the fraction of voters who choose x. This implies that $v^{*}_{xy} = v_{xy} = f_x$. In fact, any path γ from x to y starts with a link of the form xp, whose associated score is $v_{xp} = f_x$. So $v_{\gamma} \leq f_x$ and therefore $v^{*}_{xy} \leq f_x$. But on the other hand $f_x = v_{xy} \leq v^{*}_{xy}$. Consequently, we get $m^{\nu}_{xy} = v^{*}_{xy} - v^{*}_{yx} = v_{xy} - v_{yx} = f_x - f_y$, and the admissible orders are those for which the f_x are non-increasing. Owing to this non-increasing character, the intermediate projected margins are $m^{\sigma}_{xx'} = m_{xx'} = f_x - f_{x'}$. On the other hand, the intermediate projected turnovers are $t^{\sigma}_{xy} = t_{xy} = f_x + f_y$. In fact these numbers are easily seen to satisfy conditions (27–29) and they obviously minimize (31). As a consequence, $\gamma_{xx'} = [f_{x'}, f_x]$. In particular, the intervals $\gamma_{xx'}$ and $\gamma_{x'x''}$ are adjacent to each other (the right end of the latter coincides with the left end of the former). This fact entails that $\gamma_{xy} = [f_y, f_x]$ whenever $x \overset{\xi}{\succcurlyeq} y$. Finally, the projected scores are the end points of these intervals, namely $v^{\pi}_{xy} = f_x = v_{xy}$ and $v^{\pi}_{yx} = f_y = v_{yx}$. □

Proposition 9.5. *Assume that the votes are complete. Assume also that there exists a total order ξ such that $\mu(v) \subseteq \xi \subseteq \hat{\mu}(v)$ and such that the original margins satisfy*

$$m_{xz} = \max(m_{xy}, m_{yz}), \qquad \textit{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z \textit{ in } \xi. \tag{86}$$

In that case, the projected scores coincide with the original ones. Besides, condition (86) *holds also for any other total order $\widetilde{\xi}$ which satisfies $\mu(v) \subseteq \widetilde{\xi} \subseteq \hat{\mu}(v)$.*

Remark. The hypothesis that $\mu(v) \subseteq \xi \subseteq \hat{\mu}(v)$ is not the one which defines an admissible order, namely $\mu(v^*) \subseteq \xi \subseteq \hat{\mu}(v^*)$. However, in the course of the proof we will see that $v^*_{xy} = v_{xy}$. So, ξ will be after all an admissible order.

Proof. Since we are in the complete case, the scores v_{xy} and the margins m_{xy} are related to each other by the monotone increasing transformation $v_{xy} = (1 + m_{xy})/2$. Therefore, condition (86) on the margins is equivalent to the following one on the scores:

$$v_{xz} = \max(v_{xy}, v_{yz}), \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z \text{ in } \xi. \tag{87}$$

On the other hand, since $v_{xy} + v_{yx} = 1$, the preceding condition is also equivalent to the following one:

$$v_{zx} = \min(v_{zy}, v_{yx}), \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z \text{ in } \xi. \tag{88}$$

In fact, $v_{zx} = 1 - v_{xz} = 1 - \max(v_{xy}, v_{yz}) = \min(v_{zy}, v_{yx})$.

Now, we claim that these properties imply the following one:

$$v_{xz} \geq \min(v_{xy}, v_{yz}), \qquad \text{for any } x, y, z. \tag{89}$$

In order to prove (89) we will distinguish four cases depending on whether or not do xy and yz belong to ξ: (a) If $xy, yz \in \xi$, then (89) is an immediate consequence of (87). (b) Similarly, if $xy, yz \notin \xi$, then (89) is an immediate consequence of (88) with x and z interchanged with each other. (c) Consider now the case where $xy \notin \xi$ and $yz \in \xi$. In this case we have $v_{xy} \leq 1/2 \leq v_{yz}$, so $\min(v_{xy}, v_{yz}) = v_{xy}$. Now we must distinguish two subcases: If $xz \in \xi$, then $v_{xy} \leq 1/2 \leq v_{xz}$, so we get (89). If, on the contrary, $zx \in \xi$, then (88) applied to $y \succcurlyeq^{\xi} z \succcurlyeq^{\xi} x$ gives $v_{xy} = \min(v_{xz}, v_{zy}) \leq v_{xz}$ as claimed. (d) Finally, the case where $xy \in \xi$ and $yz \notin \xi$ is analogous to the preceding one.

Now we invoke Lemma 6.2, according to which (89) implies that $v^*_{xy} = v_{xy}$. In particular, ξ is ensured to be an admissible order. Let us consider any pair xy contained in ξ. By applying condition (86) we see that $m^\sigma_{xy} = m^\nu_{xy} = m_{xy}$. On the other hand, since the votes are complete we have $t^\sigma_{xy} = 1$. So, the intervals $\gamma_{pp'}$ and their unions are all of them centred at $1/2$. In this case, the union operation of (23) is equivalent to a maximum operation performed upon the margins. On account of (86), this implies that $m^\pi_{xy} = m_{xy}$. Since we also have $t^\pi_{xy} = 1 = t_{xy}$, it follows that $v^\pi_{xy} = v_{xy}$ and $v^\pi_{yx} = v_{yx}$.

Having proved that $v_{xy} = v^*_{xy} = v^\pi_{xy}$, and taking into account that this entails $m_{xy} = m^\pi_{xy}$, one easily sees that condition (86) holds also for any other total order $\widetilde{\xi}$ such that $\mu(v) \subseteq \widetilde{\xi} \subseteq \hat{\mu}(v)$. In fact, such an order is an admissible one, since $\mu(v^*) = \mu(v)$, and that condition is guaranteed by part (c) of Theorem 9.3. □

10 The rank-like rates

Let us recall that the rank-like rates r_x are given by the formula

(8)
$$r_x \;=\; N - \sum_{y \neq x} v^\pi_{xy}. \tag{90}$$

where v^π_{xy} are the projected scores. In the special case of complete votes, where $v^\pi_{xy} + v^\pi_{yx} = 1$, the preceding formula is equivalent to the following one:

(9)
$$r_x \;=\; (N + 1 - \sum_{y \neq x} m^\pi_{xy}\,)\,/\,2. \tag{91}$$

Let us remark also that in this special case the rank-like rates have the property that
$$\sum_{x \in A} r_x \;=\; N(N+1)/2. \tag{92}$$

In view of formula (90), the properties of the projected scores obtained in Theorem 9.3 imply the following facts:

Lemma 10.1.

(a) *If* $x \overset{\xi}{\succcurlyeq} y$ *in an admissible order* ξ, *then* $r_x \leq r_y$.

(b) $r_x = r_y$ *if and only if* $v^\pi_{xy} = v^\pi_{yx}$, *i. e.* $m^\pi_{xy} = 0$.

(c) *The inequalities* (80–84) *are satisfied whenever* $r_x \leq r_y$.
In particular, $v^\pi_{xy} > v^\pi_{yx}$ *implies* $r_x < r_y$.

Proof. Part (a). It is an immediate consequence of the preceding formula together with the inequalities (80) and (81.1) ensured by Theorem 9.3.

Part (b). According to the formula above,

$$r_y - r_x \;=\; (v^\pi_{xy} - v^\pi_{yx}) \;+\; \sum_{\substack{z\neq x\\ z\neq y}} (v^\pi_{xz} - v^\pi_{yz}). \tag{93}$$

Let ξ be an admissible order. By symmetry we can assume $xy \in \xi$. As a consequence, Theorem 9.3 ensures that the terms of (93) which appear in parentheses are all of them greater than or equal to zero. So the only possibility for their sum to vanish is that each of them vanishes separately, i. e. $v^\pi_{xy} = v^\pi_{yx}$ and $v^\pi_{xz} - v^\pi_{yz}$ for any $z \notin \{x, y\}$. Finally, part (b) of Theorem 9.3 ensures that all of these equalities hold as soon as the first one is satisfied.

Part (c). It suffices to use the contrapositive of (a) in the case of a strict inequality and (b) together with Theorem 9.3.(b) in the case of an equality. □

Theorem 10.2. *The rank-like rating given by* (90) *is related to the indirect comparison relation* $\nu = \mu(v^*)$ *in the following way:*

(a) $xy \in \hat\nu \;\Rightarrow\; r_x \leq r_y$.

(b) $r_x < r_y \;\Rightarrow\; xy \in \nu$.

(c) *If* ν *contains a set of the form* $X \times Y$ *with* $X \cup Y = A$,
then $r_x < r_y$ *for any* $x \in X$ *and* $y \in Y$.

(d) *If* ν *is total, i. e.* $\hat\nu = \nu$, *then* $xy \in \nu \Leftrightarrow r_x < r_y$.

Proof. Part (a). Let us begin by noticing that $xy \in \nu$ implies $r_x \leq r_y$. This follows from part (a) of Lemma 10.1 since ν is included in any admissible ordering ξ. Consider now the case $xy \in \hat\nu \setminus \nu$. This is equivalent to saying that ν contains neither xy nor yx. Now, in this case Theorem 8.2 ensures the existence of an admissible order which contains such a pair xy. So, using again the preceding proposition, we are still ensured that $r_x \le r_y$.

Part (b). It reduces to the the contrapositive of (a).

Part (c). Let $x \in X$ and $y \in Y$. Since $X \times Y \subset \nu$, part (a) ensures that $r_x \le r_y$. So, it suffices to exclude the possibility that $r_x = r_y$. This will be done by showing that this equality leads to a contradiction. By part (b) of Lemma 10.1, that equality implies $v^\pi_{xy} = v^\pi_{yx}$, or equivalently, $m^\pi_{xy} = 0$. But according to (22–24), this means that $m^\sigma_{hh'} = 0$ for all h such that $x \overset{\xi}{\succeq} h \overset{\xi}{\succ} y$. Here we are making use of an admissible order ξ. In particular we have $m^\sigma_{\ell\ell'} = 0$, where ℓ denotes the lowest element of X according to ξ, and ℓ' is the top element of Y. But this contradicts the fact that $\ell\ell' \in X \times Y \subset \nu$.

Part (d). It suffices to show that $r_x < r_{x'}$, where x' denotes the item that immediately follows x in the total order ν. This follows from part (c) by taking $X = \{p \mid p \overset{\xi}{\succeq} x\}$ and $Y = \{q \mid x' \overset{\xi}{\succeq} q\}$ and using the transitivity of ν. □

By construction, the rank-like rates are related to the projected scores in the same way as the average ranks are related to the original scores when the votes are complete rankings (§ 2.5). Therefore, if we are in the case of complete ranking votes and the projected scores coincide with the original ones, then the rank-like rates coincide with the average ranks:

Proposition 10.3. *Assume that the votes are complete rankings. Assume also that the Llull matrix satisfies the hypothesis of Proposition 9.5. In that case, the rank-like rates r_x coincide exactly with the average ranks $\bar{r}_x$.*

Proof. This is an immediate consequence of Proposition 9.5. □

11 Zermelo's method

The Llull matrix of a vote can be viewed as corresponding to a tournament between the members of A where x and y have played T_{xy} matches (the number of voters who made a comparison between x and y, even if this comparison resulted in a tie) and V_{xy} of these matches were won by x, whereas the other V_{yx} were won by y (one tied match will be counted as half a match in favour of x plus half a match in favour of y). For such a scenario, Ernst Zermelo [*40*] devised in 1929 a rating method which turns out to be quite suitable to convert our rank-like rates into fraction-like ones. This method was rediscovered later on by other autors [*3, 10*].

Zermelo's method is based upon a probabilistic model for the outcome of a match between two items x and y. This model assumes that such a match is won by x with probability $\varphi_x/(\varphi_x+\varphi_y)$ whereas it is won by y with probability $\varphi_y/(\varphi_x+\varphi_y)$, where φ_x is a non-negative parameter associated with each player x, usually referred to as its **strength**. If all matches are independent events, the probability of obtaining a particular system of values for the scores (V_{xy}) is given by

$$P = \prod_{\{x,y\}} \binom{T_{xy}}{V_{xy}} \left(\frac{\varphi_x}{\varphi_x+\varphi_y}\right)^{V_{xy}} \left(\frac{\varphi_y}{\varphi_x+\varphi_y}\right)^{V_{yx}}, \tag{94}$$

where the product runs through all unordered pairs $\{x,y\} \subseteq A$ with $x \neq y$. Notice that P depends only on the strength ratios; in other words, multiplying all the strengths by the same value has no effect on the result. On account of this, we will normalize the strengths by requiring their sum to take a fixed positive value f. In order to include certain extreme cases, one must allow for some of the strengths to vanish. However, this may conflict with P being well defined, since it could lead to indeterminacies of the type $0/0$ or 0^0. So, one should be careful in connection with vanishing strengths. With all this in mind, for the moment we will let the strengths vary in the following set:

$$Q = \{\, \varphi \in \mathbb{R}^A \mid \varphi_x > 0 \text{ for all } x \in A,\ \sum_{x\in A} \varphi_x = f \,\}. \tag{95}$$

Together with this set, in the following we will consider also its closure $\overline{Q}$, which includes vanishing strengths, and its boundary $\partial Q = \overline{Q} \setminus Q$.

In connection with our interests, it is worth noticing that Zermelo's model can be viewed as a special case of a more general one, proposed in 1959 by Robert Duncan Luce, which considers the outcome of making a choice out of multiple options [*21*]. According to Luce's 'choice axiom', the probabilities of two different choices x and y are in a ratio which does not depend on which other options are present. As a consequence, it follows that every option x can be associated a number φ_x so that the probability of choosing x out of a set $X \ni x$ is given by $\varphi_x/(\sum_{y\in X}\varphi_y)$. Obviously, Zermelo's model corresponds to considering binary choices only. It is interesting to notice that Luce's model allows to associate every ranking with a certain probability. In fact, a ranking can be viewed as the result of first choosing the winner out of the whole set A, then choosing the best of the remainder, and so on. If these successive choices are assumed to be independent events, then one can easily figure out the corresponding probability. Anyway, when the normalization condition $\sum_{x\in A}\varphi_x = f\ (\leq 1)$ is adopted, Luce's theory of choice allows to view φ_x as the first-choice probability of x, and to view $1-f$ as the probability of abstaining from making a choice out of A.

Let us mention here also that the hypothesis of independence which lies behind formula (94) is certainly not satisfied by the binary comparisons which arise out of preferential voting. In order to satisfy that hypothesis, the individual votes should be based upon independent binary comparisons, in which case they could take the form of an arbitrary binary relation, as we considered in § 3.3. However, even if the independence hypothesis is not satisfied, we will see that Zermelo's method, which we are about to discuss, has good properties for transforming our projected scores into fraction-like rates.

Zermelo's method corresponds to a maximum likelihood estimate of the parameters φ_x from a given set of actual values of V_{xy} (and of $T_{xy} = V_{xy} + V_{yx}$). In other words, given the values of V_{xy}, one looks for the values of φ_x which maximize the probability P. Since V_{xy} and T_{xy} are now fixed, this is equivalent to maximizing the following function of the φ_x:

$$F(\varphi) \;=\; \prod_{\{x,y\}} \frac{{\varphi_x}^{v_{xy}}\,{\varphi_y}^{v_{yx}}}{(\varphi_x+\varphi_y)^{t_{xy}}}, \tag{96}$$

(recall that $v_{xy} = V_{xy}/V$ and $t_{xy} = T_{xy}/V$ where V is a positive constant greater than or equal to any of the turnovers T_{xy}; going from (94) to (96) involves taking the power of exponent $1/V$ and disregarding a fixed multiplicative constant). The function F is certainly smooth on Q. Besides, it is clearly bounded from above, since the probability is always less than or equal to 1. However, generally speaking F needs not to achieve a maximum in Q, because this set is not compact. In the present situation, the only general fact that one can guarantee in this connection is the existence of maximizing sequences, i. e. sequences φ^n in Q with the property that $F(\varphi^n)$ converges to the lowest upper bound $\overline{F} = \sup\{F(\psi)\,|\,\psi\in Q\}$.

In connection with maximizing the function F defined by (96) it makes a difference whether two particular items x and y satisfy or not the inequality $v_{xy} > 0$, or more generally —as we will see— whether they satisfy $v^*_{xy} > 0$. By the definition of v^*_{xy}, the last inequality defines a transitive relation —namely the transitive closure of the one defined by the former inequality—. In the following we will denote this transitive relation by the symbol $\trianglerighteq$. Thus,

$$x \trianglerighteq y \iff v^*_{xy} > 0. \tag{97}$$

Associated with it, it is interesting to consider also the following derived relations, which keep the property of transitivity and are respectively symmetric and antisymmetric:

$$x \equiv y \iff v^*_{xy} > 0 \text{ and } v^*_{yx} > 0, \tag{98}$$

$$x \rhd y \iff v^*_{xy} > 0 \text{ and } v^*_{yx} = 0. \tag{99}$$

Therefore, $\equiv$ is an equivalence relation and $\rhd$ is a partial order. In the following, the situation where $x \rhd y$ will be expressed by saying that x **dominates** y. The equivalence classes of A by $\equiv$ are called the **irreducible components** of A (for V). If there is only one of them, namely A itself, then one says that the matrix V is irreducible. So, V is irreducible if and only if $v^*_{xy} > 0$ for any $x, y \in A$. It is not difficult to see that this property is equivalent to the following one formulated in terms of the direct scores only: there is no splitting of A into two classes X and Y so that $v_{yx} = 0$ for any $x \in X$ and $y \in Y$; in other words, there is no ordering of A for which the matrix V takes the form

$$\begin{pmatrix} \mathsf{V}_{XX} & \mathsf{V}_{XY} \\ \mathsf{O} & \mathsf{V}_{YY} \end{pmatrix}, \tag{100}$$

where V_{XX} and V_{YY} are square matrices and O is a zero matrix. Besides, a subset $X \subseteq A$ is an irreducible component if and only if X is maximal, in the sense of set inclusion, for the property of V_{XX} being irreducible. On the other hand, it also happens that the relations $\trianglerighteq$ and $\rhd$ are compatible with the equivalence relation $\equiv$, i. e. if $x \equiv \bar{x}$ and $y \equiv \bar{y}$ then $x \trianglerighteq y$ implies $\bar{x} \trianglerighteq \bar{y}$, and analogously $x \rhd y$ implies $\bar{x} \rhd \bar{y}$. As a consequence, the relations $\trianglerighteq$ and $\rhd$ can be applied also to the irreducible components of A for V. In the following we will be interested in the case where V is irreducible, or more generally, when there is a **top dominant irreducible component**, i. e. an irreducible component which dominates any other. From now on we systematically use the notation V_{RS} to mean the restriction of (v_{xy}) to $x \in R$ and $y \in S$, where R and S are arbitrary non-empty subsets of A. Similarly, φ_R will denote the restriction of (φ_x) to $x \in R$.

The next theorems collect the basic results that we need about Zermelo's method.

Theorem 11.1 (Zermelo, 1929 [*40*]; see also [*10, 15*])**.** *If V is irreducible, then:*

(a) *There is a unique $\varphi \in Q$ which maximizes F on Q.*

(b) *φ is the solution of the following system of equations:*

$$\sum_{y\neq x} t_{xy}\,\frac{\varphi_x}{\varphi_x+\varphi_y} \;=\; \sum_{y\neq x} v_{xy}, \tag{101}$$

$$\sum_{x} \varphi_x \;=\; f, \tag{102}$$

where (101) *contains one equation for every x.*

(c) *φ is an infinitely differentiable function of the scores v_{xy} as long as they keep satisfying the hypothesis of irreducibility.*

Proof. Let us begin by noticing that the hypothesis of irreducibility entails that F can be extended to a continuous function on $\overline{Q}$ by putting $F(\psi)=0$ for $\psi\in\partial Q$. In order to prove this claim we must show that $F(\psi^n)\to 0$ whenever ψ^n converges to a point $\psi\in\partial Q$. Let us consider the following sets associated with ψ: $X=\{x\,|\,\psi_x>0\}$ and $Y=\{y\,|\,\psi_y=0\}$. The second one is not empty since we are assuming $\psi\in\partial Q$, whereas the first one is not empty because the strengths add up to the positive value f. Now, for any $x\in X$ and $y\in Y$, $F(\psi^n)$ contains a factor of the form $(\psi^n_y)^{v_{yx}}$, which tends to zero as soon as $v_{yx}>0$. So, the only way for $F(\psi^n)$ not to approach zero would be $\mathsf{V}_{YX}=\mathsf{O}$, in contradiction with the irreducibility of V.

After such an extension, F is a continuous function on the compact set $\overline{Q}$. So, there exists φ which maximizes F on $\overline{Q}$. However, since $F(\psi)$ vanishes on ∂Q whereas it is strictly positive for $\psi\in Q$, the maximizer φ must belong to Q. This establishes the existence part of (a).

Maximizing F is certainly equivalent to maximizing $\log F$. According to Lagrange, any $\varphi\in Q$ which maximizes $\log F$ under the condition of a fixed sum is bound to satisfy

$$\frac{\partial \log F(\varphi)}{\partial\varphi_x} \;=\; \lambda, \tag{103}$$

for some scalar λ and every $x\in A$. Now, a straightforward computation gives

$$\frac{\partial \log F(\varphi)}{\partial\varphi_x} \;=\; \sum_{y\neq x}\left(\frac{v_{xy}}{\varphi_x}-\frac{t_{xy}}{\varphi_x+\varphi_y}\right). \tag{104}$$

On the other hand, using the fact that $v_{xy}+v_{yx}=t_{xy}$, the preceding expression is easily seen to imply that

$$\sum_{x}\frac{\partial \log F(\varphi)}{\partial\varphi_x}\,\varphi_x \;=\; 0. \tag{105}$$

In other words, the gradient of $\log F$ at φ is orthogonal to φ, which was foreseeable since $F(\varphi)$ remains constant when φ is multiplied by an arbitrary positive number. Notice that this is true for any φ. In particular, (105) entails that the above Lagrange multiplier λ is equal to zero; in fact, it suffices to plug (103) in (105) and to use the fact that $\sum_x \varphi_x = f$ is positive. So, the conditions (103) reduce finally to

$$\frac{\partial \log F(\varphi)}{\partial \varphi_x} \;=\; 0, \tag{106}$$

for every $x \in A$, which is equivalent to (101) on account of (104) and the fact that $\varphi_x > 0$. So, any maximizer must satisfy the conditions stated in (b).

Let us see now that the maximizer is unique. Instead of following the interesting proof given by Zermelo, here we will prefer to follow [*15*], which will have the advantage of preparing matters for part (c). More specifically, the uniqueness will be obtained by seeing that any critical point of $\log F$ as a function on Q, i. e. any solution of (101–102), is a strict local maximum; this implies that there is only one critical point, because otherwise one should have other kinds of critical points [*8*: §VI.6] (we are invoking the so-called mountain pass theorem; here we are using the fact that $\log F$ becomes $-\infty$ at ∂Q). In order to study the character of a critical point we will look at the second derivatives of $\log F$ with respect to φ. By differentiating (104), one obtains that

$$\frac{\partial^2 \log F(\varphi)}{\partial {\varphi_x}^2} \;=\; -\sum_{y\neq x} \left(\frac{v_{xy}}{\varphi_x^2} \;-\; \frac{t_{xy}}{(\varphi_x+\varphi_y)^2} \right), \tag{107}$$

$$\frac{\partial^2 \log F(\varphi)}{\partial \varphi_x \, \partial \varphi_y} \;=\; \frac{t_{xy}}{(\varphi_x+\varphi_y)^2}, \qquad \text{for } x \neq y. \tag{108}$$

On the other hand, when φ is a critical point, equation (101) transforms (107) into the following expression:

$$\frac{\partial^2 \log F(\varphi)}{\partial {\varphi_x}^2} \;=\; -\sum_{y\neq x} \frac{t_{xy}}{(\varphi_x+\varphi_y)^2}\,\frac{\varphi_y}{\varphi_x}. \tag{109}$$

So, the Hessian bilinear form is as follows:

$$\begin{aligned}
\sum_{x,y} \left(\frac{\partial^2 \log F(\varphi)}{\partial \varphi_x \, \partial \varphi_y} \right) \psi_x \, \psi_y \;&=\; -\sum_{x,y\neq x} \frac{t_{xy}}{(\varphi_x+\varphi_y)^2} \left(\frac{\varphi_y}{\varphi_x}\psi_x^2 - \psi_x\psi_y \right) \\
&=\; -\sum_{x,y\neq x} \frac{t_{xy}}{(\varphi_x+\varphi_y)^2\,\varphi_x\varphi_y} \,\left(\varphi_y^2\psi_x^2 - \varphi_x\varphi_y\psi_x\psi_y \right) \\
&=\; -\sum_{\{x,y\}} \frac{t_{xy}}{(\varphi_x+\varphi_y)^2\,\varphi_x\varphi_y} \,\left(\varphi_y\psi_x - \varphi_x\psi_y \right)^2,
\end{aligned} \tag{110}$$

where the last sum runs through all unordered pairs $\{x,y\} \subseteq A$ with $x \neq y$. The last expression is non-positive and it vanishes if and only if $\psi_x/\varphi_x = \psi_y/\varphi_y$ for any $x,y \in A$ (the 'only if' part is immediate when $t_{xy} > 0$; for arbitrary x and y the hypothesis of irreducibility allows to connect them through a path $x_0x_1\dots x_n$ ($x_0 = x$, $x_n = y$) with the property that $t_{x_ix_{i+1}} \geq v_{x_ix_{i+1}} > 0$ for any i, so that one gets $\psi_x/\varphi_x = \psi_{x_1}/\varphi_{x_1} = \cdots = \psi_y/\varphi_y$). So, the vanishing of (110) happens if and only if $\psi = \lambda\varphi$ for some scalar λ. However, when ψ is restricted to variations within Q, i. e. to vectors in $TQ = \{\psi \in \mathbb{R}^A \mid \sum_x \psi_x = 0\}$, the case $\psi = \lambda\varphi$ reduces to $\psi = 0$ (since $\sum_x \varphi_x = f$ is positive). So, the Hessian is negative definite on TQ. This ensures that φ is a strict local maximum of $\log F$ as a function on Q. In fact, one easily arrives at such a conclusion when Taylor's formula is used to analyse the behaviour of $\log F(\varphi+\psi)$ for small ψ in TQ.

Finally, let us consider the dependence of $\varphi \in Q$ on the matrix V. To begin with, we notice that the set $\mathcal{I}$ of irreducible matrices is open since it is a finite intersection of open sets, namely one open set for each splitting of A into two sets X and Y. The dependence of $\varphi \in Q$ on V is due to the presence of v_{xy} and $t_{xy} = v_{xy} + v_{yx}$ in the equations (101–102) which determine φ. However, we are not in the standard setting of the implicit function theorem since we are dealing with a system of $N+1$ equations whilst φ varies in a space of dimension $N-1$. In order to place oneself in a standard setting, it is convenient here to replace the condition of normalization $\sum_x \varphi_x = f$ by the alternative one $\varphi_a = 1$, where a is a fixed element of A. This change of normalization corresponds to mapping Q to $U = \{\varphi \in \mathbb{R}^A \mid \varphi_x > 0 \text{ for all } x \in A,\, \varphi_a = 1\}$ by means of the diffeomorphism $g : \varphi \mapsto \varphi/\varphi_a$, which has the property that $F(g(\varphi)) = F(\varphi)$. By taking as coordinates the φ_x with $x \in A \setminus \{a\} =: A'$, one easily checks that the function F restricted to U —i. e. restricted to $\varphi_a = 1$— has the property that the matrix $(\,\partial^2 \log F(\varphi)/\partial\varphi_x\partial\varphi_y \mid x,y \in A')$ is negative definite and therefore invertible, which entails that the system of equations $(\,\partial \log F(\varphi,\mathsf{V})/\partial\varphi_x = 0 \mid x \in A')$ determines $\varphi \in U$ as a smooth function of $\mathsf{V} \in \mathcal{I}$. $\square$

Let us recall that a maximizing sequence means a sequence $\varphi^n \in Q$ such that $F(\varphi^n)$ approaches the lowest upper bound of F on Q.

Theorem 11.2 (Statements (a) and (b) are proved in [*40*]; results related to (c) are contained in [*6*])**.** *Assume that there exists a top dominant irreducible component X. In this case:*

(a) *There is a unique $\varphi \in \overline{Q}$ such that any maximizing sequence converges to φ.*

(b) φ_X *is the solution of a system analogous to* (101–102) *where* x *and* y *vary only within* X. $\varphi_{A\setminus X} = 0$.

(c) φ *is a continuous function of the scores* v_{xy} *as long as they keep satisfying the hypotheses of the present theorem.*

Proof. The definition of the lowest upper bound immediately implies the existence of maximizing sequences. On the other hand, the compactness of $\overline{Q}$ guarantees that any maximizing sequence has a subsequence which converges in $\overline{Q}$. Let φ^n and φ denote respectively one of such convergent maximizing sequences and its limit. In the following we will see that φ must be the unique point specified in statement (b). This entails that any maximizing sequence converges itself to φ (without extracting a subsequence).

So, our aim is now statement (b). From now on we will use the following notations: a general element of Q will be denoted by ψ; we will write $Y = A \setminus X$. For convenience, in this part of the proof we will replace the condition $\sum_x \psi_x = f$ by $\sum_x \psi_x \leq f$ (and similarly for φ^n and φ); since $F(\lambda\psi) = F(\psi)$ for any $\lambda > 0$, the properties that we will obtain will be easily translated from $\widehat{Q} = \{\, \psi \in \mathbb{R}^A \mid \psi_x > 0 \text{ for all } x \in A,\ \sum_{x\in A} \psi_x \leq f \,\}$ to Q. On the other hand, it will also be convenient to consider first the case where Y is also an irreducible component. In such a case, it is interesting to rewrite $F(\psi)$ as a product of three factors:

$$F(\psi) \;=\; F_{XX}(\psi_X)\, F_{YY}(\psi_Y)\, F_{XY}(\psi_X, \psi_Y), \tag{111}$$

namely:

$$F_{XX}(\psi_X) \;=\; \prod_{\{x,\bar{x}\}\subset X} \frac{{\psi_x}^{v_{x\bar{x}}}\, {\psi_{\bar{x}}}^{v_{\bar{x}x}}}{(\psi_x + \psi_{\bar{x}})^{t_{x\bar{x}}}}, \tag{112}$$

$$F_{YY}(\psi_Y) \;=\; \prod_{\{y,\bar{y}\}\subset Y} \frac{{\psi_y}^{v_{y\bar{y}}}\, {\psi_{\bar{y}}}^{v_{\bar{y}y}}}{(\psi_y + \psi_{\bar{y}})^{t_{y\bar{y}}}}, \tag{113}$$

$$F_{XY}(\psi_X, \psi_Y) \;=\; \prod_{\substack{x\in X\\ y\in Y}} \left(\frac{\psi_x}{\psi_x + \psi_y}\right)^{v_{xy}}, \tag{114}$$

where we used that $v_{yx} = 0$ and $t_{xy} = v_{xy}$. Now, let us look at the effect of replacing ψ_Y by $\lambda\psi_Y$ without varying ψ_X. The values of F_{XX} and F_{YY} remain unchanged, but that of F_{XY} varies in the following way:

$$\frac{F_{XY}(\psi_X, \lambda\psi_Y)}{F_{XY}(\psi_X, \psi_Y)} \;=\; \prod_{\substack{x\in X\\ y\in Y}} \left(\frac{\psi_x + \psi_y}{\psi_x + \lambda\psi_y}\right)^{v_{xy}}. \tag{115}$$

In particular, for $0 < \lambda < 1$ each of the factors of the right-hand side of (115) is greater than 1. This remark leads to the following argument. First, we can see that $\varphi^n_y/\varphi^n_x \to 0$ for any $x \in X$ and $y \in Y$ such that $v_{xy} > 0$ (such pairs xy exist because of the hypothesis that X dominates Y). Otherwise, the preceding remark entails that the sequence $\widetilde{\varphi}^n = (\varphi^n_X, \lambda\varphi^n_Y)$ with $0 < \lambda < 1$ would satisfy $F(\widetilde{\varphi}^n) > KF(\varphi^n)$ for some $K > 1$ and infinitely many n, in contradiction with the hypothesis that φ^n was a maximizing sequence. On the other hand, we see also that $F_{XY}(\varphi^n)$ approaches its lowest upper bound, namely 1. Having achieved such a property, the problem of maximizing F reduces to separately maximizing F_{XX} and F_{YY}, which is solved by Theorem 11.1. For the moment we are dealing with relative strengths only, i. e. without any normalizing condition like (102). So, we see that F_{YY} gets optimized when each of the ratios $\varphi^n_y/\varphi^n_{\bar{y}}$ $(y, \bar{y} \in Y)$ approaches the homologous one for the unique maximizer of F_{YY}, and analogously with F_{XX}. Since these ratios are finite positive quantities, the statement that $\varphi^n_y/\varphi^n_x \to 0$ becomes extended to any $x \in X$ and $y \in Y$ whatsoever (since one can write $\varphi^n_y/\varphi^n_x = (\varphi^n_y/\varphi^n_{\bar{y}})\times(\varphi^n_{\bar{y}}/\varphi^n_{\bar{x}})\times(\varphi^n_{\bar{x}}/\varphi^n_x)$ with $v_{\bar{x}\bar{y}} > 0$). Let us recover now the condition $\sum_{x\in A}\varphi^n_x = f$. The preceding facts imply that $\varphi^n_Y \to 0$, whereas φ^n_X converges to the unique maximizer of F_{XX}. This establishes (b) as well as the uniqueness part of (a).

The general case where Y decomposes into several irreducible components, all of them dominated by X, can be taken care of by induction over the different irreducible components of A. At each step, one deals with an irreducible component Z with the property of being minimal, in the sense of the dominance relation $\vartriangleright$, among those which are still pending. By means of an argument analogous to that of the preceding paragraph, one sees that: (i) $\varphi^n_z/\varphi^n_x \to 0$ for any $z \in Z$ and x such that $x \vartriangleright z$ with $v_{xz} > 0$; (ii) the ratios $\varphi^n_z/\varphi^n_{\bar{z}}$ $(z, \bar{z} \in Z)$ approach the homologous ones for the unique maximizer of F_{ZZ}; and (iii) φ^n_R is a maximizing sequence for F_{RR}, where R denotes the union of the pending components, Z excluded. Once this induction process has been completed, one can combine its partial results to show that $\varphi^n_z/\varphi^n_x \to 0$ for any $x \in X$ and $z \notin X$ (it suffices to consider a path $x_0x_1\ldots x_n$ from $x_0 \in X$ to $x_n = z$ with the property that $v_{x_ix_{i+1}} > 0$ for any i and to notice that each of the factors $\varphi^n_{x_{i+1}}/\varphi^n_{x_i}$ remains bounded while at least one of them tends to zero). As above, one concludes that $\varphi^n_{A\setminus X} \to 0$, whereas φ^n_X converges to the unique maximizer of F_{XX}.

The two following remarks will be useful in the proof of part (c): (1) According to the proof above, φ_X is determined (up to a multiplicative constant) by equations (101) with x and y varying only within X:

$$\mathcal{F}_x(\varphi_X, \mathsf{V}) \;:=\; \sum_{\substack{y\in X\\ y\neq x}} t_{xy}\,\frac{\varphi_x}{\varphi_x+\varphi_y} \;-\; \sum_{\substack{y\in X\\ y\neq x}} v_{xy} \;=\; 0, \qquad \forall x\in X. \tag{116}$$

However, since $y \in A\setminus X$ implies on the one hand $\varphi_y = 0$ and on the other hand $t_{xy} = v_{xy}$, each of the preceding equations is equivalent to a similar one where y varies over the whole of $A\setminus\{x\}$:

$$\mathcal{F}'_x(\varphi, \mathsf{V}) \;:=\; \sum_{\substack{y\in A\\ y\neq x}} t_{xy}\,\frac{\varphi_x}{\varphi_x+\varphi_y} \;-\; \sum_{\substack{y\in A\\ y\neq x}} v_{xy} \;=\; 0, \qquad \forall x\in X. \tag{117}$$

(2) Also, it is interesting to see the result of adding up the equations (117) for all x in some subset W of X. Using the fact that $v_{xy} + v_{yx} = t_{xy}$, one sees that such an addition results in the following equality:

$$\sum_{\substack{x\in W\\ y\notin W}} t_{xy}\,\frac{\varphi_x}{\varphi_x+\varphi_y} \;-\; \sum_{\substack{x\in W\\ y\notin W}} v_{xy} \;=\; 0, \qquad \forall W\subseteq X. \tag{118}$$

Let us proceed now with the proof of (c). In the following, V and $\widetilde{\mathsf{V}}$ denote respectively a fixed matrix satisfying the hypotheses of the theorem and a slight perturbation of it. As we have done in similar occasions, we systematically use a tilde to distinguish between homologous objects associated respectively with V and $\widetilde{\mathsf{V}}$; in particular, such a notation will be used in connection with the labels of certain equations. Our aim is to show that $\widetilde{\varphi}$ approaches φ as $\widetilde{\mathsf{V}}$ approaches V. In this connection we will use the little-o and big-O notations made popular by Edmund Landau (who by the way is the author of a paper on the rating of chess players, namely [*18*], which inspired Zermelo's work). This notation refers here to functions of $\widetilde{\mathsf{V}}$ and their behaviour as $\widetilde{\mathsf{V}}$ approaches V; if f and g are two such functions, $f = o(g)$ means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\widetilde{\mathsf{V}} - \mathsf{V}\| \le \delta$ implies $\|f(\widetilde{\mathsf{V}})\| \le \epsilon\,\|g(\widetilde{\mathsf{V}})\|$; on the other hand, $f = O(g)$ means that there exist M and $\delta > 0$ such that $\|\widetilde{\mathsf{V}} - \mathsf{V}\| \le \delta$ implies $\|f(\widetilde{\mathsf{V}})\| \le M\,\|g(\widetilde{\mathsf{V}})\|$.

Obviously, if $\widetilde{\mathsf{V}}$ is near enough to V then $v_{xy} > 0$ implies $\widetilde{v}_{xy} > 0$. As a consequence, $x \trianglerighteq y$ implies $x \mathrel{\widetilde{\trianglerighteq}} y$. In particular, the irreducibility of V_{XX} entails that $\widetilde{\mathsf{V}}_{XX}$ is also irreducible. Therefore, X is entirely contained in some irreducible component $\widetilde{X}$ of A for $\widetilde{\mathsf{V}}$. Besides, $\widetilde{X}$ is a top dominant irreducible component for $\widetilde{\mathsf{V}}$; in fact, we have the following chain of implications for $x \in X \subseteq \widetilde{X}$: $y \notin \widetilde{X} \Rightarrow y \notin X \Rightarrow x \triangleright y \Rightarrow x \mathrel{\widetilde{\trianglerighteq}} y \Rightarrow x \mathrel{\widetilde{\triangleright}} y$, where we have used successively: the inclusion $X \subseteq \widetilde{X}$, the hypothesis that X is top dominant for V, the fact that $\widetilde{\mathsf{V}}$ is near enough to V, and the hypothesis that y does not belong to the irreducible component $\widetilde{X}$. Now, according to part (b) and remark (1) from p. 66–67, φ_X and $\widetilde{\varphi}_{\widetilde{X}}$ are determined respectively by the systems (116) and $(\widetilde{116})$, or equivalently by (117) and $(\widetilde{117})$, whereas $\varphi_{A\setminus X}$ and $\widetilde{\varphi}_{A\setminus\widetilde{X}}$ are both of them equal to zero. So we must show that $\widetilde{\varphi}_y = o(1)$ for any $y \in \widetilde{X}\setminus X$, and that $\widetilde{\varphi}_x - \varphi_x = o(1)$ for any $x \in X$. The proof is organized in three main steps.

Step (1). $\widetilde{\varphi}_y = O(\widetilde{\varphi}_x)$ *whenever* $v_{xy} > 0$. For the moment, we assume $\widetilde{\mathsf{V}}$ fixed (near enough to V so that $\widetilde{v}_{xy} > 0$) and $x, y \in \widetilde{X}$. Under these hypotheses one can argue as follows: Since $\widetilde{\varphi}_{\widetilde{X}}$ maximizes $\widetilde{F}_{\widetilde{X}\widetilde{X}}$, the corresponding value of $\widetilde{F}_{\widetilde{X}\widetilde{X}}$ can be bounded from below by any particular value of the same function. On the other hand, we can bound it from above by the factor $\widetilde{\varphi}_x/(\widetilde{\varphi}_x + \widetilde{\varphi}_y)^{\widetilde{v}_{xy}}$. So, we can write

$$\left(\frac{1}{2}\right)^{N(N-1)} \le \left(\frac{1}{2}\right)^{\sum\limits_{p,q\in\widetilde{X}} \widetilde{t}_{pq}} = \widetilde{F}_{\widetilde{X}\widetilde{X}}(\psi) \;\le\; \widetilde{F}_{\widetilde{X}\widetilde{X}}(\widetilde{\varphi}_{\widetilde{X}}) \;\le\; \left(\frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y}\right)^{\widetilde{v}_{xy}}, \tag{119}$$

where ψ has been taken so that ψ_q has the same value for all $q \in \widetilde{X}$ (and it vanishes for $q \notin \widetilde{X}$). The preceding inequality entails that

$$\widetilde{\varphi}_y \;\le\; \left(2^{\,N(N-1)\,/\,\widetilde{v}_{xy}} - 1\right)\, \widetilde{\varphi}_x. \tag{120}$$

Now, this inequality holds not only for $x, y \in \widetilde{X}$, but it is also trivially true for $y \notin \widetilde{X}$, since then one has $\widetilde{\varphi}_y = 0$. On the other hand, the case $y \in \widetilde{X}$, $x \notin \widetilde{X}$ is not possible at all, because the hypothesis that $\widetilde{v}_{xy} > 0$ would then contradict the fact that $\widetilde{X}$ is a top dominant irreducible component. Finally, we let $\widetilde{\mathsf{V}}$ vary towards V. The desired result is a consequence of (120) since $\widetilde{v}_{xy}$ approaches $v_{xy} > 0$.

Step (2). $\widetilde{\varphi}_y = o(\widetilde{\varphi}_x)$ *for any* $x \in X$ *and* $y \notin X$. Again, we will consider first the special case where $v_{xy} > 0$. In this case the result is easily obtained as a consequence of the equality $(\widetilde{118})$ for $W = X$:

$$\sum_{\substack{x\in X\\ y\notin X}} \widetilde{t}_{xy}\, \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} \;-\; \sum_{\substack{x\in X\\ y\notin X}} \widetilde{v}_{xy} \;=\; 0. \tag{121}$$

In fact, this equality implies that

$$\sum_{\substack{x\in X\\ y\notin X}} \widetilde{t}_{xy} \left(1 - \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y}\right) \;=\; \sum_{\substack{x\in X\\ y\notin X}} \widetilde{v}_{yx}. \tag{122}$$

Now, it is clear that the right-hand side of this equation is $o(1)$ and that each of the terms of the left-hand side is positive. Since $\widetilde{t}_{xy} - v_{xy} = \widetilde{t}_{xy} - t_{xy} =$

$o(1)$, the hypothesis that $v_{xy} > 0$ allows to conclude that $\widetilde{\varphi}_x/(\widetilde{\varphi}_x + \widetilde{\varphi}_y)$ approaches 1, or equivalently, $\widetilde{\varphi}_y = o(\widetilde{\varphi}_x)$. Let us consider now the case of any $x \in X$ and $y \notin X$. Since X is top dominant, we know that there exists a path $x_0 x_1 \dots x_n$ from $x_0 = x$ to $x_n = y$ such that $v_{x_i x_{i+1}} > 0$ for all i. According to step (1) we have $\widetilde{\varphi}_{x_{i+1}} = O(\widetilde{\varphi}_{x_i})$. On the other hand, there must be some j such that $x_j \in X$ but $x_{j+1} \notin X$, which has been seen to imply that $\widetilde{\varphi}_{x_{j+1}} = o(\widetilde{\varphi}_{x_j})$. By combining these facts one obtains the desired result.

Step (3). $\widetilde{\varphi}_x - \varphi_x = o(1)$ *for any* $x \in X$. Consider the equations ($\widetilde{117}$) for $x \in X$ and split the sums in two parts depending on whether $y \in X$ or $y \notin X$:

$$\sum_{\substack{y \in X \\ y \neq x}} \widetilde{t}_{xy} \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} - \sum_{\substack{y \in X \\ y \neq x}} \widetilde{v}_{xy} = \sum_{y \notin X} (\widetilde{v}_{xy} - \widetilde{t}_{xy} \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y}). \tag{123}$$

The last sum is $o(1)$ since step (2) ensures that $\widetilde{\varphi}_y = o(\widetilde{\varphi}_x)$ and we also know that $\widetilde{t}_{xy} - \widetilde{v}_{xy} = \widetilde{v}_{yx} = o(1)$ (because $x \in X$ and $y \notin X$). So $\widetilde{\varphi}$ satisfies a system of the following form, where x and y vary only within X and $\widetilde{w}_{xy}$ is a slight modification of $\widetilde{v}_{xy}$ which absorbs the right-hand side of (123):

$$\mathcal{G}_x(\widetilde{\varphi}_X, \widetilde{\mathsf{V}}, \widetilde{\mathsf{W}}) := \sum_{\substack{y \in X \\ y \neq x}} \widetilde{t}_{xy} \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \widetilde{\varphi}_y} - \sum_{\substack{y \in X \\ y \neq x}} \widetilde{w}_{xy} = 0, \qquad \forall x \in X. \tag{124}$$

Here, the second argument of $\mathcal{G}$ refers to the dependence on $\widetilde{\mathsf{V}}$ through $\widetilde{t}_{xy}$. We know that $\widetilde{t}_{xy} - t_{xy} = o(1)$ and also that $\widetilde{w}_{xy} - v_{xy} = (\widetilde{w}_{xy} - \widetilde{v}_{xy}) + (\widetilde{v}_{xy} - v_{xy}) = o(1)$. So we are interested in the preceding equation near the point $(\varphi_X, \mathsf{V}, \mathsf{V})$. Now in this point we have $\mathcal{G}(\varphi_X, \mathsf{V}, \mathsf{V}) = \mathcal{F}(\varphi_X, \mathsf{V}) = 0$, as well as $(\partial \mathcal{G}_x / \partial \widetilde{\varphi}_y)(\varphi_X, \mathsf{V}, \mathsf{V}) = (\partial \mathcal{F}_x / \partial \varphi_y)(\varphi_X, \mathsf{V})$. Therefore, the implicit function theorem can be applied similarly as in Theorem 11.1, with the result that $\widetilde{\varphi}_X = \mathcal{H}(\widetilde{\mathsf{V}}, \widetilde{\mathsf{W}})$, where $\mathcal{H}$ is a smooth function which satisfies $\mathcal{H}(\mathsf{V}, \mathsf{V}) = \varphi_X$. In particular, the continuity of $\mathcal{H}$ allows to conclude that $\widetilde{\varphi}_X$ approaches φ_X, since we know that both $\widetilde{\mathsf{V}}$ and $\widetilde{\mathsf{W}}$ approach V.

Finally, by combining the results of steps (2) and (3) one obtains $\widetilde{\varphi}_y = o(1)$ for any $y \notin X$. □

Remarks

1. The convergence of φ^n to φ is a necessary condition for φ^n being a maximizing sequence but not a sufficient one. The preceding proof shows that a necessary and sufficient condition is that the ratios $\varphi^n_y / \varphi^n_z$ tend to 0

whenever $y \rhd z$, whereas, if $y \equiv z$, i.e. if y and z belong to the same irreducible component Z, these ratios approach the homologous ones for the unique maximizer of F_{ZZ}.

2. If there is not a dominant component then the maximizing sequences can have multiple limit points. However, as we will see in the next section, the projected Llull matrices are always in the hypotheses of Theorem 11.2.

12 The fraction-like rates

Let us recall from § 2.9 that the fraction-like rates φ_x will be obtained by applying Zermelo's method to the projected Llull matrix (v^π_{xy}).

The next results show that this matrix has a very special structure in connection with irreducibility.

Lemma 12.1. *The projected Llull matrix (v^π_{xy}) has the following properties for any admissible order ξ (p' denotes the immediate successor of p in ξ):*

(a) *If $x \overset{\xi}{\succ} y$ and $v^\pi_{yx} = 0$, then $v^\pi_{p'p} = 0$ for some p such that $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$.*

(b) *If $v^\pi_{p'p} = 0$ for some p, then $v^\pi_{yx} = 0$ for all x, y such that $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$.*

(c) *If $x \overset{\xi}{\succ} y$ and $v^\pi_{xy} = 0$, then $v^\pi_{ab} = 0$ for all a, b such that $x \overset{\xi}{\succeq} a$.*

Proof. Part (a). Assume that $x \overset{\xi}{\succ} y$. Then v^π_{yx} is the left end of the interval γ_{xy}. Now, since $\gamma_{xy} = \bigcup\{\gamma_{pp'} \,|\, x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y\}$, a vanishing left end for γ_{xy} implies the same property for some of the $\gamma_{pp'}$, i.e. $v^\pi_{p'p} = 0$.

Part (b). According to Theorem 9.3.(a), $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$ implies the inequalities $v^\pi_{yx} \le v^\pi_{p'x} \le v^\pi_{p'p}$. Therefore, $v^\pi_{p'p} = 0$ implies $v^\pi_{yx} = 0$.

Part (c). For $x \overset{\xi}{\succ} y$, $v^\pi_{xy} = 0$ means that $\gamma_{xy} = [0, 0]$. This implies that $\gamma_{pp'} = [0, 0]$ for all p such that $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$. Now, according to Lemma 9.1, the barycentres of the intervals $\gamma_{qq'}$ decrease or stay the same when q moves towards the bottom. So $\gamma_{qq'} = [0, 0]$ for all q such that $x \overset{\xi}{\succeq} q$. As a consequence, we immediately get $v^\pi_{ab} = 0$ for any a, b such that $x \overset{\xi}{\succeq} a, b$. Furthermore, for $b \overset{\xi}{\succ} x \overset{\xi}{\succeq} a$, part (a) of Theorem 9.3 gives the following inequalities: $v^\pi_{ab} \le v^\pi_{ax}$ for $a \ne x$, and $v^\pi_{ab} \le v^\pi_{ay}$ for $a = x$, where the right-hand sides are already known to vanish. So v^π_{ab} vanishes also for such a and b. □

Proposition 12.2. *Let us assume that the projected Llull matrix (v^π_{xy}) is not the zero matrix. Let us consider the set*

$$X = \{x \in A \mid v^\pi_{p'p} > 0 \textit{ for all } p \textit{ such that } p \overset{\xi}{\succ} x\}, \tag{125}$$

where the right-hand side makes use of an admissible order ξ. *This set has the following properties:*

(a) *It does not depend on the admissible order* ξ.

(b) $v^\pi_{xy} > 0$ *for any* $x \in X$ *and* $y \in A$.

(c) $v^\pi_{yx} = 0$ *for any* $x \in X$ *and* $y \notin X$.

(d) $r_x < r_y$ *for any* $x \in X$ *and* $y \notin X$.

(e) X *is the top dominant irreducible component of* A *for* (v^π_{xy}).

Proof. Statement (a) will be proved at the end. The definition of X is equivalent to the following one: $X = A$ if $v^\pi_{p'p} > 0$ for any p; otherwise, $X = \{x \in A \,|\, x \overset{\xi}{\preceq} h\}$, where h is the topmost (in ξ) element of A which satisfies $v^\pi_{h'h} = 0$. In particular, X reduces to the topmost element of A when $v^\pi_{p'p} = 0$ for any p.

Statement (b). In view of Lemma 12.1.(a), the definition of X implies that $v^\pi_{yx} > 0$ for any $x, y \in X$ such that $x \overset{\xi}{\succ} y$. This statement is empty when X reduces to a single element a, but then we will make use of the fact that $v^\pi_{aa'} > 0$, which is bound to happen because otherwise Lemma 12.1.(c) would entail that the whole matrix is zero, against our hypothesis. These facts imply statement (b) by virtue of Theorem 9.3.(a).

Statement (c). If $X = A$ there is nothing to prove. Otherwise, if h is the above-mentioned topmost element of A which satisfies $v^\pi_{h'h} = 0$, then Lemma 12.1.(b) ensures that $v^\pi_{yx} = 0$ for any x, y such that $x \overset{\xi}{\preceq} h \overset{\xi}{\succ} y$, i. e. any $x \in X$ and $y \notin X$.

Statement (d). If $X = A$ there is nothing to prove. Otherwise, the result follows from parts (b) and (c) together with Lemma 10.1.(c).

Statement (e). This is an immediate consequence of (b) and (c).

Statement (a). A top dominant irreducible component is always unique because the relation of dominance between irreducible components is anti-symmetric. □

Remarks

1. In the complete case, the average ranks $\bar{r}_x$ defined by equation (6) are easily seen to satisfy already a property of the same kind as (d): if X and Y are two irreducible components of (v_{xy}) such that X dominates Y, then $\bar{r}_x < \bar{r}_y$ for all $x \in X$ and $y \in Y$ [*24*: Thm. 2.5].

2. Even in the complete case, Zermelo's rates associated with the original Llull matrix (v_{xy}) are not necessarily compatible with the average ranks $\bar{r}_x$. However, as we will see below, the projected Llull matrices will always enjoy such a compatibility.

From now on, X denotes the top dominant irreducible component whose existence is established by the preceding proposition. According to Theorem 11.2, the fraction-like rates φ_x vanish if and only if $x \in A \setminus X$ and their values for $x \in X$ are determined by the restriction of (v^π_{xy}) to $x, y \in X$. More specifically, the latter are determined by the condition of maximizing the function

$$F(\varphi) \;=\; \prod_{\{x,y\}} \frac{{\varphi_x}^{v^\pi_{xy}}\, {\varphi_y}^{v^\pi_{yx}}}{(\varphi_x + \varphi_y)^{t^\pi_{xy}}}, \tag{126}$$

under the restriction

(37) $$\sum_x \varphi_x \;=\; f. \tag{127}$$

where we will understand that x and y are restricted to X, and f denotes the fraction of non-empty votes (i. e. $f = F/V$ where F is the number of non-empty votes and V is the total number of votes). Moreover, we know that $(\varphi_x \,|\, x \in X)$ is the solution of the following system of equations besides (127):

(36) $$\sum_{y \neq x} t^\pi_{xy}\, \frac{\varphi_x}{\varphi_x + \varphi_y} \;=\; \sum_{y \neq x} v^\pi_{xy}. \tag{128}$$

where the sums extend to all $y \neq x$ in X. The next result shows that the resulting fraction-like rates are fully compatible with the rank-like ones except for the vanishing of those outside the top dominant component.

Theorem 12.3.

(a) $\varphi_x > \varphi_y \implies r_x \,<\, r_y$.

(b) $r_x \,<\, r_y \implies$ *either* $\varphi_x > \varphi_y$ *or* $\varphi_x = \varphi_y = 0$.

Proof. Let us begin by noticing that both statements hold if $\varphi_y = 0$, i. e. if $y \notin X$. In the case of statement (a), this is true because of Proposition 12.2.(d). So, we can assume that $\varphi_y > 0$, i. e. $y \in X$. But in this case, each one of the hypotheses of the present theorem implies that $\varphi_x > 0$, i. e. $x \in X$. In the case of statement (b), this is true because of Proposition 12.2.(d) (with x and y interchanged with each other) and the fact that X is a top interval for any admissible order. So, from now on we can assume that x and y are both in X, or, on account of Theorem 11.2, that $X = A$.

Statement (a): It will be proved by seeing that a simultaneous occurrence of the inequalities $\varphi_x > \varphi_y$ and $r_x \geq r_y$ would entail a contradiction with the fact that φ is the unique maximizer of $F(\varphi)$. More specifically, we will see that one would have $F(\widetilde{\varphi}) \geq F(\varphi)$ where $\widetilde{\varphi}$ is obtained from φ by interchanging the values of φ_x and φ_y, that is

$$\widetilde{\varphi}_z = \begin{cases} \varphi_y, & \text{if } z = x, \\ \varphi_x, & \text{if } z = y, \\ \varphi_z, & \text{otherwise.} \end{cases} \tag{129}$$

In fact, $\widetilde{\varphi}$ differs from φ only in the components associated with x and y, so that

$$\begin{aligned} \frac{F(\widetilde{\varphi})}{F(\varphi)} &= \left(\frac{\widetilde{\varphi}_x}{\varphi_x}\right)^{v^\pi_{xy}} \prod_{z\neq x,y} \left(\frac{\widetilde{\varphi}_x/(\widetilde{\varphi}_x+\varphi_z)}{\varphi_x/(\varphi_x+\varphi_z)}\right)^{v^\pi_{xz}} \left(\frac{\varphi_x+\varphi_z}{\widetilde{\varphi}_x+\varphi_z}\right)^{v^\pi_{zx}} \\ &\times \left(\frac{\widetilde{\varphi}_y}{\varphi_y}\right)^{v^\pi_{yx}} \prod_{z\neq x,y} \left(\frac{\widetilde{\varphi}_y/(\widetilde{\varphi}_y+\varphi_z)}{\varphi_y/(\varphi_y+\varphi_z)}\right)^{v^\pi_{yz}} \left(\frac{\varphi_y+\varphi_z}{\widetilde{\varphi}_y+\varphi_z}\right)^{v^\pi_{zy}}. \end{aligned} \tag{130}$$

More particularly, in the case of (129) this expression becomes

$$\frac{F(\widetilde{\varphi})}{F(\varphi)} = \left(\frac{\varphi_y}{\varphi_x}\right)^{v^\pi_{xy}-v^\pi_{yx}} \prod_{z\neq x,y} \left(\frac{\varphi_y/(\varphi_y+\varphi_z)}{\varphi_x/(\varphi_x+\varphi_z)}\right)^{v^\pi_{xz}-v^\pi_{yz}} \left(\frac{\varphi_y+\varphi_z}{\varphi_x+\varphi_z}\right)^{v^\pi_{zy}-v^\pi_{zx}}, \tag{131}$$

where all of the bases are strictly less than 1, since $\varphi_x > \varphi_y$, and all of the the exponents are non-positive, because of Lemma 10.1.(c). Therefore, the product is greater than or equal to 1, as claimed.

Statement (b): Since we are assuming $x, y \in X$, it is a matter of proving that $r_x < r_y \Rightarrow \varphi_x > \varphi_y$. On the other hand, by making use of the contrapositive of (a), the problem reduces to proving that $\varphi_x = \varphi_y \Rightarrow r_x = r_y$.

Similarly to above, this implication will be proved by seeing that a simultaneous occurrence of the equality $\varphi_x = \varphi_y =: \omega$ together with the inequality $r_x < r_y$ (by symmetry it suffices to consider this one) would entail a contradiction with the fact that φ is the unique maximizer of $F(\varphi)$. More specifically, here we will see that one would have $F(\widetilde{\varphi}) > F(\varphi)$ where $\widetilde{\varphi}$ is obtained from φ by slightly increasing φ_x while decreasing φ_y, that is

$$\widetilde{\varphi}_z = \begin{cases} \omega+\epsilon, & \text{if } z = x, \\ \omega-\epsilon, & \text{if } z = y, \\ \varphi_z, & \text{otherwise.} \end{cases} \tag{132}$$

This claim will be proved by checking that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \log \frac{F(\widetilde{\varphi})}{F(\varphi)}\bigg|_{\epsilon=0} > 0. \tag{133}$$

In fact, (130) entails that

$$
\begin{aligned}
\log \frac{F(\widetilde{\varphi})}{F(\varphi)} \;=\; & C \,+\, v^{\pi}_{xy} \log \widetilde{\varphi}_x + v^{\pi}_{yx} \log \widetilde{\varphi}_y \\
& + \sum_{z\neq x,y} \Big(v^{\pi}_{xz} \log \frac{\widetilde{\varphi}_x}{\widetilde{\varphi}_x + \varphi_z} + v^{\pi}_{yz} \log \frac{\widetilde{\varphi}_y}{\widetilde{\varphi}_y + \varphi_z} \Big) \\
& - \sum_{z\neq x,y} \Big(v^{\pi}_{zy} \log(\widetilde{\varphi}_y + \varphi_z) + v^{\pi}_{zx} \log(\widetilde{\varphi}_x + \varphi_z) \Big),
\end{aligned}
\tag{134}
$$

where C does not depend on ϵ. Therefore, in view of (132) we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\epsilon} \log \frac{F(\widetilde{\varphi})}{F(\varphi)} \bigg|_{\epsilon=0} \;=\; & (v^{\pi}_{xy} - v^{\pi}_{yx}) \, \frac{1}{\omega} \,+\, \sum_{z\neq x,y} (v^{\pi}_{xz} - v^{\pi}_{yz}) \, \frac{\varphi_z}{\omega(\omega+\varphi_z)} \\
& + \sum_{z\neq x,y} (v^{\pi}_{zy} - v^{\pi}_{zx}) \, \frac{1}{\omega+\varphi_z}.
\end{aligned}
\tag{135}
$$

Now, according to Lemma 10.1.(b, c), the assumption that $r_x < r_y$ implies the inequalities $v^{\pi}_{xy} > v^{\pi}_{yx}$, $v^{\pi}_{xz} \geq v^{\pi}_{yz}$ and $v^{\pi}_{zy} \geq v^{\pi}_{zx}$, which result indeed in (133). □

The next proposition establishes property H:

Proposition 12.4. *In the case of plumping votes the fraction-like rates coincide with the fractions of the vote obtained by each option.*

Proof. Proposition 9.4 ensures that the projected scores coincide with the original ones. So we have $v^{\pi}_{xy} = f_x$ and $t^{\pi}_{xy} = f_x + f_y$. In these circumstances it is obvious that equations (127–128) are satisfied if we take $\varphi_x = f_x$. So it suffices to invoke the uniqueness of solution of this system. □

13 Continuity

We claim that both the rank-like rates r_x and the fraction-like ones φ_x are continuous functions of the binary scores v_{xy}. The main difficulty in proving this statement lies in the admissible order ξ, which plays a central role in the computations. Since ξ varies in a discrete set, its dependence on the data cannot be continuous at all. Even so, we claim that the final result is still a continuous function of the data.

In this connection, one can consider as data the normalized Llull matrix (v_{xy}), its domain of variation being the set Ω introduced in § 3.3. Alternatively, one can consider as data the relative frequencies of the possible votes, i. e. the coefficients α_k mentioned also in § 3.3.

Theorem 13.1. *The following objects depend continuously on the Llull matrix* (v_{xy})*: the projected scores* v^π_{xy}*, the rank-like rates* r_x*, and the fraction-like rates* φ_x*.*

Proof. Let us begin by considering the dependence of the rank-like rates and the fraction-like rates on the projected scores. In the case of the rank-like rates, this dependence is given by formula (8), which is not only continuous but even linear (non-homogeneous). In the case of the fraction-like rates, their dependence on the projected scores is more involved, but is is still continuous. In fact, Theorem 11.2.(c) ensures such a continuity under the hypothesis that there is a top irreducible component, which hypothesis is satisfied by virtue of Proposition 12.2.(e).

So we are left with the problem of showing that the projection $P : (v_{xy}) \mapsto (v^\pi_{xy})$ is continuous. As it has been mentioned above, this is not so clear, since the projected scores are the result of certain operations which are based upon an admissible order ξ which is determined separately. However, we will see that, on the one hand, P is continuous as long as ξ remains unchanged, and on the other hand, the results of § 8, 9 allow to conclude that P is continuous on the whole of Ω in spite of the fact that ξ can change. In the following we will use the following notation: for every total order ξ, we denote by Ω_ξ the subset of Ω which consists of the Llull matrices for which ξ is an admissible order, and we denote by P_ξ the restriction of P to Ω_ξ.

We claim that the mapping P_ξ is continuous for every total order ξ. In order to check the truth of this statement, one has to go over the different mappings whose composition defines P_ξ (see § 2.8), namely: $(v_{xy}) \mapsto (v^*_{xy}) \mapsto (m^\nu_{xy})$, $(v_{xy}) \mapsto (t_{xy})$, $(m^\nu_{xy}) \mapsto (m^\sigma_{xy})$, $\Psi : ((m^\sigma_{xx'}), (t_{xy})) \mapsto (t^\sigma_{xy})$, and finally $((m^\sigma_{xx'}), (t^\sigma_{xx'})) \mapsto (v^\pi_{xy})$. Quite a few of these mappings involve the max and min operations, which are certainly continuous. For instance, the last mapping above can be written as $v^\pi_{xy} = \max\{\, (t^\sigma_{pp'} + m^\sigma_{pp'})/2 \mid x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y\}$ and $v^\pi_{yx} = \min\{\, (t^\sigma_{pp'} - m^\sigma_{pp'})/2 \mid x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y\}$ for $x \overset{\xi}{\succ} y$. Concerning the operator Ψ, let us recall that its output is the orthogonal projection of (t_{xy}) onto a certain convex set determined by $(m^\sigma_{xx'})$; a general result of continuity for such an operation can be found in [*9*].

Finally, the continuity of P (and the fact that it is well-defined) is a consequence of the following facts (see for instance [*26*: §2-7]): (a) $\Omega = \bigcup_\xi \Omega_\xi$; this is true because of the existence of ξ (Corollary 8.3). (b) Ω_ξ is a closed subset of Ω; this is true because Ω_ξ is described by a set of non-strict inequalities which concern quantities that are continuous functions of (v_{xy}) (namely the inequalities $m^\nu_{xy} \geq 0$ whenever $xy \in \xi$). (c) ξ varies over a finite set. (d) P_ξ coincides with P_η at $\Omega_\xi \cap \Omega_\eta$, as it is proved in Theorem 9.2. □

Corollary 13.2. *The rank-like rates, as well as the fraction-like ones, depend continuously on the relative frequency of each possible content of an individual vote.*

Proof. It suffices to notice that the Llull matrix (v_{xy}) is simply the center of gravity of the distribution specified by these relative frequencies (formula (43) of § 3.3). $\square$

14 Decomposition

Properties E and G are concerned with having a partition of A in two sets X and Y such that the rates for $x \in X$ can be obtained by restricting the attention to V_{XX}, i. e. the $v_{x\bar{x}}$ with $x, \bar{x} \in X$ (and similarly for $y \in Y$ in the case of property E).

More specifically, property E considers the case where the following equalities are satisfied:

$$r_x = \widetilde{r}_x, \qquad \text{for all } x \in X, \tag{136}$$

$$r_y = \widetilde{r}_y + |X|, \qquad \text{for all } y \in Y, \tag{137}$$

where $\widetilde{r}_x$ and $\widetilde{r}_y$ denote the rank-like rates which are determined respectively by the matrices V_{XX} and V_{YY}. Property E states that in the complete case these equalities are equivalent to having

$$v_{xy} = 1 \quad (\text{and therefore } v_{yx} = 0) \quad \text{whenever } \ xy \in X \times Y. \tag{138}$$

In the following we will continue using a tilde to distinguish between homologous objects associated respectively with the whole matrix V and with its submatrices V_{XX} and V_{YY}.

First of all we explore the implications of condition (138).

Lemma 14.1. *Given a partition $A = X \cup Y$ in two disjoint nonempty sets, one has the following implications:*

$$\left.\begin{array}{c} v_{xy} = 1 \\ \forall\, xy \in X \times Y \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} m^{\nu}_{xy} = 1 \\ \forall\, xy \in X \times Y \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} v^{\pi}_{xy} = 1 \\ \forall\, xy \in X \times Y \end{array}\right. \tag{139}$$

If the individual votes are complete, or alternatively, if they are transitive relations, then the converse of the first implication holds too.

Proof. Assume that $v_{xy} = 1$ for all $xy \in X \times Y$. Then $v_{yx} = 0$, for all such pairs, which implies that v_γ vanishes for any path γ which goes from Y to X. This fact, together with the inequality $v^*_{xy} \geq v_{xy}$, entails the following equalities for all $x \in X$ and $y \in Y$: $v^*_{yx} = 0$, $v^*_{xy} = 1$, and consequently $m^\nu_{xy} = 1$.

Assume now that $m^\nu_{xy} = 1$ for all $xy \in X \times Y$. Let ξ be an admissible order. As an immediate consequence of the definition, it includes the set $X \times Y$. Let ℓ be the last element of X according to ξ. From the present hypothesis it is clear that $m^\sigma_{\ell\ell'} = 1$, i. e. $\gamma_{\ell\ell'} = [0, 1]$, which entails that $\gamma_{xy} = [0, 1]$, i. e. $v^\pi_{xy} = 1$, for every $xy \in X \times Y$.

Assume now that $v^\pi_{xy} = 1$ for all $xy \in X \times Y$. Let ξ be an admissible order. Here too, we are ensured that it includes the set $X \times Y$; this is so by virtue of Theorem 9.3.(a). Let ℓ be the last element of X according to ξ. From the fact that $m^\sigma_{\ell\ell'} = m^\pi_{\ell\ell'} = 1$, one infers that $m^\nu_{xy} = 1$ for all $xy \in X \times Y$.

Finally, let us assume again that $m^\nu_{xy} = 1$ for all $xy \in X \times Y$. Since $m^\nu_{xy} = v^*_{xy} - v^*_{yx}$ and both terms of this difference belong to $[0, 1]$, the only possibility is $v^*_{xy} = 1$ and $v^*_{yx} = 0$, which implies that $v_{yx} = 0$. In the complete case, this equality is equivalent to $v_{xy} = 1$. In the case where the individual votes are transitive relations, one can reach the same conclusion in the following way: The equality $v^*_{xy} = 1$ implies the existence of a path $x_0 x_1 \ldots x_n$ from x to y such that $v_{x_i x_{i+1}} = 1$ for all i. But this means that all of the votes include each of the pairs $x_i x_{i+1}$ of this path. So, if they are transitive relations, all of them include also the pair xy, i. e. $v_{xy} = 1$. □

Lemma 14.2. *Condition* (138) *implies, for any admissible order, the following equalities:*

$$m^\sigma_{xx'} = \widetilde{m}^\sigma_{xx'}, \qquad \textit{whenever } x, x' \in X, \tag{140}$$

$$m^\sigma_{yy'} = \widetilde{m}^\sigma_{yy'}, \qquad \textit{whenever } y, y' \in Y, \tag{141}$$

$$t^\pi_{x\bar{x}} = 1, \qquad \textit{for all } x, \bar{x} \in X. \tag{142}$$

Proof. As we saw in the proof of Lemma 14.1, condition (138) implies the vanishing of v_γ for any path γ which goes from Y to X. Besides the conclusions obtained in that lemma, this implies also the following equalities:

$$v^*_{x\bar{x}} = \widetilde{v}^*_{x\bar{x}}, \qquad m^\nu_{x\bar{x}} = \widetilde{m}^\nu_{x\bar{x}}, \qquad \text{for all } x, \bar{x} \in X, \tag{143}$$

$$v^*_{y\bar{y}} = \widetilde{v}^*_{y\bar{y}}, \qquad m^\nu_{y\bar{y}} = \widetilde{m}^\nu_{y\bar{y}}, \qquad \text{for all } y, \bar{y} \in Y. \tag{144}$$

Let us fix an admissible order ξ. The second equality of (139) ensures not only that ξ includes the set $X \times Y$, but it can also be combined with (143) and (144) to obtain respectively (140) and (141). On the other hand, the third equality of (139) implies that $t^{\pi}_{xy} = 1$ for all $xy \in X \times Y$, from which the pattern of growth of the projected turnovers —more specifically, equation (83.2)— allows to obtain (142). $\square$

Theorem 14.3. *In the complete case one has the following equivalences:* (136) $\Longleftrightarrow$ (137) $\Longleftrightarrow$ (138).

Proof. Since we are considering the complete case, we can make use of the margin-based procedure (§ 2.6). The proof is organized in two parts:

Part (a): (138) $\Longrightarrow$ (136) and (137). As a consequence of the equalities (140) and (141), the margin-based procedure —more specifically, steps (13) and (14)— results in the following equalities:

$$m^{\pi}_{x\bar{x}} = \widetilde{m}^{\pi}_{x\bar{x}}, \qquad \text{for all } x, \bar{x} \in X, \tag{145}$$

$$m^{\pi}_{y\bar{y}} = \widetilde{m}^{\pi}_{y\bar{y}}, \qquad \text{for all } y, \bar{y} \in Y. \tag{146}$$

On the other hand, the third equality of (139) is equivalent to saying that

$$m^{\pi}_{xy} = 1, \qquad \text{for all } xy \in X \times Y. \tag{147}$$

When the projected margins are introduced in (9) these equalities result in (136) and (137).

Part (b): (136) $\Rightarrow$ (138); (137) $\Rightarrow$ (138). On account of formula (9), conditions (136) and (137) are easily seen to be respectively equivalent to the following equalities:

$$\sum_{\substack{y\in A\\ y\neq x}} m^{\pi}_{xy} = \sum_{\substack{\bar{x}\in X\\ \bar{x}\neq x}} \widetilde{m}^{\pi}_{x\bar{x}} + |Y|, \qquad \text{for all } x \in X, \tag{148}$$

$$\sum_{\substack{x\in A\\ x\neq y}} m^{\pi}_{yx} = \sum_{\substack{\bar{y}\in Y\\ \bar{y}\neq y}} \widetilde{m}^{\pi}_{y\bar{y}} - |X|, \qquad \text{for all } y \in Y. \tag{149}$$

Let us add up respectively the equalities (148) over $x \in X$ and the equalities (149) over $y \in Y$. Since $m^{\pi}_{pq} + m^{\pi}_{qp} = \widetilde{m}^{\pi}_{pq} + \widetilde{m}^{\pi}_{qp} = 0$, we obtain

$$\sum_{\substack{x\in X\\ y\in Y}} m^{\pi}_{xy} = |X|\,|Y|, \tag{150}$$

$$\sum_{\substack{y\in Y\\ x\in X}} m^{\pi}_{yx} = -|X|\,|Y|. \tag{151}$$

Since the projected margins belong to $[-1, 1]$, the preceding equalities imply respectively

$$m^\pi_{xy} \;=\; 1, \qquad \text{for all } x \in X \text{ and } y \in Y, \tag{152}$$

$$m^\pi_{yx} \;=\; -1, \qquad \text{for all } x \in X \text{ and } y \in Y, \tag{153}$$

(which are equivalent to each other since $m^\pi_{xy} + m^\pi_{yx} = 0$). Finally, either of these equalities implies that $v^\pi_{xy} = 1$ for all $xy \in X \times Y$, from which Lemma 14.1 allows to obtain (138). □

The following propositions do not require the votes to be complete, but they require them to be rankings, or, more generally, in the case of Proposition 14.6, to be transitive relations.

Lemma 14.4. *In the case of ranking votes, condition* (138) *implies that* $t_{xy} = 1$ *for any* $x \in X$ *and* $y \in A$.

Proof. In fact, even if we are dealing with truncated ranking votes, the rules that we are using for translating them into binary preferences —namely, rules (a–d) of § 2.1— entail the following implications: (i) $v_{xy} = 1$ for some $y \in A$ implies that x is explicitly mentioned in all of the ranking votes; and (ii) x being explicitly mentioned in all of the ranking votes implies that $t_{xy} = 1$ for any $y \in A$. □

Proposition 14.5. *In the case of ranking votes, condition* (138) *implies* (136).

Proof. Let us fix an admissible order. According to Lemma 14.2, we have $t^\pi_{x\bar{x}} = 1$ for all $x, \bar{x} \in X$. On the other hand, Lemma 14.4 ensures that $t_{x\bar{x}} = 1$ for all $x, \bar{x} \in X$, from which it follows that $\widetilde{t}^\pi_{x\bar{x}} = 1$ for all $x, \bar{x} \in X$ (since $\widetilde{t}^\pi_{x\bar{x}}$ are the turnovers obtained from the restriction to the matrix V_{XX}, which belongs to the complete case). In particular, we have $t^\sigma_{xx'} = \widetilde{t}^\sigma_{xx'} = 1$ whenever $x, x' \in X$. On the other hand, Lemma 14.2 ensures also that $m^\sigma_{xx'} = \widetilde{m}^\sigma_{xx'}$ whenever $x, x' \in X$. These equalities entail that $v^\pi_{x\bar{x}} = \widetilde{v}^\pi_{x\bar{x}}$ for all $x, \bar{x} \in X$. By Lemma 14.1 we know also that $v^\pi_{xy} = 1$ for all $xy \in X \times Y$. Therefore,

$$r_x \;=\; N - \sum_{\substack{y \neq x \\ y \in A}} v^\pi_{xy} \;=\; |X| - \sum_{\substack{\bar{x} \neq x \\ \bar{x} \in X}} \widetilde{v}^\pi_{x\bar{x}} \;=\; \widetilde{r}_x, \qquad \forall x \in X.$$

□

Proposition 14.6. *Assume that the individual votes are transitive relations. In this case, the equality*

$$\sum_{x\in X} r_x \;=\; |X|(|X|+1)/2 \tag{154}$$

implies (138) *(with* $Y = A\setminus X$ *).*

Proof. Let us introduce formula (8) for r_x into (154). By using the fact that $v^\pi_{x\bar x} + v^\pi_{\bar x x} \le 1$, one obtains that

$$\sum_{\substack{x\in X\\ y\in Y}} v^\pi_{xy} \;\ge\; |X|\,|Y|. \tag{155}$$

The only possible way to satisfy this inequality is having $v^\pi_{xy} = 1$ for all $xy \in X\times Y$. Finally, (138) follows by virtue of Lemma 14.1 since we are assuming that the individual votes are transitive relations. □

Corollary 14.7. *Assume that the votes are rankings. Then* $r_x = 1$ *if and only if all voters have put* x *into first place.*

Proof. It suffices to apply Propositions 14.5 and 14.6 with $X = \{x\}$. □

The next theorem establishes property G.

Theorem 14.8. (a) *In the complete case, or alternatively, under the hypothesis that the individual votes are rankings, one has the following implication: Assume that* $X\subset A$ *has the property that* $v_{xy} = 1$ *whenever* $x\in X$ *and* $y\in Y = A\setminus X$*, and that there is no proper subset with the same property. In that case, the fraction-like rates satisfy* $\varphi_x = \widetilde\varphi_x > 0$ *for all* $x\in X$ *and* $\varphi_y = 0$ *for all* $y\in Y$. (b) *In the complete case the converse implication holds too.*

Proof. Statement (a). Let us fix an admissible order ξ. By Lemma 14.1, the hypothesis that $v_{xy} = 1$ for all $xy\in X\times Y$ implies the following facts for all $xy\in X\times Y$: $m^\nu_{xy} = 1$, $xy\in\xi$, $v^\pi_{xy} = 1$, $v^\pi_{yx} = 0$. On the other hand, we can see that under the present hypothesis one has

$$v^\pi_{x\bar x} = \widetilde v^\pi_{x\bar x}, \qquad \text{for any } x,\bar x\in X. \tag{156}$$

In the complete case this follows from Lemma 14.2. Under the alternative hypothesis that the individual votes are rankings, it can be obtained as in the proof of Proposition 14.5 as a consequence of Lemma 14.2 and the fact that in this case $t^\pi_{x\bar{x}} = \widetilde{t}^\pi_{x\bar{x}} = 1$ for any $x, \bar{x} \in X$.

Now, according to Lemma 12.1, the matrix (v^π_{xy}) has a top dominant irreducible component $\widehat{X}$. Since $v^\pi_{yx} = 0$ for all $xy \in X \times Y$, it is clear that $\widehat{X} \subseteq X$. However, a strict inclusion $\widehat{X} \subset X$ would imply $v^\pi_{x\hat{x}} = 0$ and therefore $v^\pi_{\hat{x}x} = 1$ for any $x \in X \setminus \widehat{X}$ and $\hat{x} \in \widehat{X}$. Since we also have $v^\pi_{xy} = 1$ for $x \in X$ and $y \notin X$, we would get $v^\pi_{\hat{x}\hat{y}} = 1$ for all $\hat{x} \in \widehat{X}$ and $\hat{y} \notin \widehat{X}$, which would imply, by Lemma 14.1, that $v_{\hat{x}\hat{y}} = 1$ for all such pairs. This would contradict the supposed minimality of X. So, X itself is the top dominant irreducible component of the matrix (v^π_{xy}).

By making use of Theorem 11.2, it follows that $\varphi_x = \widetilde{\varphi}_x > 0$ for all $x \in X$ and $\varphi_y = 0$ for all $y \in Y$. In principle, $\widetilde{\varphi}_x$ are here the fraction-like rates computed from the restriction of (v^π_{xy}) to the set X. However, (156) allows to view them also as the fraction-like rates computed from the matrix $(\widetilde{v}^\pi_{xy})$, which by definition has been worked out from the restriction of (v_{xy}) to $x, y \in X$.

Statement (b). Let us begin by noticing that the hypothesis that $\varphi_x > 0$ for all $x \in X$ and $\varphi_y = 0$ for all $y \in Y = A \setminus X$ implies that X is the top dominant irreducible component of the matrix (v^π_{xy}). In fact, otherwise Theorem 11.2 would imply the existence of some $x \in X$ with $\varphi_x = 0$ or some $y \in Y$ with $\varphi_y > 0$. In particular, we have $v^\pi_{yx} = 0$ for all $xy \in X \times Y$. Because of the completeness assumption, this implies that $v^\pi_{xy} = 1$ and —by Lemma 14.1— $v_{xy} = 1$ for all those pairs. Finally, let us see that X is minimal for this property: If we had $\widehat{X} \subset X$ satisfying $v_{\hat{x}\hat{y}} = 1$ for all $\hat{x}\hat{y} \in \widehat{X} \times \widehat{Y}$ with $\widehat{Y} = A \setminus \widehat{X}$, then Lemma 14.1 would give $v^\pi_{\hat{x}\hat{y}} = 1$ and therefore $v^\pi_{\hat{y}\hat{x}} = 0$ for all such pairs, so X could not be the top dominant irreducible component of the matrix (v^π_{xy}). □

15 The majority principle

Theorem 15.1. *The relation $\mu(v^*)$ complies with the majority principle: Let A be partitioned in two sets X and Y with the property that $v_{xy} > 1/2$ whenever $x \in X$ and $y \in Y$; in that case, $\mu(v^*)$ includes any pair xy with $x \in X$ and $y \in Y$.*

Proof. Assume that $x \in X$ and $y \in Y$. Since $v^*_{xy} \geq v_{xy}$, the hypothesis of the theorem entails that $v^*_{xy} > 1/2$. On the other hand, let γ be a path from y to x such that $v^*_{yx} = v_\gamma$; since it goes from Y to X, this path must contain at least one link $y_i y_{i+1}$ with $y_i \in Y$ and $y_{i+1} \in X$; now, for this link we have $v_{y_i y_{i+1}} \leq 1 - v_{y_{i+1} y_i} < 1/2$, which entails that $v^*_{yx} = v_\gamma < 1/2$. Therefore, we get $v^*_{yx} < 1/2 < v^*_{xy}$, i.e. $xy \in \mu(v^*)$. □

Corollary 15.2. *The social ranking determined by the rank-like rates complies with the majority principle: Let A be partitioned in two sets X and Y with the property that $v_{xy} > 1/2$ whenever $x \in X$ and $y \in Y$; in that case, the inequality $r_x < r_y$ holds for any $x \in X$ and $y \in Y$.*

Proof. It follows from Theorem 15.1 by virtue of part (c) of Theorem 10.2. □

Corollary 15.3. *In the complete case the social ranking determined by the rank-like rates complies with the Condorcet principle: If x has the property that $v_{xy} > v_{yx}$ for any $y \neq x$, then $r_x < r_y$ for any $y \neq x$.*

Proof. In the complete case $v_{xy} > v_{yx}$ implies $v_{xy} > 1/2$. So, it suffices to apply the preceding result with $X = \{x\}$ and $Y = A \setminus X$. □

16 Clone consistency

The notion of a cluster (of clones) was defined in §5 in connection with a binary relation: A subset $C \subseteq A$ is said to be a cluster for a relation ρ when, for any $x \notin C$, having $ax \in \rho$ for some $a \in C$ implies $bx \in \rho$ for any $b \in C$, and similarly, having $xa \in \rho$ for some $a \in C$ implies $xb \in \rho$ for any $b \in C$.

Here we will extend the notion of a cluster in the following way: $C \subseteq A$ is said to be a cluster for a system of binary scores (v_{xy}) when

$$v_{ax} = v_{bx}, \quad v_{xa} = v_{xb}, \qquad \text{whenever } a, b \in C \text{ and } x \notin C. \tag{157}$$

This definition can be viewed as an extension of the preceding one because of the following obvious fact:

Lemma 16.1. *C is a cluster for a relation ρ if and only if C is a cluster for the corresponding system of binary scores, which is defined in* (42).

In particular, the extended notion allows the following results to include the case where the individual votes belong to the general class considered in § 3.3.

In this section we will prove the clone consistency property J: If a set of options is a cluster for each of the individual votes, then: (a) it is a cluster for the social ranking; and (b) contracting it to a single option in all of the individual votes has no other effect in the social ranking than getting the same contraction.

In the remainder of this section we assume the following **standing hypothesis**:

C is a cluster for all of the individual votes.

Since the collective binary scores are obtained by adding up the individual ones (equation (43)), the preceding hypothesis immediately implies that

C is a cluster for the collective binary scores v_{xy}.

In the following we will see that this property of being a cluster is maintained throughout the whole procedure which defines the social ranking.

Lemma 16.2. *Assume that either x or y, or both, lie outside C. In this case*

$$v^*_{xy} = \max \{ v_\gamma \mid \gamma \textit{ contains no more than one element of } C \}$$

Proof. It suffices to see that any path $\gamma = x_0 \dots x_n$ from $x_0 = x$ to $x_n = y$ which contains more than one element of C can be replaced by another one $\widetilde{\gamma}$ which contains only one such element and satisfies $v_{\widetilde{\gamma}} \geq v_\gamma$. Consider first the case where $x, y \notin C$. In this case it will suffice to take $\widetilde{\gamma} = x_0 \dots x_{j-1} x_k \dots x_n$, where $j = \min \{ i \mid x_i \in C \}$ and $k = \max \{ i \mid x_i \in C \}$, which obviously satisfy $0 < j < k < n$. Since $x_{j-1} \notin C$ and $x_j, x_k \in C$, we have $v_{x_{j-1}x_j} = v_{x_{j-1}x_k}$, so that

$$\begin{aligned} v_\gamma &= \min \left(v_{x_0x_1}, \dots, v_{x_{n-1}x_n} \right) \\ &\leq \min \left(v_{x_0x_1}, \dots, v_{x_{j-1}x_j}, v_{x_kx_{k+1}}, \dots, v_{x_{n-1}x_n} \right) \\ &= \min \left(v_{x_0x_1}, \dots, v_{x_{j-1}x_k}, v_{x_kx_{k+1}}, \dots, v_{x_{n-1}x_n} \right) \\ &= v_{\widetilde{\gamma}}. \end{aligned}$$

The case where $x \notin C$ but $y \in C$ can be dealt with in a similar way by taking $\widetilde{\gamma} = x_0 \dots x_{j-1} x_n$, and analogously, in the case where $x \in C$ and $y \notin C$ it suffices to take $\widetilde{\gamma} = x_0 x_{k+1} \dots x_n$. □

Proposition 16.3. *C is a cluster for the indirect scores v^*_{xy}.*

Proof. Consider $a, b \in C$ and $x \notin C$. Let $\gamma = x_0 x_1 x_2 \dots x_n$ be a path from a to x such that $v^*_{ax} = v_\gamma$. By Lemma 16.2, we can assume that a is the only element of γ that belongs to C. In particular, $x_1 \notin C$, so that $v_{ax_1} = v_{bx_1}$, which allows to write

$$\begin{aligned} v^*_{ax} \;=\; v_\gamma \;&=\; \min\,\big(v_{ax_1}, v_{x_1x_2}, \dots, v_{x_{n-1}x}\big) \\ &=\; \min\,\big(v_{bx_1}, v_{x_1x_2}, \dots, v_{x_{n-1}x}\big) \\ &\le\; v^*_{bx}. \end{aligned}$$

By interchanging a and b, one gets the reverse inequality $v^*_{bx} \le v^*_{ax}$ and therefore the equality $v^*_{ax} = v^*_{bx}$. An analogous argument shows that $v^*_{xa} = v^*_{xb}$. □

Proposition 16.4. *C is a cluster for the indirect comparison relation $\nu = \mu(v^*)$.*

Proof. This is an immediate consequence of the preceding proposition. □

Proposition 16.5. *There exists an admissible order ξ such that C is a cluster for ξ.*

Proof. This result is given by Theorem 8.4 of p. 43. □

Theorem 16.6. *C is a cluster for the ranking defined by the rank-like rates (i. e. for the relation $\sigma = \{xy \in \Pi \mid r_x < r_y\}$).*

Proof. We must show that, for any $x \notin C$ and any $a, b \in C$, $r_a < r_x$ implies $r_b < r_x$ and $r_x < r_a$ implies $r_x < r_b$ (from which it follows that $r_a = r_x$ implies $r_b = r_x$). Equivalently, it suffices to show that: (a) $r_a < r_x$ implies $r_b \le r_x$; (b) $r_x < r_a$ implies $r_x \le r_b$; and (c) $r_a = r_x$ implies $r_b = r_x$. The proof will make use of an admissible order ξ with the property that C is a cluster for ξ (whose existence is ensured by Proposition 16.5).

Parts (a) and (b) are then a straightforward consequence of part (a) of Lemma 10.1.(a). In fact, by combining this result, and its contrapositive, with the fact that C is a cluster for ξ, we have the following implications: $r_a < r_x \Rightarrow ax \in \xi \Rightarrow bx \in \xi \Rightarrow r_b \le r_x$, and similarly, $r_x < r_a \Rightarrow xa \in \xi \Rightarrow xb \in \xi \Rightarrow r_x \le r_b$.

Part (c): $r_a = r_x$ implies $r_b = r_x$ (for $x \notin C$ and $a, b \in C$). Since ξ is a total order, we must have either $ax \in \xi$ or $xa \in \xi$; in the following we assume $ax \in \xi$ (the other possibility admits of a similar treatment). In order to deal with this case we will consider the last element of C according to ξ, which we will denote as ℓ, and its immediate successor ℓ', which does not belong to C. Since $a \succeq^{\xi} \ell \succ^{\xi} \ell' \succeq^{\xi} x$ and $r_a = r_x$, we must have $r_\ell = r_{\ell'}$. Now, according to part (b) of Lemma 10.1, $m^\pi_{\ell\ell'} = 0$; in other words, $m^\sigma_{\ell\ell'} = 0$. By the definition of $m^\sigma_{\ell\ell'}$, this means that there exist p and q with $p \succeq^{\xi} \ell \succ^{\xi} q$ such that $m^\nu_{pq} = v^*_{pq} - v^*_{qp} = 0$. Obviously, $q \notin C$, whereas p either belongs to C or it precedes all elements of C. In the latter case, we immediately get $m^\sigma_{cc'} = 0$ for all $c \in C$ (by the definition of $m^\sigma_{cc'}$). If $p \in C$, we arrive at the same conclusion thanks to Proposition 16.3, which ensures that $m^\nu_{cq} = m^\nu_{pq}$. So, the intervals $\gamma_{cc'}$ with $c \in C$ are all of them reduced to a point. Since C is a cluster for the total order ξ, this implies that γ_{ab} is also reduced to the same point (this holds for any $a, b \in C$). According to part (b) of Lemma 10.1, this implies that $r_a = r_b$, as it was claimed. □

Finally, we consider the effect of contracting C to a single element. So we consider a new set $\widetilde{A} = (A \setminus C) \cup \{\widetilde{c}\}$ together with the scores $\widetilde{v}_{xy}$ $(x, y \in \widetilde{A})$ defined by the following equalities, where $p, q \in A \setminus C$ and c is an arbitrary element of C: $\widetilde{v}_{pq} = v_{pq}$, $\widetilde{v}_{p\widetilde{c}} = v_{pc}$ and $\widetilde{v}_{\widetilde{c}q} = v_{cq}$ (the definition is not ambiguous since C is a cluster for the scores v_{xy}). In the following, a tilde is systematically used to distinguish between homologous objects associated respectively with (A, v) and $(\widetilde{A}, \widetilde{v})$. We will also make use of the following notation: for every $x \in A$, $\widetilde{x}$ denotes the element of $\widetilde{A}$ defined by $\widetilde{x} = \widetilde{c}$ if $x \in C$ and by $\widetilde{x} = x$ if $x \notin C$; in terms of this mapping, the preceding equalities say simply that $\widetilde{v}_{\widetilde{x}\widetilde{y}} = v_{xy}$ whenever $\widetilde{x} \neq \widetilde{y}$.

Theorem 16.7. *The ranking $\widetilde{\sigma} = \{xy \in \widetilde{\Pi} \mid \widetilde{r}_x < \widetilde{r}_y\}$ coincides with the contraction of $\sigma = \{xy \in \Pi \mid r_x < r_y\}$ by the cluster C.*

Proof. We begin by noticing that the indirect scores $\widetilde{v}^*_{xy}$ $(x, y \in \widetilde{A})$ coincide with those obtained by contraction of the v^*_{xy} $(x, y \in A)$, i.e. $\widetilde{v}^*_{\widetilde{x}\widetilde{y}} = v^*_{xy}$ whenever $\widetilde{x} \neq \widetilde{y}$. This follows from the analogous equality between the direct scores because of Lemma 16.2. As a consequence, $\widetilde{\nu} = \mu(\widetilde{v}^*)$ coincides with the contraction of $\nu = \mu(v^*)$ by C. From this fact, parts (a) and (b) of Theorem 10.2, allow to derive that $r_x < r_y$ implies $\widetilde{r}_{\widetilde{x}} \leq \widetilde{r}_{\widetilde{y}}$ whenever $\widetilde{x} \neq \widetilde{y}$, and that $\widetilde{r}_{\widetilde{x}} < \widetilde{r}_{\widetilde{y}}$ implies $r_x \leq r_y$.

In order to complete the proof, we must check that $r_x = r_y$ is equivalent to $\widetilde{r}_{\widetilde{x}} = \widetilde{r}_{\widetilde{y}}$ whenever $\widetilde{x} \neq \widetilde{y}$. According to part (b) of Lemma 10.1, it suffices to see that $m^\pi_{xy} = 0$ is equivalent to $\widetilde{m}^\pi_{\widetilde{x}\widetilde{y}} = 0$ whenever $\widetilde{x} \neq \widetilde{y}$. In order to prove this equivalence, we need to look at the way that m^π_{xy} and $\widetilde{m}^\pi_{\widetilde{x}\widetilde{y}}$ are obtained, which requires certain admissible orders ξ and $\widetilde{\xi}$; in this connection, it will be useful that ξ be one of the admissible orders for which C is a cluster (Proposition 16.5), and that $\widetilde{\xi}$ be the corresponding contraction, which is admissible as a consequence of Proposition 16.3. Now, that proposition entails not only that C is a cluster for the indirect margins m^{ν}_{pq}, but also that their contraction by C coincides with the margins of the contracted indirect scores, i. e. $\widetilde{m}^{\nu}_{\widetilde{p}\widetilde{q}} = m^{\nu}_{pq}$ whenever $\widetilde{p} \neq \widetilde{q}$. Moreover, by the definition of the intermediate projected margins, namely equation (21.1), it follows that C is also a cluster for the intermediate projected margins m^{σ}_{pq} and that their contraction by C coincides with the homologous quantities obtained from the contracted indirect margins, i. e. $\widetilde{m}^{\sigma}_{\widetilde{p}\widetilde{q}} = m^{\sigma}_{pq}$ whenever $\widetilde{p} \neq \widetilde{q}$. On the other hand, it is also clear from equation (21.1) that the intermediate projected margins behave in the following way:

$$m^{\sigma}_{pq} \leq m^{\sigma}_{ab} \qquad \text{whenever } a \overset{\xi}{\succeq} p \overset{\xi}{\succ} q \overset{\xi}{\succeq} b. \tag{158}$$

After these remarks, we proceed with showing that $m^{\pi}_{xy} = 0$ is equivalent to $\widetilde{m}^{\pi}_{\widetilde{x}\widetilde{y}} = 0$ whenever $\widetilde{x} \neq \widetilde{y}$. By symmetry, we can assume that $xy \in \xi$, which entails that $\widetilde{x}\widetilde{y} \in \widetilde{\xi}$. In view of (22–24), the equality $m^{\pi}_{xy} = 0$ is equivalent to saying that $m^{\sigma}_{hh'} = 0$ for all h such that $x \overset{\xi}{\succeq} h \overset{\xi}{\succ} y$, and similarly, the equality $\widetilde{m}^{\pi}_{\widetilde{x}\widetilde{y}} = 0$ is equivalent to $\widetilde{m}^{\sigma}_{\eta\eta'} = 0$ for all η such that $\widetilde{x} \overset{\xi}{\succeq} \eta \overset{\xi}{\succ} \widetilde{y}$. By considering a path $x_0x_1 \ldots x_n$ from $x_0 = x$ to $x_n = y$ with x_ix_{i+1} consecutive in ξ, it is clear that the problem reduces to proving the following implications, where ℓ denotes the last element of C by ξ, f denotes the first one, and $'h$ denotes the element that immediately precedes h in ξ: (a) $m^{\sigma}_{\ell\ell'} = 0 \Rightarrow \widetilde{m}^{\sigma}_{\widetilde{c}\ell'} = 0$; (b) $\widetilde{m}^{\sigma}_{\widetilde{c}\ell'} = 0 \Rightarrow m^{\sigma}_{cc'} = 0$ for any $c \in C$; (c) $m^{\sigma}_{'ff} = 0 \Rightarrow \widetilde{m}^{\sigma}_{'f\widetilde{c}} = 0$; and (d) $\widetilde{m}^{\sigma}_{'f\widetilde{c}} = 0 \Rightarrow m^{\sigma}_{'cc} = 0$ for any $c \in C$. Now, (a) and (c) are immediate consequences of the fact that $\widetilde{m}^{\sigma}_{\widetilde{p}\widetilde{q}} = m^{\sigma}_{pq}$ whenever $\widetilde{p} \neq \widetilde{q}$. On the other hand, (b) and (d) follow from the same equality together with the inequality (158). In fact, these facts allow us to write $m^{\sigma}_{cc'} \leq m^{\sigma}_{c\ell'} = \widetilde{m}^{\sigma}_{\widetilde{c}\ell'}$, which gives (b), and similarly, $m^{\sigma}_{'cc} \leq m^{\sigma}_{'fc} = \widetilde{m}^{\sigma}_{'f\widetilde{c}}$, which gives (d). □

17 Approval voting

In approval voting, each voter is asked for a list of approved options, without any expression of preference between them, and each option x is then rated by the number of approvals for it [5]. In the following we will refer to this number as the **approval score** of x, and its value relative to V will be denoted by α_x.

From the point of view of paired comparisons, an individual vote of approval type can be viewed as a truncated ranking where all of the options that appear in it are tied. In this section, we will see that the margin-based variant orders the options exactly in the same way as the approval scores. In other words, the method of approval voting agrees with ours under interpretation (d′) of § 3.2, i. e. under the interpretation that the non-approved options of each individual vote are tied.

Having said that, the preliminary results 17.1–17.3 will hold not only under interpretation (d′) but also under interpretation (d), i. e. that there is no information about the preference of the voter between two non-approved options, and also under the analogous interpretation that there is no information about his preference between two approved options. Interpretation (d′) does not play an essential role until Theorem 17.4, where we use the fact that it always brings the problem into the complete case.

In the following, $\lambda(\alpha)$ denotes the relation defined by

$$xy \in \lambda(\alpha) \quad \equiv \quad \alpha_x > \alpha_y. \tag{159}$$

Proposition 17.1. *In the approval voting situation, the following equality holds:*

$$v_{xy} - v_{yx} \;=\; \alpha_x - \alpha_y. \tag{160}$$

In particular, $\mu(v) = \lambda(\alpha)$*.*

Proof. Obviously, the possible ballots are in one-to-one correspondence with the subsets X of A. In the following, v_X denotes the relative number of votes that approved exactly the set X. With this notation it is obvious that

$$\alpha_x \;=\; \sum_{X \ni x} v_X \;=\; \sum_{\substack{X \ni x \\ X \not\ni y}} v_X + \sum_{\substack{X \ni x \\ X \ni y}} v_X. \tag{161}$$

On the other hand, one has

$$v_{xy} \;=\; \sum_{\substack{X \ni x \\ X \not\ni y}} v_X \;\left(+ \tfrac{1}{2} \sum_{\substack{X \ni x \\ X \ni y}} v_X + \tfrac{1}{2} \sum_{\substack{X \not\ni x \\ X \not\ni y}} v_X \right), \tag{162}$$

where the terms in brackets are present or not depending on which interpretation is used. Anyway, the preceding expressions, together with the analogous ones where x and y are interchanged with each other, result in the equality (160) independently of those alternative interpretations. □

Corollary 17.2. *In the approval voting situation, a path $x_0x_1 \ldots x_n$ is contained in $\mu(v)$ (resp. $\hat{\mu}(v)$) if and only if the sequence α_{x_i} $(i = 0, 1, \ldots n)$ is decreasing (resp. non-increasing).*

Proposition 17.3. *In the approval voting situation, one has* $\mu(w^*) = \lambda(\alpha)$.

Proof. Let us begin by proving that

$$\alpha_x > \alpha_y \implies w^*_{xy} > w^*_{yx}. \tag{163}$$

We will argue by contradiction. So, let us assume that $w^*_{yx} \geq w^*_{xy}$. According to Proposition 17.1, the hypothesis that $\alpha_x > \alpha_y$ is equivalent to $v_{xy} > v_{yx}$, which entails that $w_{xy} > 0$ (by the definition of w_{xy} together with the strict inequality $v_{xy} > v_{yx}$). Now, since $w^*_{xy} \geq w_{xy}$ and we are assuming that $w^*_{yx} \geq w^*_{xy}$, it follows that $w^*_{yx} > 0$. This implies the existence of a path from y to x which is contained in $\hat{\mu}(v)$ (by the definitions of w^*_{yx} and w_{pq}). Finally, Corollary 17.2 produces a contradiction with the present hypothesis that $\alpha_x > \alpha_y$.

Let us see now that

$$\alpha_x = \alpha_y \implies w^*_{xy} = w^*_{yx}. \tag{164}$$

Again, we will argue by contradiction. So, let us assume that $w^*_{xy} \neq w^*_{yx}$. Obviously, it suffices to consider the case $w^*_{xy} > w^*_{yx}$. Now, this inequality implies that $w^*_{xy} > 0$, which tells us that $w^*_{xy} = w_\gamma$ for a certain path $\gamma : x_0x_1 \dots x_n$ which goes from $x_0 = x$ to $x_n = y$ and is contained in $\hat{\mu}(v)$. According to Corollary 17.2, we are ensured that the sequence α_{x_i} $(i = 0, 1, \dots n)$ is non-increasing. However, the hypothesis that $\alpha_x = \alpha_y$ leaves no other possibility than α_{x_i} being constant. So, the reverse path $\gamma' : x_nx_{n-1} \dots x_1x_0$ is also contained in $\hat{\mu}(v)$. Besides, Proposition 17.1 ensures that $v_{x_{i+1}x_i} = v_{x_ix_{i+1}}$, so that $w_{\gamma'} = w_\gamma$. Since $w^*_{yx} \geq w_{\gamma'}$, it follows that $w^*_{yx} \geq w^*_{xy}$, which contradicts the hypothesis that $w^*_{xy} > w^*_{yx}$.

Finally, one easily checks that the preceding implications entail that $\operatorname{sgn}(\alpha_x - \alpha_y)$ is always equal to $\operatorname{sgn}(w^*_{xy} - w^*_{yx})$. This is equivalent to the equality of the relations $\lambda(\alpha)$ and $\mu(w^*)$. □

Theorem 17.4. *In the approval voting situation, the margin-based variant results in a full compatibility relation between the rank-like rates* r_x *and the approval scores* α_x*:* $r_x < r_y \Leftrightarrow \alpha_x > \alpha_y$.

Proof. Recall that the margin-based variant amounts to using interpretation (d′), which always brings the problem into the complete case (when the terms in brackets are included, equation (162) has indeed the property that $v_{xy} + v_{yx} = 1$). So we can invoke Theorem 7.3. By combining it with Proposition 17.3 we see that the inequality $\alpha_x > \alpha_y$ is equivalent to saying that $xy \in \nu$. In the following we will keep this equivalence in mind.

The implication $r_x < r_y \Rightarrow \alpha_x > \alpha_y$ is then an immediate consequence of part (b) of Theorem 10.2. The converse implication $\alpha_x > \alpha_y \Rightarrow r_x < r_y$ can be proved in the following way: Let ξ be an admissible order. By definition, it contains ν. So, the inequality $\alpha_x > \alpha_y$ implies $xy \in \xi$. On the other hand, that inequality implies also the existence of a consecutive pair hh' with $x \overset{\xi}{\succeq} h$ and $h' \overset{\xi}{\succeq} y$ such that $\alpha_h > \alpha_{h'}$. As a consequence, one has $\alpha_p > \alpha_q$ whenever $p \overset{\xi}{\succeq} h$ and $h' \overset{\xi}{\succeq} q$. So, the sets $X = \{p \mid p \overset{\xi}{\succeq} h\}$ and $Y = \{q \mid h' \overset{\xi}{\succeq} q\}$ are in the hypotheses of part (c) of Theorem 10.2, which ensures the desired inequality $r_x < r_y$. □

Remark

So in this case we get a converse of Theorem 10.2.(b). By following the same arguments as in the preceding proof, one can see that such a converse holds whenever there exists a function $s : A \ni x \mapsto s_x \in \mathbb{R}$, such that $xy \in \nu \Leftrightarrow s_x < s_y$.

Summing up, the standard approval voting procedure is always in full agreement with the margin-based variant of the CLC method. In the approval voting situation, this variant amounts to treat all of the candidates which are missing in an approval ballot as equally 'unpreferred' (in the same way that all approved candidates are treated as equally preferred). This is quite reasonable if one can assume that the voters are well acquainted with all of the options.

18 About monotonicity

In this section we consider the effect of raising a particular option a to a more preferred status in the individual ballots without any change in the preferences about the other options. More generally, we consider the case where the scores v_{xy} are modified into new values $\widetilde{v}_{xy}$ such that

$$\widetilde{v}_{ay} \geq v_{ay}, \quad \widetilde{v}_{xa} \leq v_{xa}, \quad \widetilde{v}_{xy} = v_{xy}, \qquad \forall x, y \neq a. \tag{165}$$

In such a situation, one would expect the social rates to behave in the following way, where y is an arbitrary element of $A \setminus \{a\}$:

$$\widetilde{r}_a < r_a, \tag{166}$$

$$r_a < r_y \Longrightarrow \widetilde{r}_a < \widetilde{r}_y, \qquad r_a \leq r_y \Longrightarrow \widetilde{r}_a \leq \widetilde{r}_y, \tag{167}$$

where the tilde indicates the objects associated with the modified scores. Unfortunately, the rating method proposed in this paper does not satisfy these conditions, but generally speaking it satisfies only the following weaker ones:

$$r_a < r_y \implies \widetilde{r}_a \le \widetilde{r}_y. \tag{168}$$

$$(r_a < r_y,\ \forall y \neq a) \implies (\widetilde{r}_a < \widetilde{r}_y,\ \forall y \neq a). \tag{169}$$

In particular, (169) is saying that if a was the only winner for the scores v_{xy}, then it is still the only winner for the scores $\widetilde{v}_{xy}$.

Let us remark that in the case of ranking votes, situation (165) includes the following ones: (a) the option a is raised to a better position in some of the ranking votes without any change in the preferences between the other options; (b) the option a is appended to some ballots which did not previously contain it; (c) some ballots are added which plump for option a. However, the third part of (165) leaves out certain situations which are sometimes considered the matter of other “monotonicity” conditions [*38*].

18.1 This section is devoted to giving a proof of properties (168) and (169).

Theorem 18.1. *Assume that* (v_{xy}) *and* $(\widetilde{v}_{xy})$ *are related to each other in accordance with* (165). *In this case, the following properties are satisfied for any* $x, y \neq a$*:*

$$\widetilde{v}^*_{ay} \ge v^*_{ay}, \qquad \widetilde{v}^*_{xa} \le v^*_{xa}, \tag{170}$$

$$\mathsf{P}_a(\widetilde{\nu}) \subseteq \mathsf{P}_a(\nu), \qquad \mathsf{S}_a(\widetilde{\nu}) \supseteq \mathsf{S}_a(\nu), \tag{171}$$

(168)
$$r_a < r_y \implies \widetilde{r}_a \le \widetilde{r}_y, \tag{172}$$

where $\nu = \mu(v^*)$ *and* $\widetilde{\nu} = \mu(\widetilde{v}^*)$

Proof. Let us begin by seeing that (172) will be a consequence of (171). In fact, we have the following chain of implications: $r_a < r_y \Rightarrow y \in \mathsf{S}_a(\nu) \Rightarrow y \in \mathsf{S}_a(\widetilde{\nu}) \Rightarrow \widetilde{r}_a \le \widetilde{r}_y$, where the central one is provided by (171.2) and the other two are guaranteed by Theorem 10.2.

The proof of (170–171) is organized in three steps. In the first one, we look at the special case where one increases the score of a single pair ab. After this, we will consider the case where an increase in the score of ab is combined with a decrease in the score of ba. Finally, the third step deals with the general situation (165).

Special case 1. Assume that

$$\widetilde{v}_{ab} > v_{ab}, \quad \widetilde{v}_{xy} = v_{xy}, \qquad \forall\, xy \neq ab. \tag{173}$$

In this case, the following properties are satisfied:

$$\widetilde{v}^*_{xy} \geq v^*_{xy}, \qquad \forall x, y \tag{174}$$

$$\widetilde{v}^*_{xa} = v^*_{xa}, \qquad \forall x \neq a \tag{175}$$

$$\widetilde{v}^*_{by} = v^*_{by}, \qquad \forall y \neq b \tag{176}$$

(171)

$$\mathsf{P}_a(\widetilde{\nu}) \subseteq \mathsf{P}_a(\nu), \qquad \mathsf{S}_a(\widetilde{\nu}) \supseteq \mathsf{S}_a(\nu), \tag{177}$$

$$\mathsf{P}_b(\widetilde{\nu}) \supseteq \mathsf{P}_b(\nu), \qquad \mathsf{S}_b(\widetilde{\nu}) \subseteq \mathsf{S}_b(\nu). \tag{178}$$

In fact, under the hypothesis (173) it is obvious that $\widetilde{v}_\gamma \geq v_\gamma$ and that the strict inequality happens only when the path $\gamma = x_0 \ldots x_n$ contains the pair ab and the latter realizes the minimum of the scores $v_{x_i x_{i+1}}$. As a consequence, the indirect scores satisfy the inequality (174). Furthermore, a strict inequality in (174) implies that the maximum which defines $\widetilde{v}^*_{xy}$ is realized by a path γ which satisfies $\widetilde{v}_\gamma > v_\gamma$ and therefore contains the pair ab.

Now, in order to obtain the indirect score for a pair of the form xa it is useless to consider paths involving ab, since such paths contains cycles whose omission results in paths not involving ab and having a better or equal score. So, the maximum which defines $\widetilde{v}^*_{xa}$ is realized by a path which does not involve ab. According to the last statement of the preceding paragraph, this implies (175). An entirely analogous argument establishes (176).

Finally, (177) is obtained in the following way: $x \in \mathsf{P}_a(\widetilde{\nu})$ means that $\widetilde{v}^*_{xa} > \widetilde{v}^*_{ax}$, from which (175) and (174) allow to derive that $v^*_{xa} = \widetilde{v}^*_{xa} > \widetilde{v}^*_{ax} \geq v^*_{ax}$, i. e. $x \in \mathsf{P}_a(\nu)$. Similarly, $x \in \mathsf{S}_a(\nu)$ implies $x \in \mathsf{S}_a(\widetilde{\nu})$ because one has $\widetilde{v}^*_{ax} \geq v^*_{ax} > v^*_{xa} = \widetilde{v}^*_{xa}$. An analogous argument establishes (178).

Special case 2. Properties (170–171) *are satisfied in the following situation:*

$$\widetilde{v}_{ab} \geq v_{ab}, \quad \widetilde{v}_{ba} \leq v_{ba}, \quad \widetilde{v}_{xy} = v_{xy}, \qquad \forall\, xy \neq ab, ba. \tag{179}$$

This result will be obtained from the preceding one by going through an intermediate Llull matrix $\widetilde{\widetilde{v}}$ defined in the following way

$$\begin{cases} \widetilde{\widetilde{v}}_{ab} = v_{ab} \\ \widetilde{\widetilde{v}}_{ba} = \widetilde{v}_{ba} \\ \widetilde{\widetilde{v}}_{xy} = \widetilde{v}_{xy} = v_{xy}, \quad \forall\, xy \neq ab, ba \end{cases} \tag{180}$$

If the hypothesis $\widetilde{v}_{ab} \geq v_{ab}$ is satisfied with strict inequality, then $\widetilde{v}$ and $\widetilde{\widetilde{v}}$ are in the hypotheses of the special case 1 (they play respectively the roles of $\widetilde{v}$ and v). In particular, we get

$$\widetilde{v}^*_{xy} \geq \widetilde{\widetilde{v}}^*_{xy}, \quad \widetilde{v}^*_{xa} = \widetilde{\widetilde{v}}^*_{xa}, \quad \mathsf{P}_a(\widetilde{\nu}) \subseteq \mathsf{P}_a(\widetilde{\widetilde{\nu}}), \quad \mathsf{S}_a(\widetilde{\nu}) \supseteq \mathsf{S}_a(\widetilde{\widetilde{\nu}}). \tag{181}$$

On the other hand, if $\widetilde{v}_{ab} = v_{ab}$ then $\widetilde{\widetilde{v}} = \widetilde{v}$ and the preceding relations hold as equalities.

Similarly, if the hypothesis $\widetilde{v}_{ba} \le v_{ba}$ is satisfied with strict inequality, then v and $\widetilde{\widetilde{v}}$ are in the hypotheses of the special case 1 with ab replaced by ba (they play respectively the roles of $\widetilde{v}$ and v). In particular, we get

$$v^*_{xy} \ge \widetilde{\widetilde{v}}^*_{xy}, \quad v^*_{ay} = \widetilde{\widetilde{v}}^*_{ay}, \quad \mathsf{P}_a(\nu) \supseteq \mathsf{P}_a(\widetilde{\widetilde{\nu}}), \quad \mathsf{S}_a(\nu) \subseteq \mathsf{S}_a(\widetilde{\widetilde{\nu}}). \tag{182}$$

As before, if $\widetilde{v}_{ba} = v_{ba}$ then $\widetilde{\widetilde{v}} = v$ and the preceding relations hold as equalities.

Finally, (170–171) are obtained by combining (181) and (182):

$$\begin{aligned} \widetilde{v}^*_{ay} &\ge \widetilde{\widetilde{v}}^*_{ay} = v^*_{ay}, \\ \widetilde{v}^*_{xa} &= \widetilde{\widetilde{v}}^*_{xa} \le v^*_{xa}, \\ \mathsf{P}_a(\widetilde{\nu}) &\subseteq \mathsf{P}_a(\widetilde{\widetilde{\nu}}) \subseteq \mathsf{P}_a(\nu), \\ \mathsf{S}_a(\widetilde{\nu}) &\supseteq \mathsf{S}_a(\widetilde{\widetilde{\nu}}) \supseteq \mathsf{S}_a(\nu). \end{aligned}$$

General case. In the general situation (165), properties (170–171) are a direct consequence of the successive application of the special case 2 to every pair ay. ☐

Corollary 18.2. *Under the hypothesis of Theorem 18.1 one has also*

$$\varphi_a > \varphi_y \;\Rightarrow\; \widetilde{\varphi}_a \ge \widetilde{\varphi}_y. \tag{183}$$

Proof. It suffices to combine (172) with Theorem 12.3. ☐

Corollary 18.3 ([2])**.** *Under the hypothesis of Theorem 18.1 one has also the property* (169)*.*

Proof. According to Theorem 10.2.(b), the left-hand side of (169) implies the strict inequality $v^*_{ay} > v^*_{ya}$ for all $y \ne a$. Now, this inequality can be combined with (170) to derive that $\widetilde{v}^*_{ay} > \widetilde{v}^*_{ya}$ for all $y \ne a$. Finally, Theorem 10.2.(c) with $X = \{a\}$ and $Y = A \setminus \{a\}$ guarantees that the right-hand side of (169) is satisfied. ☐

[2] We thank Markus Schulze for pointing out this fact.

18.2 The statements (166) and (167) can fail even in the complete case. Next we give an example of it, with 5 options (it seems to be the minimum for the failure of (167)) and 10 voters. The only change from left to right is one inversion in one of the votes; more specifically, the eighth ballot changes from the order $\mathsf{d} \succ \mathsf{b} \succ \mathsf{c} \succ \mathsf{a} \succ \mathsf{e}$ to the new one $\mathsf{b} \succ \mathsf{d} \succ \mathsf{c} \succ \mathsf{a} \succ \mathsf{e}$. In spite of this change, favourable to b and disadvantageous to d, the rank-like rate of b is worsened from 2.90 to 3.00, whereas that of d is improved from 3.10 to 3.00. This contradicts (166) for $a = \mathsf{b}$, as well as (167.1) for $a = \mathsf{b}$ and $y = \mathsf{d}, \mathsf{c}$, and also (167.2) for $a = \mathsf{d}$ and $y = \mathsf{a}, \mathsf{b}$ (when one goes from right to left). However, it complies with (168).

x	*Ranking votes*									
a	1	2	2	2	2	3	3	4	4	5
b	3	1	3	5	5	1	5	**2**	1	1
c	5	3	4	3	3	2	2	3	3	2
d	2	5	1	4	4	4	4	**1**	5	3
e	4	4	5	1	1	5	1	5	2	4

x	*Ranking votes*									
a	1	2	2	2	2	3	3	4	4	5
b	3	1	3	5	5	1	5	**1**	1	1
c	5	3	4	3	3	2	2	3	3	2
d	2	5	1	4	4	4	4	**2**	5	3
e	4	4	5	1	1	5	1	5	2	4

	V_{xy}				
x	a	b	c	d	e
a	$*$	5	5	7	5
b	5	$*$	7	**4**	7
c	5	3	$*$	7	5
d	3	**6**	3	$*$	5
e	5	3	5	5	$*$

	$\widetilde{V}_{xy}$				
x	a	b	c	d	e
a	$*$	5	5	7	5
b	5	$*$	7	**5**	7
c	5	3	$*$	7	5
d	3	**5**	3	$*$	5
e	5	3	5	5	$*$

	V^*_{xy}				
x	a	b	c	d	e
a	$*$	6	6	7	6
b	5	$*$	7	7	7
c	5	6	$*$	7	6
d	5	6	6	$*$	6
e	5	5	5	5	$*$

	$\widetilde{V}^*_{xy}$				
x	a	b	c	d	e
a	$*$	5	5	7	5
b	5	$*$	7	7	7
c	5	5	$*$	7	5
d	5	5	5	$*$	5
e	5	5	5	5	$*$

	M^ν_{xy}				
x	a	b	c	d	e
a	$*$	1	1	2	1
b	$*$	$*$	1	1	2
c	$*$	$*$	$*$	1	1
d	$*$	$*$	$*$	$*$	1
e	$*$	$*$	$*$	$*$	$*$

	$\widetilde{M}^\nu_{xy}$				
x	a	b	c	d	e
a	$*$	0	0	2	0
b	$*$	$*$	2	2	2
c	$*$	$*$	$*$	2	0
d	$*$	$*$	$*$	$*$	0
e	$*$	$*$	$*$	$*$	$*$

x	M^{π}_{xy} a	b	c	d	e	r_x
a	$*$	1	1	1	1	2.80
b	$*$	$*$	1	1	1	2.90
c	$*$	$*$	$*$	1	1	3.00
d	$*$	$*$	$*$	$*$	1	3.10
e	$*$	$*$	$*$	$*$	$*$	3.20

x	$\widetilde{M}^{\pi}_{xy}$ a	b	c	d	e	$\widetilde{r}_x$
a	$*$	0	0	0	0	3.00
b	$*$	$*$	0	0	0	3.00
c	$*$	$*$	$*$	0	0	3.00
d	$*$	$*$	$*$	$*$	0	3.00
e	$*$	$*$	$*$	$*$	$*$	3.00

As one can see, the multiple zeroes present in $\widetilde{M}^{\nu}_{xy}$ force a complete tie of the rank-like rates $\widetilde{r}_x$ in spite of the fact that $\widetilde{\nu}$ is not empty. Not only the latter contains the pair bd, but in fact $\widetilde{M}^{\nu}_{\mathsf{bd}} = 2 > 1 = M^{\nu}_{\mathsf{bd}}$.

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